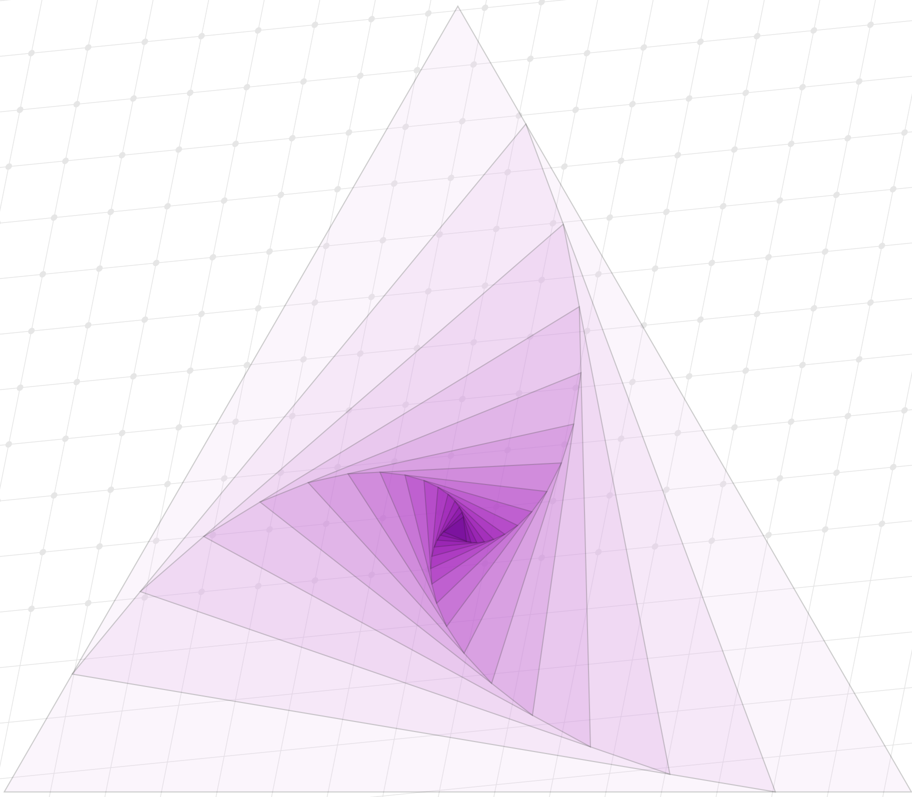
University *of Ljubljana*
Faculty *of Mathematics and Physics*

Department of Physics

Quantum Field Theory

Unofficial notes



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1 Introduction

1.1 Notation

In these notes we will work in *God-given* units where:

$$\hbar = c = 1 \, . \tag{1.1.1}$$

In this system of units:

$$[\text{length}] = [\text{time}] = [\text{mass}]^{-1} = [\text{energy}]^{-1} \, . \tag{1.1.2}$$

This is a common choice in Quantum Field Theory and is also the system used in the book by Peskin and Schroeder [1], which heavily influenced and inspired this text. The mass of a particle is thus given by its rest energy or its inverse Compton wavelength:

$$m = E_0 = \frac{1}{\lambda} \, . \tag{1.1.3}$$

As is also common in QFT, we will use the metric signature $(+, -, -, -)$. This means that the Minkowski metric is given by:

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \, . \tag{1.1.4}$$

4-vectors are thus marked as $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x^i)$. Using the metric, we can raise and lower indices:

$$x_\mu = g_{\mu\nu} x^\nu \qquad x^\mu = g^{\mu\nu} x_\nu \, . \tag{1.1.5}$$

Covariant derivatives ∂_μ and contravariant derivatives ∂^μ are defined as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla \right) \, , \tag{1.1.6}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial x_0}, -\nabla \right) \, . \tag{1.1.7}$$

$$\frac{\partial}{\partial a_\mu}(a_\nu b^\nu) = \delta^\mu{}_\nu b^\nu = b^\mu \, . \tag{1.1.8}$$

In Minkowski space it also holds that $g^\mu{}_\nu = \delta^\mu{}_\nu$. We will also use the Einstein summation convention, which means that whenever an index appears twice in a term, it is summed over. For example:

$$A^\mu B_\mu = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \, . \tag{1.1.9}$$

Vectors will be denoted by boldface, e.g. $\boldsymbol{p}$, and the magnitude of a vector will be denoted by $|\boldsymbol{p}|$. The Hermitian adjoint of an operator $\hat{A}$ will be denoted by $\hat{A}^\dagger = (\hat{A}^*)^\mathsf{T}$. Same goes for a column vector v, where $v^\dagger = (v^*)^\mathsf{T}$ is a row vector with all components complex conjugated. The complex conjugate of a scalar quantity a will be denoted by a^*.

1.2 Why Quantum Field Theory?

We use Quantum Field Theory (QFT) to describe the fundamental interactions of particles. This theory successfully describes the electromagnetic, weak and strong forces. It is often called the *Basic Theory of Nature* because it is the most fundamental theory we have. It correctly joins the concepts of Quantum Mechanics and Special Relativity.

QFT is also very practical since it uniquely describes the possible interactions of particles as opposed to one-particle Quantum Mechanics where we have complete freedom in choosing the potential. In QFT, the interactions are fixed by the symmetries of the theory. For example in Quantum Electrodynamics (QED), we add the gauge symmetry $U(1)$ to the free theory to get the full interacting theory. The standard model of particle physics is a QFT with the gauge symmetries $SU(3)_C \times SU(2)_L \times U(1)_Y$.

1.3 Fields

Fields are the basic building blocks of Quantum Field Theory. We're already familiar with them from classical physics, where they describe the state of a system at every point in space and time. In Quantum Field Theory, fields are promoted to operators, which act on a quantum state and create or destroy particles. Fields in QFT differ with respect to spin. Scalar fields have spin 0, vector fields have spin 1, spinor fields have spin 1/2 and so on. Particles are then excitations of these fields.

Classical scalar fields

- Temperature field $T(\boldsymbol{x}, t)$
- Pressure field $p(\boldsymbol{x}, t)$
- Density field $\rho(\boldsymbol{x}, t)$

Classical vector fields

- Velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$
- Electric field $\boldsymbol{E}(\boldsymbol{x}, t)$
- Deformation field $\boldsymbol{u}(\boldsymbol{x}, t)$

Quantum fields

- Scalar field $\phi(\boldsymbol{x}, t)$
- Vector field $A^\mu(\boldsymbol{x}, t)$
- Spinor field $\psi(\boldsymbol{x}, t)$

In Quantum Mechanics we can describe a particle with $a^\dagger|0\rangle$, where $a^\dagger$ is the creation operator. In QFT the same representation is given by $\hat{\phi}^\dagger(\boldsymbol{x}, t)|0\rangle$, where $\hat{\phi}^\dagger(\boldsymbol{x}, t)$ is the creation operator of the field $\phi(\boldsymbol{x}, t)$.
Friendly Reminder: Remember that in QFT, fields are operators which in general do not commute.

1.4 Why fields and not particles?

This question is analogous to asking why we prefer to use fields instead of Relativistic Quantum Mechanics. We cannot use one-particle Quantum Mechanics to describe the interactions of particles. Even if we were to describe the process $e^+e^- \to \gamma \to \mu^+\mu^-$, we can not localize the particles in space and time such that pairs of particle-antiparticle are not spontaneously created due the uncertainty principle. The second reason is **Causality**. Measurements of quantities are not allowed to be casually related. Relativistic Quantum Mechanics does not satisfy this requirement.

Example for Non-Relativistic Quantum Mechanics: Say we want to evolve a state $|\boldsymbol{x}_0\rangle$ to a state $|\boldsymbol{x}\rangle$ at time t. We can do this by applying the time evolution operator $\hat{U}(t)$:

$$U(t) \sim \langle \boldsymbol{x}|e^{-i\hat{H}t}|\boldsymbol{x}_0\rangle \,. \tag{1.4.1}$$

We can insert the momentum completeness relation into this expression, which on its own is equal to the identity operator:

$$\mathbb{1} = \int \mathrm{d}^3p \, |\boldsymbol{p}\rangle\langle\boldsymbol{p}| \,. \tag{1.4.2}$$

Taking into account that we're observing a free particle where $\hat{H} = \hat{\boldsymbol{p}}^2/2m$, we can write the following expression:

$$\begin{aligned} \mathbb{1} &\propto \int \mathrm{d}\boldsymbol{p} \, e^{-i\frac{\boldsymbol{p}^2}{2m}t}\langle\boldsymbol{x}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\boldsymbol{x}_0\rangle \\ &\propto \int \mathrm{d}\boldsymbol{p} \, e^{-i\frac{\boldsymbol{p}^2}{2m}t}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_0)} \,. \end{aligned} \tag{1.4.3}$$

Looking at only the x coordinate we get the following:

$$\mathbb{1}_x \propto \int_{-\infty}^{\infty} \mathrm{d}p_x \, e^{-i\frac{p_x^2}{2m}t+ip_x(x-x_0)} \propto \frac{1}{\sqrt{t}} \exp \frac{i(x-x_0)^2 m}{2t} \, . \tag{1.4.4}$$

Thus the time evolution operator is given by:

$$U(t) \propto \frac{1}{t^{3/2}} \exp \frac{i(\boldsymbol{x}-\boldsymbol{x}_0)^2 m}{2t} \, . \tag{1.4.5}$$

This means that $|U| \neq 0$ for every combination of $\boldsymbol{x}$, $\boldsymbol{x}_0$ and t which violates causality.

Example for Relativistic Quantum Mechanics: Analogous to the non-relativistic case, we can write the time evolution operator as:

$$U(t) \sim \langle \boldsymbol{x} | e^{-i\hat{H}t} | \boldsymbol{x}_0 \rangle \, , \tag{1.4.6}$$

however in this case the Hamiltonian is given by $\hat{H} = \sqrt{\hat{\boldsymbol{p}}^2 + m^2}$. After inserting the identity operator in the form of the momentum completeness relation we get:

$$U(t) \propto \int \mathrm{d}\boldsymbol{p} \, e^{-i\sqrt{\boldsymbol{p}^2+m^2}t} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}_0)} \, . \tag{1.4.7}$$

Integrating over the expression we get:

$$U(t) \propto \int_{-1}^{1} \mathrm{d}(\cos\theta) \, 2\pi \int_0^{\infty} p^2 \mathrm{d}p e^{-\cdots} e^{ip|\boldsymbol{x}-\boldsymbol{x}_0|\cos\theta} \, . \tag{1.4.8}$$

This integral is not easy to solve but using something like Wolfram Mathematica we can come to the conclusion that $|U| > 0$ also for $|\boldsymbol{x}| > ct$ which violates causality.

1.5 Comparison of Classical Mechanics and Field Theory

Classical Mechanics

In classical mechanics we can describe the state of a system using generalized coordinates q_i and generalized momenta p_i. These quantities are functions of (for example) time. The Lagrangian of the system is some function of the generalized coordinates, their time derivatives and possibly time. The generalized momenta are given by the partial derivative of the Lagrangian with respect to time derivatives of the generalized coordinates. Mathematically formulated:

$$L(q_i, \dot{q}_i, t) \, , \tag{1.5.1}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \, . \tag{1.5.2}$$

Classical Field Theory

In classical field theory we can describe the state of a discreet system using fields $\phi(\boldsymbol{x}_j, t)$. We can think of ϕ as the generalized coordinates which are functions of both space aand time. The Lagrangian density of the system is some function of the field of generalized coordinates and their space and time derivatives. The conjugated momenta are given by the partial derivative of the Lagrangian density with respect to the time derivatives of the field. Mathematically formulated:

$$\mathcal{L}(\phi(\boldsymbol{x}_j, t), \partial_\mu \phi(\boldsymbol{x}, t)) \, , \tag{1.5.3}$$

$$\Pi(\boldsymbol{x}_j, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(\boldsymbol{x}_j, t))} \, . \tag{1.5.4}$$

where we have used the notation $\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \nabla)$.

To describe a continuum of particles we essentially have an infinite number of generalized coordinates. Thus we use the arguments $\boldsymbol{x}$ and t to describe the field $\phi(\boldsymbol{x}, t)$ and the previous relations become:

$$\mathcal{L}(\phi(\boldsymbol{x}, t), \partial_\mu \phi(\boldsymbol{x}, t)) \, , \tag{1.5.5}$$

$$\Pi(\boldsymbol{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(\boldsymbol{x}, t))} \, . \tag{1.5.6}$$

Quantum Mechanics

In Quantum Mechanics we promote the generalized coordinates and momenta to operators $\hat{x}_i(t)$ and $\hat{p}_i(t)$. These operators act on a quantum state and create or destroy particles. The Hamiltonian of the system is some function of the operators and possibly time. The commutation relation between the operators is given by:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \ . \tag{1.5.7}$$

$$[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \ . \tag{1.5.8}$$

Quantum Field Theory

In Quantum Field Theory we promote the fields to operators. As such we have the field operator $\hat{\phi}(\boldsymbol{x}, t)$ and the conjugated momentum operator $\hat{\Pi}(\boldsymbol{x}, t)$.

$$[\hat{\phi}_a(\boldsymbol{x}_j, t), \hat{\Pi}_b(\boldsymbol{y}_k, t)] = i\hbar\delta_{ab}\delta_{jk} \ . \tag{1.5.9}$$

$$[\hat{\phi}_a(\boldsymbol{x}_j, t), \hat{\phi}_b(\boldsymbol{y}_k, t)] = [\hat{\Pi}_a(\boldsymbol{x}_j, t), \hat{\Pi}_b(\boldsymbol{y}_k, t)] = 0 \ . \tag{1.5.10}$$

Again in the case of a continuum we use an infinite number of generalized coordinates. This is called the **Canonical Quantization** of the field. The field operator is then $\hat{\phi}(\boldsymbol{x}, t)$ and the conjugated momentum operator is $\hat{\Pi}(\boldsymbol{x}, t)$. The commutation relations are given by:

$$[\hat{\phi}_a(\boldsymbol{x}, t), \hat{\Pi}_b(\boldsymbol{y}, t)] = i\hbar\delta_{ab}\delta(\boldsymbol{x} - \boldsymbol{y}) \ . \tag{1.5.11}$$

$$[\hat{\phi}_a(\boldsymbol{x}, t), \hat{\phi}_b(\boldsymbol{y}, t)] = [\hat{\Pi}_a(\boldsymbol{x}, t), \hat{\Pi}_b(\boldsymbol{y}, t)] = 0 \ . \tag{1.5.12}$$

1.6 Pictures in Quantum Field Theory

1.6.1 Schrödinger Picture

Schrödinger's picture is the most common formulation of Quantum Mechanics in which states evolve in time while operators remain constant (that is, time-independent). The state of a system is given by a state vector $|\psi\rangle$. This state can be evolved in time by the time evolution operator $\hat{U}(t)$. In the case of a time-independent Hamiltonian, the time evolution operator is given by:

$$\hat{U}(t) = e^{-i\hat{H}t} \ , \tag{1.6.1}$$

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \ . \tag{1.6.2}$$

However if our Hamiltonian is time-dependent, where Hamiltonians at different times do not commute, the time evolution operator is given by:

$$\hat{U}(t) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}(t')\mathrm{d}t'\right) \ , \tag{1.6.3}$$

where $\mathcal{T}$ is the time-ordering operator (which we will introduce later). This means that in the Schrödinger picture the field operators are time-independent while the states evolve in time.

$$\hat{x},\ \hat{p} \quad \rightarrow \quad \hat{\phi}(\boldsymbol{x}),\ \hat{\Pi}(\boldsymbol{x}) \ . \tag{1.6.4}$$

1.6.2 Heisenberg Picture

In the Heisenberg picture, states are time-independent while operators evolve in time. The state of a system is given by a state vector $|\psi\rangle$. The time evolution of an operator $\hat{A}$ is given by:

$$\hat{A}(t) = \hat{U}^\dagger(t)\hat{A}\hat{U}(t) \ , \tag{1.6.5}$$

where $\hat{U}(t)$ is the time evolution operator. The time evolution operator is given by:

$$\hat{U}(t) = e^{-i\hat{H}t} \ . \tag{1.6.6}$$

Field operators in the Heisenberg picture are time dependent:

$$\hat{x}(t),\ \hat{p}(t) \quad \rightarrow \quad \hat{\phi}(\boldsymbol{x}, t),\ \hat{\Pi}(\boldsymbol{x}, t). \tag{1.6.7}$$

1.6.3 Interaction/Dirac Picture

The Interaction Picture, also known as the Dirac Picture, is an intermediate representation between the Schrödinger and Heisenberg pictures. In this picture, states as well as operators are time-dependent. The interaction picture is useful when dealing with changes to both the states and operators due to interactions. To switch into the interaction picture we split the Schrodinger picture Hamiltonian into two parts:

$$\hat{H} = \hat{H}_0 + \hat{H}_I(t) \, , \tag{1.6.8}$$

where $\hat{H}_0$ is the free Hamiltonian which we know how to solve and $\hat{H}_I$ is the interaction Hamiltonian which is harder to analyze. Another common choice is to split the Hamiltonian into a time-independent and a time-dependent part. Let $|\psi_S(t)\rangle = e^{-iH_s t}|\psi(0)\rangle$ be the time-dependent state vector in the Schrödinger picture. The state vector in the interaction picture is defined with an additional time-dependent unitary transformation as such:

$$|\psi_I(t)\rangle = e^{iH_{0,S}t}|\psi_S(t)\rangle \, . \tag{1.6.9}$$

An operator in the interaction picture is defined as:

$$\hat{A}_I(t) = e^{iH_{0,S}t}\hat{A}_S(t)e^{-iH_{0,S}t} \, . \tag{1.6.10}$$

Since we take $\hat{A}_S$ from the Schrödinger picture, it is generally time-independent. In the case of a free Hamiltonian the interaction picture is the same as the Schrödinger picture, as there is no included interactions:

$$H_{0,I} = e^{iH_{0,S}t}H_{0,S}e^{-iH_{0,S}t} = H_{0,S} \, . \tag{1.6.11}$$

This holds true since operators commute with differentiable functions of themselves. For the interaction Hamiltonian we generally have:

$$H_{I,I} = e^{iH_{0,S}t}H_{I,S}e^{-iH_{0,S}t} \, . \tag{1.6.12}$$

The operators are the same between the Schrödinger and interaction picture if $[H_{0,S}, H_{I,S}] = 0$. Transforming Schrodinger's equation into the interaction picture we get:

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \hat{H}_I(t)|\psi_I(t)\rangle \, . \tag{1.6.13}$$

If the operator $\hat{A}_S$ is time-independent then the corresponding time evolution for the operator in the interaction picture is given by:

$$i\hbar\frac{\partial}{\partial t}\hat{A}_I(t) = [\hat{A}_I(t), \hat{H}_{0,S}] \, . \tag{1.6.14}$$

In the interaction picture operators evolve in time like operators in the Heisenberg picture with the Hamiltonian $\hat{H}_{0,S}$. Just as a sanity check, we can see that expectation values of operators in the interaction picture are the same as in the Schrödinger picture:

$$\langle\hat{A}_I(t)\rangle = \langle\psi_I(t)|\hat{A}_I(t)|\psi_I(t)\rangle = \langle\psi_S(t)|e^{-iH_{0,S}t}e^{iH_{0,S}t}\hat{A}_S(t)e^{-iH_{0,S}t}e^{iH_{0,S}t}|\psi_S(t)\rangle \, . \tag{1.6.15}$$

2 Real Scalar Fields

Excitations of the real scalar field represent electrically neutral scalar particles. Given that the field is real, we have the property that $\phi(\boldsymbol{x}, t) = \phi^*(\boldsymbol{x}, t)$. We want to find the equations of motion for such fields where we must make sure that solutions are **Lorentz invariant** and that they obey relativistic kinematics. Solutions are Lorentz invariant if, for a Lorentz transformation Λ, it holds that $\phi(\Lambda\boldsymbol{x}, \Lambda t)$ also solves the equations of motion if $\phi(\boldsymbol{x}, t)$ does. We'll see that the **Klein-Gordon equation** satifies these requirements. It's given by:

$$\left(\Box + m^2\right)\phi(\boldsymbol{x}, t) = \partial_\mu\partial^\mu\phi(\boldsymbol{x}, t) + m^2\phi(\boldsymbol{x}, t) = 0 \, , \tag{2.0.1}$$

where $\Box = \partial_\mu\partial^\mu$ is the d'Alembert operator. The Klein-Gordon equation is the relativistic generalization of the Schrödinger equation. We can quickly see that the Klein-Gordon equation obeys relativistic kinematics by considering the energy-momentum relation for a free particle. Assuming that we have solutions in the form of plane waves $\phi(\boldsymbol{x},t) = Ae^{i(p\cdot x)}$, we can find the energy-momentum relation by inserting the solution into the Klein-Gordon equation:

$$A(\pm i\boldsymbol{x})^2 e^{\pm ip\cdot x} + Am^2 e^{\pm ip\cdot x} = 0\,, \tag{2.0.2}$$

$$-p^2 + m^2 = 0 \quad \Rightarrow \quad p = (E, \boldsymbol{p}) \quad p^2 = p_\mu p^\mu\,, \tag{2.0.3}$$

$$\Rightarrow E^2 = \boldsymbol{p}^2 + m^2\,. \tag{2.0.4}$$

Note: Here $p \cdot x$ denotes the 4-vector dot product $p_\mu x^\mu = Et - \boldsymbol{p}\cdot\boldsymbol{x}$.

2.1 Basics of Classical Field Theory

Before we get into the details of the real scalar field, we need to understand the basics of classical field theory. Lets start with the Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$. We will derive the equations of motion using the Euler-Lagrange equations. The Euler-Lagrange equations come from the principle of least action, where we slightly perturb the field $\phi \to \phi + \delta\phi$ and require that the action is stationary. The action is defined as:

$$S = \int_{t_1}^{t_2} \mathrm{d}t \int \mathrm{d}\boldsymbol{x}\mathcal{L}\,. \tag{2.1.1}$$

To keep the action stationary we require:

$$\delta\phi|_{t_1,t_2} = 0\,, \tag{2.1.2}$$

$$\delta\phi|_{|\boldsymbol{x}|\to\infty} = 0\,. \tag{2.1.3}$$

$$\text{which results in} \quad \delta S = 0\,. \tag{2.1.4}$$

The Euler-Lagrange equations are derived by considering the variation of the action:

$$\delta S = \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\partial_\mu\phi\right]\,, \tag{2.1.5}$$

where we can substitute the second term from the expression for the full derivative:

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right] = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right]\delta\phi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\mu\delta\phi\,. \tag{2.1.6}$$

Thus we are left with:

$$\delta S = 0 = \int \mathrm{d}t \int \mathrm{d}\boldsymbol{x} \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\delta\phi\right)\right]\,, \tag{2.1.7}$$

where the last term becomes an integral over the edge of the volume. Since we demand that $\delta\phi = 0$ when $|\boldsymbol{x}| \to \infty$, this term vanishes. Thus $\delta S = 0$ for all $\delta\phi$ when it holds that:

$$\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right] = 0\,, \tag{2.1.8}$$

which are called the **Euler-Lagrange equations**.

The Hamiltonian and the Conjugated Momentum

We are still in the classical regime where from regular classical mechanics we know that we can define the generalized momenta as:

$$p_i = \frac{\partial L}{\partial\dot{q}_i} \tag{2.1.9}$$

where L is the Lagrangian. The Hamiltonian is then defined as:

$$H = \sum_i p_i\dot{q}_i - L\,. \tag{2.1.10}$$

In the context of field theory the Lagrangian is replaced by the Lagrangian density $\mathcal{L}(\phi(\boldsymbol{x}_j), \partial_\mu\phi(\boldsymbol{x}_j))$ as follows:

$$L = \int \mathrm{d}\boldsymbol{x}\ \mathcal{L} = \sum_j \Delta\boldsymbol{x}\mathcal{L}(\boldsymbol{x}_j)\ . \tag{2.1.11}$$

Taking this change into account we can find the conjugated momentum just like in the case of classical mechanics:

$$p(\boldsymbol{x}_i) = \frac{\partial L}{\partial\dot{\phi}(\boldsymbol{x}_i)} = \frac{\partial\sum_j \Delta\boldsymbol{x}\mathcal{L}(\phi(\boldsymbol{x}_j))}{\partial\dot{\phi}(\boldsymbol{x}_i)} = \Delta\boldsymbol{x}\ \frac{\partial\mathcal{L}}{\partial(\dot{\phi}(\boldsymbol{x}_i))}\ . \tag{2.1.12}$$

From here we can read that the conjugated momentum is defined as:

$$\Pi(\boldsymbol{x}_i) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\boldsymbol{x}_i)}\ . \tag{2.1.13}$$

As in the case of classical mechanics we can now write the Hamiltonian as:

$$H = \sum_i \Delta\boldsymbol{x}\Pi(\boldsymbol{x}_i)\dot{\phi}(\boldsymbol{x}_i) - \sum_i \Delta\boldsymbol{x}\mathcal{L}(\boldsymbol{x}_i) = \sum_i \Delta\boldsymbol{x}\left(\Pi(\boldsymbol{x}_i)\dot{\phi}(\boldsymbol{x}_i) - \mathcal{L}(\boldsymbol{x}_i)\right)\ , \tag{2.1.14}$$

from which we can read the Hamiltonian density as:

$$\mathcal{H} = \Pi(\boldsymbol{x}_i)\dot{\phi}(\boldsymbol{x}_i) - \mathcal{L}(\boldsymbol{x}_i)\ . \tag{2.1.15}$$

2.1.1 Exercise: Classical Quantization of the Deformation Field in a Crystal Lattice

Consider a 1D crystal lattice with atoms of mass m and lattice spacing a. We can denote the displacement of the n-th atom from its equilibrium position as $\tilde{\phi}_n(t)$. We want to find the Lagrangian of the system and quantize it. From classical mechanics we know that:

$$L = T - V\ , \tag{2.1.16}$$

$$T = \sum_n \frac{m}{2}\dot{\tilde{\phi}}_n^2\ , \tag{2.1.17}$$

$$V = \sum_n \frac{k_s}{2}(\tilde{\phi}_{n+1} - \tilde{\phi}_n)^2\ , \tag{2.1.18}$$

where k_s is the spring constant. To generalize the displacements $\tilde{\phi}_n(t)$ to a field $\phi(x_n)$ lets denote:

$$\tilde{\phi}_n(t) \to a^{1/2}\phi(x_n = na, t)\ , \tag{2.1.19}$$

where a is the lattice spacing. The Lagrangian then reads:

$$L = \sum_n \left[\frac{ma}{2}\dot{\phi}^2(x_n) - \frac{k_s a^3}{2}\left(\frac{\partial\phi(x_n)}{\partial x_n}\right)^2\right]\ , \tag{2.1.20}$$

where we've introduces the spatial derivative as:

$$\tilde{\phi}_{n+1} - \tilde{\phi}_n = a^{1/2}(\phi(x_n + a) - \phi(x_n)) \approx a^{3/2}\frac{\partial\phi(x_n)}{\partial x_n}\ . \tag{2.1.21}$$

We can rewrite this Lagrangian as a sum over a Lagrangian density (discreet analogue of an integral):

$$L = \sum_n \Delta\ \mathcal{L}(x_n) = \sum_n a\left[\frac{m}{2}\dot{\phi}^2(x_n) - \frac{k_s a^2}{2}\left(\frac{\partial\phi(x_n)}{\partial x_n}\right)^2\right]\ , \tag{2.1.22}$$

from which we read out the Lagrangian density:

$$\mathcal{L}(x_n) = \left[\frac{m}{2}\dot{\phi}^2(x_n) - \frac{k_s a^2}{2}\left(\frac{\partial\phi(x_n)}{\partial x_n}\right)^2\right]\ . \tag{2.1.23}$$

Now to find the equations of motion we apply the Euler-Lagrange equations (2.1.8) to get:

$$\begin{aligned} 0 &= -\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial}{\partial x_n}\left(\frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial x_n})}\right) \\ &= -\frac{\partial}{\partial t}\left(m\dot{\phi}(x_n)\right) - \frac{\partial}{\partial x_n}\left(-k_s a^2 \frac{\partial \phi(x_n)}{\partial x_n}\right) , \end{aligned} \tag{2.1.24}$$

which gives us the following equation of motion:

$$\frac{\partial^2 \phi(x_n)}{\partial t^2} = \frac{k_s a^2}{m}\frac{\partial^2 \phi(x_n)}{\partial x_n^2} , \tag{2.1.25}$$

which is a wave equation for the deformations with the speed $c^2 = k_s a^2/m$. Now lets write $\phi(x_n, t)$ as a combination of plane waves:

$$\phi(x_n, t) = f(x_n - vt) + g(x_n + vt) = \sum_k A_k \left(e^{-i(\omega_k t - k x_n)} + e^{i(\omega_k t - k x_n)}\right) . \tag{2.1.26}$$

Inserting this into the wave equation we find the dispersion relation:

$$\omega_k = \pm ck . \tag{2.1.27}$$

We can also derive the Hamiltonian density using (2.1.15):

$$\mathcal{H}(x_n) = \frac{1}{2}\left(m\dot{\phi}^2(x_n) + k_s a^2 \left(\frac{\partial \phi(x_n)}{\partial x_n}\right)^2\right) , \tag{2.1.28}$$

where we used the conjugated momentum $\Pi(x_n) = \partial \mathcal{L}/\partial \dot{\phi}(x_n) = m\dot{\phi}(x_n)$. If we insert the plane wave expansion into the Hamiltonian density we find that:

$$E \propto |A_k|^2 \cdots \Rightarrow \text{(E is continuous)} . \tag{2.1.29}$$

Now to restrict the values of k, let us imagine that we have periodic boundary conditions $\phi(x_n) = \phi(x_n + Na)$ where N is the number of atoms in the 1D lattice. We find that:

$$k_n a N = 2\pi n \quad \Rightarrow \quad k_n = n\frac{2\pi}{aN}, \quad n = 0, 1, \ldots, N-1 . \tag{2.1.30}$$

For a given k, A_k can take any value, which means that energy is continuous. With PBC we've discretized modes $k \to k_n$ but not their amplitudes. Going forward we will drop the index of k_n to keep equations cleaner but do keep in mind that **PBC limit us to discreet modes**. To quantize the field we need to impose commutation relations between the field and its conjugated momentum. Lets first promote the field and its conjugated momentum to operators:

$$\hat{\phi}(x_n, t) = \sum_k A_k \left[a_k e^{-i(\omega_k t - k x_n)} + a_k^\dagger e^{i(\omega_k t - k x_n)}\right] , \tag{2.1.31}$$

$$\hat{\Pi}(x_n, t) = m\sum_k A_k \left[-i\omega_k a_k e^{-i(\omega_k t - k x_n)} + i\omega_k a_k^\dagger e^{i(\omega_k t - k x)}\right] , \tag{2.1.32}$$

where a_k and $a_k^\dagger$ are the annihilation and creation operators for a mode with wavevector k. Now we postulate the canonical commutation relation:

$$[\hat{\phi}(x_n, t), \hat{\Pi}(x_m, t)] = i\delta_{nm} . \tag{2.1.33}$$

Lets write this commutator out explicitly for $t = 0$:

$$\begin{aligned} [\hat{\phi}(x_n, t), \hat{\Pi}(x_m, t)] &= \sum_{k,k'} A_k A_{k'} (-i\omega_{k'}) e^{i(k x_n + k' x_m)} \left(-2[a_k, a_{k'}^\dagger]\right) m \\ &= \sum_{k,k'} A_k A_{k'} (-i\omega_{k'}) e^{i(k x_n + k' x_m)} \left(-2\delta_{kk'}\right) m \\ &= \sum_k 2mi\omega_k A_k^2 e^{ik(x_n - x_m)} \\ &= i\delta_{nm} , \end{aligned} \tag{2.1.34}$$

where we used the commutation relation $[a_k, a^\dagger_{k'}] = \delta_{kk'}$ and defined the normalization constant A_k such that:

$$A_k = \frac{1}{\sqrt{2m\omega_k Na}} \,. \tag{2.1.35}$$

Plugging this back into the Hamiltonian density and summing over all lattice sites we get:

$$\hat{H} = \sum_n \left[\frac{m}{2}\dot{\hat{\phi}}^2(x_n,t) + \frac{k_s a^2}{2}(\partial_{x_n}\hat{\phi}(x_n,t))^2\right] = \sum_n \left[\frac{1}{2m}\hat{\Pi}^2(x_n,t) + \frac{mc^2}{2}(\partial_{x_n}\phi(x_n,t))^2\right] \,. \tag{2.1.36}$$

We can better understand this Hamiltonian if we write it in terms of ladder operators. Besides the expression from Equation (2.1.32), we'll also need the spatial derivative of the field operator:

$$\partial_{x_n}\hat{\phi}(x_n,t) = \sum_k A_k \left(ika_k e^{-i(\omega_k t - kx_n)} - ika^\dagger_k e^{i(\omega_k t - kx_n)}\right) \,. \tag{2.1.37}$$

Lets simplify the squares of these two operators. For the conjugated momentum operator we get:

$$\begin{aligned}\hat{\Pi}^2(x_n,0) = \sum_{k,k'} m^2\omega_k\omega_{k'}A_kA_{k'} \Big[&-a_ka_{k'}e^{i(k+k')x_n} + a_ka^\dagger_{k'}e^{i(k-k')x_n} \\ &+ a^\dagger_k a_{k'}e^{-i(k-k')x_n} - a^\dagger_k a^\dagger_{k'}e^{-i(k+k')x_n}\Big] \,.\end{aligned} \tag{2.1.38}$$

Now we sum over all lattice sites using PBC orthogonality:

$$a\sum_{n=0}^{N-1} e^{i(k-k')x_n} = aN\delta_{kk'} \,, \tag{2.1.39}$$

$$a\sum_{n=0}^{N-1} e^{i(k+k')x_n} = aN\delta_{k',-k} \,. \tag{2.1.40}$$

This means that the $a_ka_{k'}e^{i(k+k')x_n}$ term becomes:

$$\begin{aligned}&a\sum_n\sum_{k,k'} m^2\omega_k\omega_{k'}A_kA_{k'}(-a_ka_{k'}e^{i(k+k')x_n}) = \\ &= -aN\sum_{k,k'} m^2\omega_k\omega_{k'}A_kA_{k'}a_ka_{k'}\delta_{k',-k} \\ &= -aN\sum_k m^2\omega_k^2A_k^2 a_ka_{-k} \,,\end{aligned} \tag{2.1.41}$$

where we used $\omega_{k'} = \omega_{-k}$ and the usual real-mode normalization $A_{k'} = A_{-k} = A_k$. The $a_ka^\dagger_{k'}e^{i(k-k')x_n}$ term becomes:

$$a\sum_n\sum_{k,k'} m^2\omega_k\omega_{k'}A_kA_{k'}a_ka^\dagger_{k'}e^{i(k-k')x_n} = aN\sum_k m^2\omega_k^2A_k^2a_ka^\dagger_k \,. \tag{2.1.42}$$

The other two terms are simplified in a similar manner as:

$$a\sum_n\sum_{k,k'} m^2\omega_k\omega_{k'}A_kA_{k'}a^\dagger_ka_{k'}e^{-i(k-k')x_n} = aN\sum_k m^2\omega_k^2A_k^2a^\dagger_ka_k \,, \tag{2.1.43}$$

$$a\sum_n\sum_{k,k'} m^2\omega_k\omega_{k'}A_kA_{k'}(-a^\dagger_ka^\dagger_{k'}e^{-i(k+k')x_n}) = -aN\sum_k m^2\omega_k^2A_k^2a^\dagger_ka^\dagger_{-k} \,. \tag{2.1.44}$$

Combining all these results we get:

$$a\sum_{n=0}^{N-1}\hat{\Pi}^2(x_n,0) = aN\sum_k m^2\omega_k^2A_k^2\left[-a_ka_{-k} + a_ka^\dagger_k + a^\dagger_ka_k - a^\dagger_ka^\dagger_{-k}\right] \,. \tag{2.1.45}$$

Performing the same steps for the spatial derivative of the field operator we get:

$$a\sum_{n=0}^{N-1}(\partial_{x_n}\hat{\phi}(x_n,0))^2 = aN\sum_k k^2A_k^2\left[a_ka_{-k} + a_ka^\dagger_k + a^\dagger_ka_k + a^\dagger_ka^\dagger_{-k}\right] \,. \tag{2.1.46}$$

With these two pieces we can now assemble the Hamiltonian:

$$\begin{aligned}\hat{H} &= \sum_n \left[\frac{1}{2m}\hat{\Pi}^2(x_n,0) + \frac{mc^2}{2}(\partial_{x_n}\hat{\phi}(x_n,0))^2\right] \\ &= \sum_k \frac{aNA_k^2 m}{2}\left\{\left[-\omega_k^2 + c^2k^2\right]\left[a_k a_{-k} + a_k^\dagger a_{-k}^\dagger\right] + \left[\omega_k^2 + c^2k^2\right]\left[a_k a_k^\dagger + a_k^\dagger a_k\right]\right\} . \end{aligned} \tag{2.1.47}$$

Since our dispersion relation is $\omega_k^2 = c^2k^2$ the first term vanishes and we are left with:

$$\hat{H} = \sum_k \frac{aNA_k^2 m}{2} 2\omega_k^2 \left(a_k a_k^\dagger + a_k^\dagger a_k\right) . \tag{2.1.48}$$

Using the commutation relation $[a_k, a_{k'}^\dagger] = \delta_{kk'}$ and the definition of our renormalization constant A_n we can rewrite this as:

$$\hat{H} = \sum_k \omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right) . \tag{2.1.49}$$

This is simply the Hamiltonian of a Linear Harmonic Oscillator (LHO), which means that different Fourier modes k_n are independent from each other. From this we can see that the ground state is:

$$\hat{H}|0\rangle = E_0|0\rangle \quad \Rightarrow \quad E_0 = \sum_k \frac{\omega_k}{2} , \tag{2.1.50}$$

which is the zero-point energy of the quantized deformation field in the crystal lattice. The excited states are given by:

$$a_k^\dagger|0\rangle : \quad E = \omega_k\left(\frac{1}{2} + 1\right) , \tag{2.1.51}$$

$$(a_k^\dagger)^n|0\rangle : \quad E = \omega_k\left(\frac{1}{2} + n\right) . \tag{2.1.52}$$

This means that for a specified Fourier mode k we can have any number of quanta n in that mode. We can interpret these quanta as quasi-particles called phonons. Phonons are bosons which means that they obey Bose-Einstein statistics, hence why a single mode can have any number of quanta.

2.2 The Klein-Gordon Equation's Lagrangian

At this point we can *guess* the Lagrangian density for the Klein-Gordon equation. It must be a Lorentz scalar and reproduce the Klein-Gordon equation when we insert it into the Euler-Lagrange equations. Since the Klein-Gordon equation is linear and second-order in derivatives, the Lagrangian should be at most quadratic in ϕ and only involve first-order derivatives. Thus we expect the form to be:

$$\mathcal{L} = a(\partial_\mu\phi)(\partial^\mu\phi) + b\phi^2 , \tag{2.2.1}$$

where a and b are constants. If we apply the Euler-Lagrange equations (2.1.8) to this ansatz we can find the constants to be $a = 1/2$ and $b = -m^2/2$. Thus the Lagrangian density for the Klein-Gordon equation is:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 . \tag{2.2.2}$$

This corresponds to the Lagrangian density of a **free real scalar field** (where free means no interaction terms) which we can rewrite in terms of time and spatial derivatives as:

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2) . \tag{2.2.3}$$

We can easily find the Hamiltonian density by using the conjugated momentum, which we can calculate from the Lagrangian density:

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} \quad \rightarrow \quad \Pi = \dot{\phi} , \tag{2.2.4}$$

and thus using the definition of the Hamiltonian density we get:

$$\mathcal{H} = \Pi\dot{\phi} - \mathcal{L} = \dot{\phi}^2 - \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right) . \tag{2.2.5}$$

2.3 Noether's Theorem

Noether's theorem is a powerful tool in theoretical physics which was first formulated by Emmy Noether in 1915.

Theorem: Noether's Theorem

If equations of motion are invariant under a continuous transformation of the fields $\phi_a \to \phi_a + \alpha\Delta\phi_a$ and it holds that the Lagrangian density stays the same up to a 4-divergence $\mathcal{L} \to \mathcal{L} + \alpha\partial_\mu J^\mu$, then there exists a conserved current j^μ and a conserved charge Q which are given by:

$$j^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\Delta\phi_a - J^\mu \,,$$

$$Q = \int j^0(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \,,$$

for which it holds that:

$$\text{(i)}\;\; \partial_\mu j^\mu(\boldsymbol{x}, t) = 0$$

$$\text{(ii)}\;\; \frac{\mathrm{d}}{\mathrm{d}t}Q = 0$$

In other words: For every continuous symmetry of the action there is a conserved current and a conserved charge.

Lets talk a little more about the transformations in question. We said the theorem applies to continuous transformations of the fields. This excludes transformations such as charge conjugation or parity inversion. Continuous transformations transformations are transformations that can be parametrized by a continuous parameter α. Notice that this parameter is not dependent on the fields or rather their spatial coordinates. This means that the transformations that Noether's theorem applies to are global transformations. Likewise this does not mean that $\Delta\phi_a|_{\text{boundary}} = 0$ always holds.

Proof of Noether's Theorem

We can prove Noether's theorem by considering the variation of the action under the transformation $\phi_a \to \phi_a + \alpha\Delta\phi_a$. We assume that ϕ_a satisfies the Euler-Lagrange equations. To keep the action stationary we require that the Lagrangian either doesn't change $\Delta\mathcal{L} = 0$ or that it changes by a total derivative $\Delta\mathcal{L} = \alpha\partial_\mu J^\mu$. That means that in the first order we have:

$$\Delta\mathcal{L} = \alpha\partial_\mu J^\mu = \sum_a \left(\frac{\partial\mathcal{L}}{\partial\phi_a}\Delta\phi_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\partial_\mu\Delta\phi_a \right) \alpha \,, \tag{2.3.1}$$

the second term can be further expanded so that the expression reads:

$$\Delta\mathcal{L} = \sum_a \left(\frac{\partial\mathcal{L}}{\partial\phi_a}\Delta\phi_a + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\Delta\phi_a \right] - \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a} \right] \Delta\phi_a \right) \alpha \,. \tag{2.3.2}$$

The first and last term here can be combined to give us the Euler-Lagrange equations (2.1.8) which yield zero as ϕ_a satisfies them. Thus we are left with:

$$\partial_\mu \left(\sum_a \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\Delta\phi_a - J^\mu \right) = 0 \,, \tag{2.3.3}$$

where we can identify the conserved current j^μ as the contents of the parenthesis.

$$j^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\Delta\phi_a - J^\mu \,. \tag{2.3.4}$$

From here we can see that the first relation in the theorem holds if:

$$\partial_\mu j^\mu = 0 = \frac{\mathrm{d}j^0}{\mathrm{d}t} + \nabla \boldsymbol{j}\,, \tag{2.3.5}$$

$$\Rightarrow \frac{\mathrm{d}j^0}{\mathrm{d}t} = -\nabla \boldsymbol{j}(\boldsymbol{x}, t)\,. \tag{2.3.6}$$

The second relation in the theorem holds if we integrate the first relation over all space:

$$\int \frac{\mathrm{d}j^0(\boldsymbol{x}, t)}{\mathrm{d}t}\mathrm{d}\boldsymbol{x} = -\int \nabla \boldsymbol{j}(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x}\,. \tag{2.3.7}$$

Turning the right side integral into a integral over the boundary we can see that the second relation holds if the current j^μ goes to zero at infinity. Mathematically this is expressed as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int j^0(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x} = 0 = -\boldsymbol{j}_{\text{boundary}} \quad \Rightarrow \quad |\boldsymbol{j}(\boldsymbol{x}, t)|_{|\boldsymbol{x}|\to\infty} = 0\,. \tag{2.3.8}$$

2.4 Derivation of the Energy-Momentum Tensor

The energy-momentum tensor is a very important quantity in field theory. It is a tensor that describes the energy and momentum density of a field. We can derive the energy-momentum by considering the change of the action under a Lorentz transformation, more specifically a translation.

Let us consider a transformation $\boldsymbol{x}^\nu \to \boldsymbol{x}^\nu + a^\nu$ where a^ν is a constant. The action changes as:

$$S = \int \mathcal{L}(\boldsymbol{x})\mathrm{d}^4x \quad \to \quad S' = \int \mathcal{L}(\phi(\boldsymbol{x} + a))\mathrm{d}^4x\,. \tag{2.4.1}$$

Our transformation is a translation, which means that the field ϕ changes as $\phi(x) \to \phi(x + a) = \phi(x) + a^\nu\partial_\nu\phi = \phi(x) + \alpha\Delta\mathcal{L}$, where $\alpha = a^\nu$ and $\Delta\mathcal{L} = \partial_\nu\phi$. With this the Lagrangian density changes as:

$$\mathcal{L}(x) \quad \to \quad \mathcal{L}(x + a) = \mathcal{L}(x) + a^\nu\partial_\nu\mathcal{L}\,, \tag{2.4.2}$$

where we can represent $\partial_\nu\mathcal{L}$ as $\partial_\mu(\delta_\nu{}^\mu\mathcal{L})$. From this we can identify the quantity $J^\mu{}_\nu = \delta_\nu{}^\mu\mathcal{L}$.

Remember: Energy-Momentum Tensor

Now we can use Noether's theorem (2.3) to find the conserved current j^μ_ν which represents **the energy-momentum tensor**.

$$j^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\partial_\nu\phi - \delta_\nu{}^\mu\mathcal{L} = T^\mu{}_\nu$$

As is stated in Noether's theorem $\partial_\mu T^\mu{}_\nu(\boldsymbol{x}, t) = 0$ which means that the energy-momentum tensor is conserved. We can now check the components of the energy-momentum tensor by calculating the conserved charges. Let's first start with $\nu = 0$:

$$Q = \int j^0_0 \,\mathrm{d}\boldsymbol{x} = \int \left(\frac{\partial\mathcal{L}}{\partial\partial_0\phi}\partial_0\phi - \mathcal{L}\right) \mathrm{d}\boldsymbol{x} = \tag{2.4.3}$$

$$= \int \left(\Pi(\boldsymbol{x})\dot{\phi}(\boldsymbol{x}) - \mathcal{L}\right) \mathrm{d}\boldsymbol{x} = \int \mathcal{H}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,, \tag{2.4.4}$$

where we have identified the Hamiltonian (2.1.15) as the integrand which is constant according to Noether's theorem $\mathrm{d}E/\mathrm{d}t = 0$. The Hamiltonian represents the energy density of the field. Now lets calculate the conserved charge for $\nu = i$ where $i = 1, 2, 3$:

$$Q = \int j^0_i \,\mathrm{d}\boldsymbol{x} = \int \left(\Pi(\boldsymbol{x})\partial_i\phi(\boldsymbol{x})\right) \mathrm{d}\boldsymbol{x}\,, \tag{2.4.5}$$

where we can identify the integrand as the momentum density of the field $p^i = \Pi(\boldsymbol{x})\partial_i\phi(\boldsymbol{x})$. Again according to Noether's theorem the momentum density is conserved $\mathrm{d}p^i/\mathrm{d}t = 0$.

Exercise: Invariance and conserved current for a given Lagrangian

Consider a Lagrangian in 2D field theory given by:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2 \,,$$

where $\phi = (\phi_1,\, \phi_2)^\mathsf{T}$. We want to show that the Lagrangian is invariant under the transformation $\phi \to \phi + i\alpha^a \sigma^a \phi$ where σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,,$$

and $\alpha^a = (\alpha,\, 0,\, 0)^\mathsf{T}$. We also want to find the conserved current. Remember that on conjugation of the transformation the order of the matrices changes.

$$\phi + i\alpha^a \underline{\sigma^a \phi} \xrightarrow{\dagger} \phi^\dagger - i\alpha^a \underline{\phi^\dagger \sigma^a}$$

Solution: Let's directly calculate the change of the Lagrangian under the transformation term by term:

$$\begin{aligned} \partial_\mu \phi'^\dagger \partial^\mu \phi' &= \partial_\mu \left(\phi^\dagger - i\alpha^a \phi^\dagger \sigma^a\right) \partial^\mu (\phi + i\alpha^a \sigma^a \phi) = \\ &= \partial_\mu \phi^\dagger \partial^\mu \phi + i\alpha^a \left(-(\partial_\mu \phi^\dagger)\sigma^a \partial^\mu \phi + \phi^\dagger \sigma^a \partial^\mu \phi\right) + \mathcal{O}({\alpha^a}^2) = \\ &= \partial_\mu \phi^\dagger \partial^\mu \phi + i\alpha^a \left(\partial_\mu \phi^\dagger \partial^\mu \sigma^a \phi - \partial_\mu \phi^\dagger \partial^\mu \sigma^a \phi\right) + \mathcal{O}({\alpha^a}^2) = \\ &= \partial_\mu \phi^\dagger \partial^\mu \phi \,, \end{aligned}$$

which is therefore invariant under the transformation. For the second and third term we can prove invariance by considering the invariance of $\phi^\dagger \phi$ under the transformation:

$$\begin{aligned} \phi'^\dagger \phi' &= (\phi^\dagger - i\alpha^a \phi^\dagger \sigma^a)(\phi + i\alpha^a \sigma^a \phi) = \\ &= \phi^\dagger \phi + \phi^\dagger i\alpha^a \sigma^a \phi - i\alpha^a \phi^\dagger \sigma^a \phi + \mathcal{O}({\alpha^a}^2) = \\ &= \phi^\dagger \phi + i\alpha^a \left[\phi^\dagger \sigma^a \phi - \phi^\dagger \sigma^a \phi\right] + \mathcal{O}({\alpha^a}^2) = \\ &= \phi^\dagger \phi \,, \end{aligned}$$

with which we have shown that the Lagrangian is invariant under the transformation. For the second part of the exercise we need to find the conserved current j^μ. Generally we can think of the transformation of the field as $\phi \to \phi + \alpha \Delta\phi$ where in our case $\Delta\phi = i\sigma^a \phi$. From this we can calculate the changes given the Pauli matrix σ^1 as an example:

$$\Delta\phi = i\sigma^1 \phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \phi_2 \\ \phi_1 \end{bmatrix} \,.$$

From this we get:

$$\begin{aligned} \Delta\phi_1 &= i\phi_2 \quad \Delta\phi_1^* = -i\phi_2^* \,, \\ \Delta\phi_2 &= i\phi_1 \quad \Delta\phi_2^* = -i\phi_1^* \,. \end{aligned}$$

In the first part of the exercise we proved $\Delta\mathcal{L} = 0$ which means that $J^\mu = 0$. Thus the conserved current is given by:

$$j^\mu = \sum_{i=1}^{2} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \Delta\phi_i + \sum_{i=1}^{2} \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i^*} \Delta\phi_i^* \,.$$

It is now convenient to write the Lagrangian in terms of dependance on the derivatives of the fields:

$$\begin{aligned} \mathcal{L} &= \partial_\mu (\phi_1^* \phi_2^*) + \partial^\mu \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + f(\phi) = \\ &= \partial_\mu \phi_1^* \partial^\mu \phi_1 + \partial_\mu \phi_2^* \partial^\mu \phi_2 + f(\phi) \,, \end{aligned}$$

where we've considered that $\partial_\mu \phi_i \partial^\mu \phi_j = \partial^\mu \phi_i \partial_\mu \phi_j$. We've marked the dependece on just the fields (not its derivatives) as $f(\phi)$. Now we can calculate the conserved current:

$$\begin{aligned}
j^\mu = {} & i\frac{\partial}{\partial \partial_\mu \phi_1}\left[\partial_\nu \phi_1^* \partial^\nu \phi_1\right]\phi_2 + i\frac{\partial}{\partial \partial_\mu \phi_2}\left[\partial_\nu \phi_2^* \partial^\nu \phi_2\right]\phi_1 + \\
& -i\frac{\partial}{\partial \partial_\mu \phi_1^*}\left[\partial_\nu \phi_1^* \partial^\nu \phi_1\right]\phi_2^* - i\frac{\partial}{\partial \partial_\mu \phi_2^*}\left[\partial_\nu \phi_2^* \partial^\nu \phi_2\right]\phi_1^* = \\
& = i\partial^\nu \phi_1^* \phi_2 + i\partial^\nu \phi_2^* \phi_1 - i\partial^\nu \phi_1 \phi_2^* - i\partial^\nu \phi_2 \phi_1^* = \\
& = i\left[\partial^\nu \phi_1^* \phi_2 + \partial^\nu \phi_2^* \phi_1 - \partial^\nu \phi_1 \phi_2^* - \partial^\nu \phi_2 \phi_1^*\right] ,
\end{aligned}$$

which if we reverse the effect of the Pauli matrix σ^1 gives us the conserved current:

$$j^\mu = i\left[\partial^\mu \phi^\dagger \sigma^1 \phi - \phi^\dagger \sigma^1 \partial^\mu \phi\right]$$

2.5 Quantization of the Real Scalar Field

We can quantize the real scalar field by promoting the field and the conjugated momentum to operators $\hat{\phi}(\boldsymbol{x}, t)$, $\hat{\Pi}(\boldsymbol{x}, t)$. We demand that the field operator satisfies the classic equations of motion (the Klein-Gordon equation (2.0.1)). We also demand that the field operator and its conjugated momentum satisfy the canonical commutation relations.

Remember: Canonical Commutation Relations

The canonical commutation relations for the field and its conjugated momentum are given by:

$$[\hat{\phi}(\boldsymbol{x}, t), \hat{\Pi}(\boldsymbol{y}, t)] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}) . \tag{2.5.1}$$

The commutator of the field with itself is zero $[\hat{\phi}(\boldsymbol{x}, t), \hat{\phi}(\boldsymbol{y}, t)] = 0$. Likewise the commutator of the conjugated momentum with itself is zero $[\hat{\Pi}(\boldsymbol{x}, t), \hat{\Pi}(\boldsymbol{y}, t)] = 0$.

Classic Fields

Classically the field must satisfy the Klein-Gordon equation (2.0.1). For straight plane wave solutions $\phi(\boldsymbol{x}, t) = Ae^{\pm i(p\cdot x)}$ we have:

$$\phi = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} A_{\boldsymbol{p}}\left(e^{-ip\cdot x} + e^{ip\cdot x}\right) = \phi^* . \tag{2.5.2}$$

These satisfy the Klein-Gordon equation. We see that we can choose $A_{\boldsymbol{p}}$ freely and since $E \propto A_{\boldsymbol{p}}^2$ we have a continuous spectrum of energies. The conjugated momentum is given by:

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}(-iE_{\boldsymbol{p}})\left(e^{-ip\cdot x} - e^{ip\cdot x}\right) . \tag{2.5.3}$$

Quantum Fields

Similarly to the classical case we can write the field operator with creation and annihilation operators:

$$\hat{\phi}(\boldsymbol{x}, t) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} A_{\boldsymbol{p}}\left(a_{\boldsymbol{p}} e^{-ip\cdot x} + a_{\boldsymbol{p}}^\dagger e^{ip\cdot x}\right) , \tag{2.5.4}$$

where the creation and annihilation operators satisfy the commutation relations:

$$[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}^\dagger] = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}') , \tag{2.5.5}$$

$$[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}] = 0 , \quad [a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{p}'}^\dagger] = 0 , \tag{2.5.6}$$

and $a_{\boldsymbol{p}}|0\rangle = 0$. Now using the Klein-Gordon equation:

$$\partial_\mu \partial^\mu \hat{\phi}(\boldsymbol{x}, t) + m^2 \hat{\phi}(\boldsymbol{x}, t) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} A_{\boldsymbol{p}}\left[a_{\boldsymbol{p}}\left((-ip)^\mu (-ip)_\mu + m^2\right) e^{-ip\cdot x} + \dots\right] . \tag{2.5.7}$$

We'd like to show that the commutation relations we postulated hold. If we take a look at the conjugated momentum operator:

$$\hat{\Pi}(\boldsymbol{x}, t) = \frac{\partial \hat{\mathcal{L}}}{\partial \dot{\hat{\phi}}} = \dot{\hat{\phi}} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} A_{\boldsymbol{p}} \left(-iE_{\boldsymbol{p}}\right) \left[a_{\boldsymbol{p}} e^{-ip\cdot x} - a^\dagger_{\boldsymbol{p}} e^{ip\cdot x}\right] . \tag{2.5.8}$$

We claim that $A_{\boldsymbol{p}} = 1/\sqrt{2E_{\boldsymbol{p}}}$ for the commutation relations to hold. Lets check if this is true at $t = 0$ since the commutation relations must hold at all times. Our operators ar then:

$$\hat{\phi}(\boldsymbol{x}, t) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}\right) e^{ip\cdot x} , \tag{2.5.9}$$

$$\hat{\Pi}(\boldsymbol{y}, t) = \int \frac{\mathrm{d}\boldsymbol{p'}}{(2\pi)^3} \frac{(-iE_{\boldsymbol{p'}})}{\sqrt{2E'_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p'}} - a^\dagger_{-\boldsymbol{p'}}\right) e^{ip'\cdot y} . \tag{2.5.10}$$

Now we can calculate the commutator:

$$[\hat{\phi},\, \hat{\Pi}] = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \int \frac{\mathrm{d}\boldsymbol{p'}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \frac{(-iE_{\boldsymbol{p'}})}{\sqrt{2E_{\boldsymbol{p'}}}} e^{i(p\cdot x + p'\cdot x)} \left[-[a_{\boldsymbol{p}},\, a^\dagger_{-\boldsymbol{p'}}] + [a^\dagger_{-\boldsymbol{p}},\, a_{\boldsymbol{p}}]\right] . \tag{2.5.11}$$

Using the comutation relations for the annihilation and creation operators we can simplify the expression:

$$[\hat{\phi},\, \hat{\Pi}] = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{2E_{\boldsymbol{p}} i}{2E_{\boldsymbol{p}}} e^{ip\cdot(x-y)} = i\delta(\boldsymbol{x} - \boldsymbol{y}) . \tag{2.5.12}$$

Thus we have shown that the commutation relations hold for the field and its conjugated momentum. Using a similar method we can show that the commutation relations for the field with itself and the conjugated momentum with itself hold. From all of this we see that the energy spectrum we get is quantized because $A_{\boldsymbol{p}}$ are exactly defined.

2.6 Operators for Observables in QFT

In quantum field theory we can define operators for observables. For example the operator $\hat{A}$ measures the expected value of the observable A. Remember that the expected value of an observable can be found by calculating eigenvalues of the operator:

$$\hat{A}|\psi_n\rangle = A_n|\psi_n\rangle ,$$
$$|\psi\rangle = \sum_n |c_n||\psi_n\rangle ,$$
$$p(A_n) = |c_n|^2 .$$

What we'd like to do is find operators in the form $\hat{A} = f(\hat{\phi}, \partial_\mu \hat{\phi})$ which are hermitian $\hat{A}^\dagger = \hat{A}$.

2.6.1 The Momentum Operator

The momentum operator is found by using the momentum density as we found in the derivation of the energy-momentum tensor (2.4):

$$\hat{p}^i = \int \hat{\Pi}(\boldsymbol{x}) \partial^i \hat{\phi}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} . \tag{2.6.1}$$

Inserting the operators we've found so far we get:

$$\hat{\boldsymbol{p}} = \int \mathrm{d}\boldsymbol{x} \int \frac{\mathrm{d}\boldsymbol{p'}}{(2\pi)^3} \frac{-iE_{\boldsymbol{p'}}}{\sqrt{2E_{\boldsymbol{p'}}}} (a_{\boldsymbol{p'}} - a^\dagger_{-\boldsymbol{p'}}) \int \underset{\cdots\cdots}{(-i\boldsymbol{p})} \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} (a_{\boldsymbol{p}} + a^\dagger_{-\boldsymbol{p}}) e^{ip'\cdot x + ip\cdot x} , \tag{2.6.2}$$

where the dotted line indicates what we got by applying the derivative. Taking a further look at the given integrals we can apply the comutation relations to the end exponential:

$$\int \mathrm{d}\boldsymbol{x} \int \frac{\mathrm{d}\boldsymbol{p'}}{(2\pi)^3} e^{i(\boldsymbol{p'}+\boldsymbol{p})\cdot\boldsymbol{x}} f(\boldsymbol{p}, \boldsymbol{p'}) = \int \mathrm{d}\boldsymbol{p'} \delta(\boldsymbol{p} + \boldsymbol{p'}) f(\boldsymbol{p}, \boldsymbol{p'}) = f(\boldsymbol{p}, -\boldsymbol{p}) . \tag{2.6.3}$$

From this we can reduce our previous expression to:

$$\hat{\boldsymbol{p}} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}(-i\boldsymbol{p})\left(-\frac{i}{2}\right)\left[\underline{a_{-\boldsymbol{p}}a_{\boldsymbol{p}}} - \underline{a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}} - \underline{a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}} + \underline{a_{-\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}}\right] , \tag{2.6.4}$$

where the dotted lines indicate terms that vanish due to commutation relations. Notice that the second dashed line term is not of the proper order, for which we can use the commutation relations to rewrite it as $a_{-\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}} = a^\dagger_{-\boldsymbol{p}}a_{-\boldsymbol{p}} + (2\pi)^3\delta(0)$. This gives us:

$$\hat{\boldsymbol{p}} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\boldsymbol{p}\left(-\frac{1}{2}\right)\left[a^\dagger_{-\boldsymbol{p}}a_{-\boldsymbol{p}} + (2\pi)^3\delta(0) - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\right] . \tag{2.6.5}$$

Many texts skip what I deem to be two crucial spets in this derivation. The first is that as we are integrating over over all space the integral containing $(\boldsymbol{p}\,\delta(0))$ will vanish. Second is that there exists an identity which links the creation and annihilation operators as such: $a^\dagger_{-\boldsymbol{p}}a_{-\boldsymbol{p}} = -a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}$. This finally brings us to the expression for the momentum operator.

Remember: Momentum Operator

The momentum operator is given by:

$$\hat{\boldsymbol{p}} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\boldsymbol{p}\, a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} . \tag{2.6.6}$$

Eigenstates: Lets have a quick look at the eigenstates of the momentum operator. We'd like to know if the states $a^\dagger_{\boldsymbol{p}}|0\rangle$ are eigenstates:

$$\hat{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}|0\rangle = \underline{[\hat{\boldsymbol{p}},\, a^\dagger_{\boldsymbol{p}}]}|0\rangle + a^\dagger_{\boldsymbol{p}}\hat{\boldsymbol{p}}|0\rangle ,$$

The underlined commutator can be calculated as:

$$[\hat{\boldsymbol{p}},\, a^\dagger_{\boldsymbol{p}}] = \int \frac{\mathrm{d}\boldsymbol{p}'}{(2\pi)^3}\boldsymbol{p}'\,[a^\dagger_{\boldsymbol{p}'}a_{\boldsymbol{p}'},\, a^\dagger_{\boldsymbol{p}'}] = \boldsymbol{p}\, a^\dagger_{\boldsymbol{p}} , \tag{2.6.7}$$

where we used the commutation relations for the creation and annihilation operators $[a^\dagger_{\boldsymbol{p}'}a_{\boldsymbol{p}'},\, a^\dagger_{\boldsymbol{p}'}] = a^\dagger_{\boldsymbol{p}'}(2\pi)^3\delta(\boldsymbol{p}'-\boldsymbol{p})$. Since $\hat{\boldsymbol{p}}|0\rangle = 0$ we get:

$$\hat{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}|0\rangle = \boldsymbol{p}\, a^\dagger_{\boldsymbol{p}}|0\rangle , \tag{2.6.8}$$

which means that the states $a^\dagger_{\boldsymbol{p}}|0\rangle$ are eigenstates of the momentum operator. We claim that this state can be raised to the power of n. It is quickly shown that:

$$\hat{\boldsymbol{p}}a^{\dagger^n}_{\boldsymbol{p}}|0\rangle = n\boldsymbol{p}a^{\dagger^n}_{\boldsymbol{p}}|0\rangle . \tag{2.6.9}$$

The n in the eigenvalue corresponds to the number of particles in the same state. What we've recieved here is the **Bose-Einstein statistics** where one Fourier mode can have any number of excitations.

2.6.2 The Hamiltonian Operator

To find the expression for the Hamiltonian operator we can use the Hamiltonian density we found earlier (2.1.15) and insert the field operators:

$$\hat{H} = \int \hat{\mathcal{H}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int \left[\hat{\Pi}^2(\boldsymbol{x}) + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right]\mathrm{d}\boldsymbol{x} . \tag{2.6.10}$$

Now our professor described the simplification of this expression as a *fun exercise*. I might add the solution later on. After simplification we get the expression:

$$\hat{H} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}E_{\boldsymbol{p}}\frac{1}{2}\left[a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}\right] , \tag{2.6.11}$$

where all that is left is to switch the order of the operators in the second term as we've done before when looking at the eigenstates of the momentum operator. Thus we find the expression for the Hamiltonian operator.

Remember: Hamiltonian Operator

The Hamiltonian operator is given by:

$$\hat{H} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}} \left[a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + \frac{1}{2}(2\pi)^3 \delta(0) \right] . \tag{2.6.12}$$

The second term represents the sum over all modes of the zero-point energies which is expected but Peskin does comment that he finds it disturbing. It is this term that represents the 1/2 in the Hamiltonian of a harmonic oscillator. This term has an intresting consequence which is that the energy of the vacuum is infinite. We can see this mathematically as:

$$\hat{H}|0\rangle = E_0|0\rangle \quad \to \quad E_0 = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} E_{\boldsymbol{p}} \frac{1}{2}(2\pi)^3 \delta(0) = \infty . \tag{2.6.13}$$

Lets have a look at the other energies just in case, so $E - E_0 > 0$:

$$\hat{H} a^\dagger_{\boldsymbol{p}}|0\rangle = [a^\dagger_{\boldsymbol{p}},\ \hat{H}]|0\rangle + a^\dagger_{\boldsymbol{p}}\hat{H}|0\rangle = \tag{2.6.14}$$

$$= E_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}}|0\rangle \ + E_0 a^\dagger_{\boldsymbol{p}}|0\rangle = (E_{\boldsymbol{p}} + E_0) a^\dagger_{\boldsymbol{p}}|0\rangle . \tag{2.6.15}$$

from which we can see that $E - E_0 = E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. We've now laid out the necessary groundwork to see that the excitation $a^\dagger_{\boldsymbol{p}}|0\rangle$ represents a scalar particale with energy $E_{\boldsymbol{p}}$ and momentum $\boldsymbol{p}$. For multiple creation operators at different momenta we get a multi-particle state like so:

$$a^{\dagger n}_{\boldsymbol{p}_1 1} \dots a^{\dagger m}_{\boldsymbol{p}_n n}|0\rangle \quad \text{with} \quad E - E_0 = n_1 E_1 + \dots + m_n E_n, \quad \boldsymbol{p} = n_1 \boldsymbol{p}_1 + \dots + m_n \boldsymbol{p}_n .$$

2.7 Vacuum Energy*

In QFT the vacuum energy is infinite for two reasons. We've already seen that the zero-point energy is infinite. From this we can find the volume of the space over which we are integrating:

$$(2\pi)^3\delta(0) = \int_{-L/2}^{L/2} \mathrm{d}\boldsymbol{x} \int \mathrm{d}\boldsymbol{p}'\, e^{i(\boldsymbol{p}+\boldsymbol{p}')\cdot\boldsymbol{x}} \delta(\boldsymbol{p}+\boldsymbol{p}') = V .$$

We can try and find the energy density of the vacuum by dividing the vacuum energy by the volume:

$$\frac{E_0}{V} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{1}{2} E_{\boldsymbol{p}} = \infty .$$

This can be thought of as a harmonic oscillator per each Fourier mode. We can also estimate the reach of our theory:

$$\frac{E_0}{V} = \int \frac{p^2}{2}\sqrt{p^2+m^2}\,\mathrm{d}p \sim \frac{(1\ \mathrm{TeV})^4}{4} ,$$

We can speculate about the origin of the infinite vacuum energy. One possible explaination would be for example gravity that could feel the energy of the vacuum. It might make sense to remember that only 4% of the universe is made up of visible matter. The rest is dark matter and dark energy. It is also interesting to note that the observed cosmological constant is of the order of $\Lambda_{\mathrm{obs}} = (10^{-3})^4\ \mathrm{eV}^4$.

2.8 Normalization of the Momentum Operator

It is worthwhile to adopt a convention for normalizing one-particle states $|\boldsymbol{p}\rangle \propto a^\dagger_{\boldsymbol{p}}|0\rangle$. The issue that arises is that the normalization which is often used is not Lorentz invariant. The normalization condition is given by:

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q}) . \tag{2.8.1}$$

We'd like to find the missing factor $B_{\boldsymbol{p}}$ that makes the normalization Lorentz invariant. We can start by defining:

$$|\boldsymbol{p}\rangle = B_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}}|0\rangle \,. \tag{2.8.2}$$

We demand the following to be true:

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = B_{\boldsymbol{p}} B_{\boldsymbol{q}} \langle 0|a_{\boldsymbol{q}} a^\dagger_{\boldsymbol{p}}|0\rangle = B^2_{\boldsymbol{p}}(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q}) \,. \tag{2.8.3}$$

Normalization comes from the fact that the dot product must be conserved for all Lorentz transformations, $\langle \boldsymbol{q}|\boldsymbol{p}\rangle = \langle \boldsymbol{q}'|\boldsymbol{p}'\rangle$. Let's consider a boost in the z direction and observe the changes. We get:

$$p'_3 = \gamma(p_3 - \beta E), \quad p'_{1,2} = p_{1,2}, \quad E' = \gamma(E + \beta p_3) \,.$$

Here we can use an identity for the Dirac delta function:

$$\delta(f(x) - f(x_0)) = \frac{\delta(x - x_0)}{\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)} \,, \tag{2.8.4}$$

which stems from the defining property of the delta function for any test function $g(x)$:

$$\int g(x)\delta(f(x)) = \sum_i \frac{g(x_i)}{|f'(x_i)|} \quad \text{where} \quad f(x_i) = 0 \,. \tag{2.8.5}$$

Applied to the z component which we boost:

$$\delta(p_3 - q_3) = \delta(p'_3(p_3) - q'_3(q_3)) \left|\frac{\mathrm{d}p'_3}{\mathrm{d}p_3}\right| \,. \tag{2.8.6}$$

From this we can see how the delta function transforms under such a Lorentz boost:

$$\begin{aligned}
\delta(\boldsymbol{p} - \boldsymbol{q}) &= \delta(\boldsymbol{p}' - \boldsymbol{q}') \cdot \frac{\mathrm{d}p'_3}{\mathrm{d}p_3} \\
&= \delta(\boldsymbol{p}' - \boldsymbol{q}')\gamma\left(1 + \beta\frac{\mathrm{d}E}{\mathrm{d}p_3}\right) \\
&= \delta(\boldsymbol{p}' - \boldsymbol{q}')\frac{\gamma}{E}(E + \beta p_3) \\
&= \delta(\boldsymbol{p}' - \boldsymbol{q}')\frac{2E'}{2E} \,.
\end{aligned}$$

Notice that the factor of 2 here is unnecessary but is conveniant since it cancels out a factor that is used in the definition of the field operators (2.5.9). This problem of non-invariance of the normalization is a consequence of Lorentz contractions when in a boosted frame. A box whose volume is V in the rest frame will have a volume of V/γ in the boosted frame. From this we can find the missing factor $B_{\boldsymbol{p}}$:

$$B_{\boldsymbol{p}} = \sqrt{2E_{\boldsymbol{p}}} \quad \Rightarrow \quad |\boldsymbol{p}\rangle = \sqrt{2E_{\boldsymbol{p}}}a^\dagger_{\boldsymbol{p}}|0\rangle \,. \tag{2.8.7}$$

2.9 Wave Packets

Here are some notes on wave packets. We can write the field operator as:

$$|\phi(\boldsymbol{x})\rangle = \int \frac{\mathrm{d}\boldsymbol{x}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\,\varphi(\boldsymbol{p})|\boldsymbol{p}\rangle \,, \tag{2.9.1}$$

where we demand $\langle\phi|\phi\rangle = 1$ which we can prove like so:

$$\langle\phi|\phi\rangle = \int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\frac{1}{\sqrt{2E_{\boldsymbol{q}}}}\varphi^*(\boldsymbol{p})\varphi(\boldsymbol{q})\langle\boldsymbol{p}|\boldsymbol{q}\rangle = \int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\,\varphi^2(\boldsymbol{p}) = 1 \,. \tag{2.9.2}$$

where we used the commutation relation $\langle\boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3\sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{q}}}\delta(\boldsymbol{p}-\boldsymbol{q})$. We can now use this like in the wave packet from Quantum Mechanics, where we do the following:

$$\psi_{\boldsymbol{q}}(\boldsymbol{x}, t) = \langle\boldsymbol{x}|\boldsymbol{q}\rangle = e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \,. \tag{2.9.3}$$

Since we do not have the operator $\hat{\boldsymbol{x}}$ we use ladder operators of the field:

$$\phi^\dagger(\boldsymbol{x},t)|0\rangle = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\, e^{-ip\cdot x}a^\dagger_{\boldsymbol{p}}|0\rangle \; . \tag{2.9.4}$$

Taking the non-relativistic limit we can find that this is analagous to a state with a well defined position in Quantum Mechanics:

$$\overset{\text{N.R.}}{\Longrightarrow} \quad \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{2m}e^{imt}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}|\boldsymbol{p}\rangle = |\boldsymbol{x}\rangle \; . \tag{2.9.5}$$

Thus we can say that the ladder operator $a^\dagger_{\boldsymbol{p}=\boldsymbol{q}}$ in $\phi^\dagger(\boldsymbol{x},t)$ creates a particle with the momentum $\boldsymbol{q}$. The wave function of such a state is $\psi_{\boldsymbol{q}}(\boldsymbol{x},t) = e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$. This can be seen in action like so:

$$\langle \boldsymbol{q}|\phi^\dagger(\boldsymbol{x},t)|0\rangle = \langle \boldsymbol{q}|\boldsymbol{x}\rangle = \psi^*_{\boldsymbol{q}}(\boldsymbol{x},t) = e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \; . \tag{2.9.6}$$

Similarly we can say that the ladder operator $a_{\boldsymbol{p}=\boldsymbol{q}}$ in $\phi(\boldsymbol{x},t)$ annihilates a particle with the momentum $\boldsymbol{q}$. The wave function of such a state is $\psi_{\boldsymbol{q}}(\boldsymbol{x},t) = e^{i\boldsymbol{q}\cdot\boldsymbol{x}}$. We can see this from the following:

$$\psi_{\boldsymbol{q}}(\boldsymbol{x},t) = \langle \boldsymbol{x}|\boldsymbol{q}\rangle = \langle \boldsymbol{p}|\hat{\phi}(\boldsymbol{x},t)|\boldsymbol{q}\rangle = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\langle \boldsymbol{p}|\boldsymbol{q}\rangle = e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \; . \tag{2.9.7}$$

2.10 Completness Relations

Mathematicians tell us that a Hilbert space $\mathcal{H}$ (not to confuse with the Hamiltonian) is **complete** if every *Cauchy sequence* of vectors admits a limit in the space itself. This gives rise to **complete orthonormal systems** of vectors in such a space. A set of vectors $\{\psi_i\}_{i\in I} \subset \mathcal{H}$ is called an **orthonormal system** if $\langle\psi_i|\psi_j\rangle = \delta_{ij}$. The set is additionally called **complete** if every vector in the space can be written as a linear combination of the vectors in the set:

$$\langle\phi|\phi\rangle = \sum_{i\in I}\langle\psi|\psi_i\rangle\langle\psi_i|\phi\rangle \quad \forall\psi,\phi\in\mathcal{H}. \tag{2.10.1}$$

2.10.1 Completness relations in QM

In Quantum Mechanics we have the following completeness relation:

$$\mathbb{1} = \sum_n |n\rangle\langle n| \; , \tag{2.10.2}$$

$$\mathbb{1}|m\rangle = \sum_n |n\rangle\langle n|m\rangle = |m\rangle \; . \tag{2.10.3}$$

where we sum over all possible states.

2.10.2 Completness relations in QFT

In Quantum Field Theory we have the following completeness relation:

Remember: Completness Relation in QFT

The completeness relation in Quantum Field Theory is given by:

$$\mathbb{1} = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}|\boldsymbol{p}\rangle\langle\boldsymbol{p}| \; , \tag{2.10.4}$$

$$\mathbb{1}|\boldsymbol{q}\rangle = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|\boldsymbol{q}\rangle = |\boldsymbol{q}\rangle \; , \tag{2.10.5}$$

where we once again used the relation $\langle\boldsymbol{p}|\boldsymbol{q}\rangle = (2\pi)^3\sqrt{2E_{\boldsymbol{p}}}\sqrt{2E_{\boldsymbol{q}}}\delta(\boldsymbol{p}-\boldsymbol{q})$. The delta function ensures that we get the correct energy factor of $2E_{\boldsymbol{p}}$.

2.11 Causality and Correlation

Note that causality is not the same as correlation. Causality is the principle that an effect cannot occur before its cause. Correlation is the principle that two events are related. Consider two measurements $\hat{\mathcal{O}}_1(x)$ and $\hat{\mathcal{O}}_2(y)$. They must not effect each other if x and y are spacelike separated $(x-y)^2 < 0$. This is achived if the two operators commute $[\hat{\mathcal{O}}_1(x),\, \hat{\mathcal{O}}_2(y)] = 0$.

Lets consider that the two measurements are spacelike seperated. That means that there exists a frame of reference where the two events happen at the same time $x^0 = y^0 = t$. In this frame the commutators are:

$$\begin{gathered}[\hat{\phi}(\boldsymbol{x},t),\, \hat{\phi}(\boldsymbol{y},t)] = 0\,, \\ [\hat{\Pi}(\boldsymbol{x},t),\, \hat{\Pi}(\boldsymbol{y},t)] = 0\,, \\ [\hat{\phi}(\boldsymbol{x},t),\, \hat{\Pi}(\boldsymbol{y},t)] = i\delta(\boldsymbol{x}-\boldsymbol{y}) = 0\,.\end{gathered}$$

The last commutator is zero because the field and its conjugated momentum are at different points in space. We can see that causality is trivially satisfied for real scalar fields where $\hat{\phi} = \hat{\phi}^\dagger$. When we have complex scalar fields or fermions we will need to introduce ladder operators for the creation and annihilation of anti-particles to satisfy causality.

2.11.1 Quantum Entanglement

Quantum entanglement is a phenomenon where two particles are connected in such a way that the state of one particle is dependent on the state of the other. This is an example of correlated events. Let's consider an example from Quantum Mechanics. Let's have a two particle system with the following state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2\right) = |\mathrm{s}=1,\, \mathrm{s}_z=0\rangle\,. \tag{2.11.1}$$

This state is a superposition of two spin states. We can prove that causality is satisfied for this state by showing that the commutator of the two spins is zero:

$$\hat{S}_{1,z}\hat{S}_{2,z}|\psi\rangle = \hat{S}_{2,z}\hat{S}_{1,z}|\psi\rangle\,. \tag{2.11.2}$$

Measurements of the spins will be correlated. If we measure the spin of particle 1 to be up, the spin of particle 2 will be down. This is an example of quantum entanglement.

2.11.2 Scalar Correlators

We can define the correlation function between two states as:

$$\langle\boldsymbol{x},\, x_0|\boldsymbol{y},\, y_0\rangle = \langle 0|\hat{\phi}(\boldsymbol{x})\hat{\phi}^\dagger(\boldsymbol{y})|0\rangle\,. \tag{2.11.3}$$

If $x_0 > y_0$ this represents a propagator.

2.11.3 Propagators

In QFT we write interactions such as:

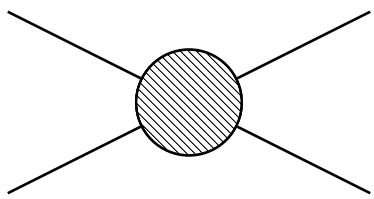


Figure 2.1: A Feynman diagram representing an unknown interaction.

using propagators. The propagator is the correlation function of the field operators. We can write the propagator as follows.

Remember: Scalar Propagator

The scalar propagator is given by:

$$\Delta_s(\boldsymbol{x},\boldsymbol{y}) = \langle 0|\mathcal{T}\left[\hat{\phi}(\boldsymbol{x})\hat{\phi}^\dagger(\boldsymbol{y})\right]|0\rangle = \begin{cases}\langle 0|\hat{\phi}(\boldsymbol{x})\hat{\phi}^\dagger(\boldsymbol{y})|0\rangle & \text{if } x^0 > y^0\,,\\ \langle 0|\hat{\phi}^\dagger(\boldsymbol{y})\hat{\phi}(\boldsymbol{x})|0\rangle & \text{if } y^0 > x^0\,,\end{cases} \tag{2.11.4}$$

where $\mathcal{T}$ is the time ordering operator.

Cluster Decomposition Principle: In physics the cluster decomposition principle states that the correlation function of two spacelike separated events factorizes. Something like this:

$$\lim_{|\boldsymbol{x}|\to\infty}\langle\mathcal{O}_1(\boldsymbol{x})|\mathcal{O}_2(0)\rangle \to \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle\,. \tag{2.11.5}$$

The propagator is actually the Green's function for a given equation of motion.

$$\hat{A}\mathcal{G}(\boldsymbol{r}-\boldsymbol{r}_0) = \delta(\boldsymbol{r}-\boldsymbol{r}_0)\,. \tag{2.11.6}$$

The scalar propagator Δ_s is the Green's function for the Klein-Gordon equation. We will derive that the expression for the scalar propagator is given by:

$$\Delta_s(x-y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\underline{\frac{i}{p^2-m^2+i\varepsilon}}e^{-ip\cdot(x-y)}\,. \tag{2.11.7}$$

The dashed line indicates the expression for the scalar propagator. We can check that this is the Green's function for the Klein-Gordon equation by inserting it into the equation:

$$\begin{aligned}\left(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}+m^2\right)\Delta_s(x-y) &= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}((-ip^\mu)(-ip_\mu)+m^2)e^{-ip\cdot(x-y)}\\ &\propto \delta^{(4)}(x-y)\,.\end{aligned} \tag{2.11.8}$$

Derivation of the Scalar Propagator: We can seperate the problem into two cases. The first case is when $x^0 > y^0$. This will give us a pole at p^0:

$$\begin{aligned}p^2-m^2+i\varepsilon &= p_0^2-\boldsymbol{p}^2-m^2+i\varepsilon\\ &= (p_0-E_{\boldsymbol{p}})(p_0+E_{\boldsymbol{p}})+i\varepsilon\\ &= (p_0-(E_{\boldsymbol{p}}-i\varepsilon'))(p_0+(E_{\boldsymbol{p}}-i\varepsilon'))\\ &= (p_0-E_{\boldsymbol{p}})(p_0-E_{\boldsymbol{p}})+\frac{i\varepsilon'}{p_0+E_{\boldsymbol{p}}+E_{\boldsymbol{p}}-p_0}\\ &= (p_0-E_{\boldsymbol{p}})(p_0-E_{\boldsymbol{p}})+i\varepsilon\,,\end{aligned} \tag{2.11.9}$$

where we took into account that $-\boldsymbol{p}^2-m^2=-E_{\boldsymbol{p}}^2$. From this we can see that there are two poles, one at $p^0=(E_{\boldsymbol{p}}-i\varepsilon)$ and the second at $p^0=-(E_{\boldsymbol{p}}-i\varepsilon)$. We can now use the residue theorem to find the scalar propagator. We need to solve:

$$\Delta_s(x-y) = \int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\int\frac{\mathrm{d}p^0}{2\pi}\frac{ie^{-ip\cdot(x-y)}}{(p_0-(E_{\boldsymbol{p}}-i\varepsilon))(p_0+(E_{\boldsymbol{p}}-i\varepsilon))}\,. \tag{2.11.10}$$

Here we can make use of a property of the exponential function:

$$\exp(-i|p_0|(|a|-i|b|)(x_0-y_0)) \propto \exp(-i|p_0||b|(x_0-y_0)) \overset{|y_0|\to\infty}{\longrightarrow} 0\,, \tag{2.11.11}$$

which states why an integral over an infinite semicircle in the upper half plane will vanish if $x^0 > y^0$. From that we can integrate. Remember the residue theorem:

$$\oint_C f(z)\,\mathrm{d}z = -2\pi i\sum\mathrm{Res}(f(z),z_i)\,. \tag{2.11.12}$$

Which we now apply to our expression for the scalar propagator. We integrate around an infinite semicircle in the upper half plane:

$$\begin{aligned}\Delta_s(x-y) &= \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^4}(-2\pi i)\frac{ie^{-ip\cdot(x-y)}}{2(E_{\boldsymbol{p}}-i\varepsilon)}\Bigg|_{\substack{p^0 = E_{\boldsymbol{p}}-i\varepsilon \\ \varepsilon\to 0}} \\ &= \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{-ip\cdot(x-y)} = \Delta_s(x-y)\,.\end{aligned} \tag{2.11.13}$$

Similarly we can find the expression for the scalar propagator when $y^0 > x^0$, where we integrate around an infinite semicircle in the lower half plane. This gives us:

$$\Delta_s(x-y) = \int \frac{\mathrm{d}^4p'}{(2\pi)^4}\frac{e^{-ip'\cdot(y-x)}}{p'^2-m^2+i\varepsilon} = \cdots = \Delta_s(y-x)\,, \tag{2.11.14}$$

where $p = -p'$.

3 Lorentz Transformations

As we've seen with real scalar fields we now have a method using different fields. This is as follows:

1. Perform Lorentz transformations on the field.
2. Find equations of motion that are invariant under Lorentz transformations.
3. Solve the equations of motion to find the field operators.
4. Quantization: impose commutation relations on the field operators.

If we have a solution to the equations of motion ϕ then the equations of motion are invariant under Lorentz transformations if the transformed field $\phi'(x')$ is also a solution to the equations of motion.

Lorentz transformations form a group, meaning that if we have two Lorentz transformations $L, L' \in \mathrm{SO}(1,3)$ then the product $L'L = L'' \in \mathrm{SO}(1,3)$ is also a Lorentz transformation. The identity $\mathbb{1}$ is also a Lorentz transformation and for every Lorentz transformation L there exists an inverse L^{-1} such that $LL^{-1} = \mathbb{1}$.

Lorentz transformations can be represented in different ways. In the case of vector fields we'll mark the representation of a Lorentz transformation L as $M(L) = \Lambda^\mu{}_\nu$, where this is mathematically a 4×4 matrix. Thus vector fields transform as:

$$A^\mu \quad\longrightarrow\quad \Lambda^\mu{}_\nu A^\nu = A'^\mu\,. \tag{3.0.1}$$

More generally, any Lorentz group element $L \in \mathrm{SO}(1,3)$ has various representations $M(L)$ depending on the type of object it acts on. In the above example $M(L) = \Lambda$ is the representation of L that acts on 4-vectors.

3.1 Properties of Lorentz Transformations

Lorentz transformations have some interesting properties that we can use. These appear due to the metric tensor $g^{\mu\nu}$. The identities are as follows:

$$x'\cdot y' = x\cdot y \Rightarrow \cdots \Rightarrow \Lambda_{\mu\alpha}\Lambda_{\nu\beta} = g_{\alpha\beta}\,, \tag{3.1.1}$$

$$\Lambda_{\nu\alpha}\Lambda^\nu{}_\beta = g_{\alpha\beta} \Rightarrow \cdots \Rightarrow \Lambda_\nu{}^\alpha = (\Lambda^{-1})^\nu{}_\alpha\,. \tag{3.1.2}$$

3.2 Transformations of the Basis Vectors and Vector Components

A vector $\boldsymbol{v}$ can be expressed in a basis $\{\hat{e}_i\}$ as a linear combination of the basis vectors:

$$\boldsymbol{v} = \sum_i c_i \hat{e}_i \,. \tag{3.2.1}$$

If we perform a liner transformation (rotation, or Lorentz boost, etc.), the basis vectors themselves transform. To keep the overall vector $\boldsymbol{v}$ unchanged, the components c_i must also transform. For example, under a rotation R in 3D space, the basis vectors transform as:

$$\hat{e}_i = R_{ij}\hat{e}_j \,, \quad R \in \mathrm{SO}(3) \,, \tag{3.2.2}$$

$$c'_j = (R^{-1})_{ji} c_i \,. \tag{3.2.3}$$

The same is true if we perform a Lorentz transformation, with $M(L)$ being the matrix representation of the Lorentz transformation L that operates on infinite-dimensional function space basis functions ϕ_a.

$$\phi = \sum_a c_a \phi_a \to \phi^a = M^a{}_b \phi_b \,; \quad c'_a = (M^{-1})_a{}^b c_b \,. \tag{3.2.4}$$

3.3 Lorentz Transformations of Scalar Fields

Applying a Lorentz transformation to a scalar field does not change the field itself. It only changes its argument ie. the coordinates at which the field is evaluated. The transformation of scalar fields is given by:

$$\phi(x) \quad \longrightarrow \quad \phi'(x) = \phi(x') \qquad x' = \Lambda^{-1} x \,. \tag{3.3.1}$$

This means that the transformed field evaluated at a new point x gives the same value as the original field evaluated at the corresponding pre-boost coordinate $x' = \Lambda^{-1}x$. To show this very explicitly consider for example that our original field $\phi(x)$ has a maximum at some point x_m. After a boost $x'_m = \Lambda x_m$ which we can plug into ϕ' to get:

$$\phi'(x'_m) = \phi(\Lambda^{-1}(\Lambda x_m)) = \phi(x_m) = \max. \,, \tag{3.3.2}$$

which states that the transformed field ϕ' has a maximum at the boosted point x'_m, ie. the location of the maximum changes under a boost but the value of the maximum does not. This can be generalized to any coordinate x which is boosted to $x' = \Lambda x$. We have:

$$\phi'(x') = \phi'(\Lambda x) = \phi(\Lambda^{-1}(\Lambda x)) = \phi(x) \,. \tag{3.3.3}$$

Thus no value changes, only spacetime locations of these values change.

3.3.1 Example: Lorentz Transformation of the Klein-Gordon equation

As an exercise we can show that the Klein-Gordon equation from Equation (2.0.1) is invariant under Lorentz transformations. Let us consider a general Lorentz transformation Λ. Our problem is:

$$\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi(x) + m^2\phi(x)\right) = 0 \quad \overset{?}{\longrightarrow} \quad \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi'(x) + m^2\phi'(x)\right) = 0 \,. \tag{3.3.4}$$

We solve this simply by using the chain rule to evaluate the derivative:

$$\frac{\partial}{\partial x^\mu}\phi'(x) = \frac{\partial}{\partial x^\mu}\phi(\Lambda^{-1}x) = \frac{\partial \phi(\tilde{x})}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \frac{\partial \phi(\tilde{x})}{\partial \tilde{x}^\alpha}\left(\Lambda^{-1}\right)^\alpha{}_\mu \,, \tag{3.3.5}$$

where we have defined $\tilde{x}^\alpha = (\Lambda^{-1})^\alpha{}_\mu x^\mu$. So for two derivatives we have:

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi'(x) = \left(\Lambda^{-1}\right)^\alpha{}_\mu \left(\Lambda^{-1}\right)^{\beta\mu}\frac{\partial}{\partial \tilde{x}^\alpha}\frac{\partial}{\partial \tilde{x}^\beta}\phi(\tilde{x}) = g^{\alpha\beta}\frac{\partial}{\partial \tilde{x}^\alpha}\frac{\partial}{\partial \tilde{x}^\beta}\phi(\tilde{x}) \,. \tag{3.3.6}$$

where we made use of the properties from Equations (3.1.1) and (3.1.2). We can insert this into the Klein-Gordon equation to get:

$$\frac{\partial}{\partial \tilde{x}^\alpha}\frac{\partial}{\partial \tilde{x}_\alpha}\phi(\tilde{x}) + m^2\phi(\tilde{x}) = 0 \,, \tag{3.3.7}$$

which is exactly the same equation as before, just with a different argument $\tilde{x}$. Thus the Klein-Gordon equation is invariant under Lorentz transformations.

Let's also quickly have a look at what happens to the Lagrangian (see Equation (2.2.2)) upon a Lorentz transformation:

$$\mathcal{L} \quad \to \quad \mathcal{L}' = \mathcal{L}(\Lambda^{-1}x)\,. \tag{3.3.8}$$

Action must be invariant if we want our equations of motion to be invariant:

$$S \quad \to \quad S' = \int \mathcal{L}(\Lambda^{-1}x)\mathrm{d}^4x = \int \mathrm{d}^4\tilde{x}\; \mathcal{L}(\tilde{x})\det(\Lambda) = S\,, \tag{3.3.9}$$

which checks out since $\det(\Lambda) = 1$ for Lorentz transformations. This argument generalizes beyond only scalar fields. For Lagrangians that involve spinor fields ψ or vector fields A^μ the combination of fields and derivatives must be such that the action is invariant under Lorentz transformations. In this sense, the Lagrangian density behaves as a scalar under Lorentz transformations, even though the fields themselves transform non-trivially.

3.4 Lorentz Transformations of Vector Fields

In QFT we usually mark vector fields with $A^\mu(x)$ (usually marking the photon or gluon field) or $V^\mu(x)$ (which marks a general vector field). The transformation of vector fields is given by the following relations.

Remember: Lorentz Transformations of Vector Fields

The transformation of vector fields is given by:

$$V^i(x) \quad \to \quad V^{i'}(x) = R^{ij}V^j(\Lambda^{-1}x)\,, \tag{3.4.1}$$

$$V^\mu(x) \quad \to \quad V^{\mu'}(x) = \Lambda^{\mu'}{}_\nu V^\nu(\Lambda^{-1}x)\,. \tag{3.4.2}$$

3.4.1 Example: Invariance of Maxwell's Equations

The relativistic form of the first and second Maxwell's equations are given by:

$$\partial_\mu F^{\mu\nu} = 0 \quad \to \quad \frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x_\mu}A^\nu(x) - \frac{\partial}{\partial x_\nu}A^\mu(x)\right) = 0\,, \tag{3.4.3}$$

The third and fourth equations (the ones that contain the rotors) are given by the Hodge dual of $\star F^{\mu\nu}$, but that is outside the scope of this course. Using the transformation of the vector field as stated above one can show that it holds that:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x_\mu}A^{\nu'}(x) - \frac{\partial}{\partial x_\nu}A^{\mu'}(x)\right) = 0\,, \tag{3.4.4}$$

where we used the transformation of the vector field as stated above $A^\mu(x) \to A^{\mu'}(x) = \Lambda^{\mu'}{}_\nu A^\nu(\Lambda^{-1}x)$. For posterity here is the Lagrangian density of Maxwell's equations. We get this result from taking Gauge invariance into account, which we're not going to do now but is a part of prof. Kamenik's course *Gauge Field Theory*. The Lagrangian density is given by:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x)\,. \tag{3.4.5}$$

By applying Euler-Lagrange equations (2.1.8) we directly get back Maxwell's equations.

3.5 Representations of Lorentz Transformations

We've stated earlier that representations $M(L)$ of a Lorentz group element $L \in \mathrm{SO}(1,3)$ depend on the spin of the field we're operating on. For a general field we have said that the following holds:

$$\phi^a(x) \quad \to \quad \phi^{a'}(x) = M^{a'}{}_b(L)\phi^b(\Lambda^{-1}x)\,, \tag{3.5.1}$$

where $M(L)$ is the representation of L that acts on the field ϕ and Λ is the representation of the same group element L that acts on 4-vectors. Think of $M(L)$ as a function that returns the representation of the Lorentz transformation L appropriate for the field it acts on. We've found via the examples above that $M(L)$ is:

- $M(L) = \mathbb{1}_{1\times 1}$ (a trivial matrix) for **scalar fields** with spin $s = 0$.
- $M(L) = \Lambda$ (a 4×4 matrix) for **vector fields** with spin $s = 1$.
- $M(L) = ?$ for **spinor fields** with spin $s = 1/2$.

Since we also want to describe fermions in QFT we need to find the representation $M(L)$ for spinor fields with spin $s = 1/2$. That is our motivation to study the representations of Lorentz transformations. Representations of a group are quite a complex topic (and group theory in general is out of the scope of this subject), but for us it will be sufficient if we think of of them as matrices $M^a{}_b(L)$ as we have done above.

Peskin states that it can be shown that most general nonlinear transformation laws can be built from these linear transformations so it makes no sense to consider transformations more general than as stated above in Equation (3.5.1). To simplify further we can forget about the change in the field argument at transformation. With that we can write the previous equation as:

$$\phi \quad \rightarrow \quad \phi' = M(L)\phi \,. \tag{3.5.2}$$

Since Lorentz transformations form a group we have restrictions on the form of the matrices $M(L)$. As said in the introduction to this section, if we consider two successive Lorentz transformations L and L' the net result must be a new Lorentz transformation L'', which is also a group element. This gives us a simple condition (group multiplication structure) that must be satisfied by the matrices $M(L)$:

$$\phi \quad \rightarrow \quad M(L')M(L)\phi = M(L'')\phi \,, \tag{3.5.3}$$

where we've marked $L'' = L'L$. This means that the correspondence between the matrices $M(L)$ and the Lorentz transformation elements L must be preserved under multiplication. Mathematically this means that the matrices $M(L)$ form an n-dimensional **representation** of the Lorentz group.

3.5.1 Example: Finite-dimensional Representations for $s = 1/2$

To find the representation $M(L)$ for spinor fields with spin $s = 1/2$ we can start by considering infinitesimal transformations. Let us for example consider a special case of a Lorentz transformation L which corresponds to a **pure spatial rotation**. In this case $L \in \mathrm{SO}(3) \subset \mathrm{SO}(1,3)$ and the corresponding representation on spin-s fields is given by:

$$M(L) = \exp\left(\sum_k -i\Theta^k J^k_{(s)}\right) \,, \tag{3.5.4}$$

where $J^k_{(s)}$ are the rotation generators of the Lorentz Group, ie. matrices that are dependent on spin. The parameters Θ^k are the angles of rotation about the three spatial axes. In general the Lorentz group also contains generators for boosts $K^k_{(s)}$, but as we've limited ourselves to pure rotations $L \in \mathrm{SO}(3)$ we do not need to think about them. Rotation generators obey the commutation relation:

$$\left[J^l_{(s)},\, J^m_{(s)}\right] = i\varepsilon^{lmk} J^k_{(s)} \,. \tag{3.5.5}$$

For a Dirac spinor field with spin $s = 1/2$ the rotation generators are:

$$J^k_{(s=1/2)} = \frac{1}{2}\sigma^k \,, \tag{3.5.6}$$

where σ^k are the Pauli matrices. This means that the representation of a pure rotation $L \in \mathrm{SO}(3)$ on a spin-1/2 field is given by:

$$M_{(s=1/2)}(L) = \exp\left(-i\frac{\boldsymbol{\Theta}\cdot\boldsymbol{\sigma}}{2}\right) = \mathbb{1}\cos\frac{|\boldsymbol{\Theta}|}{2} - i\sin\frac{|\boldsymbol{\Theta}|}{2}\frac{\boldsymbol{\Theta}\cdot\boldsymbol{\sigma}}{|\boldsymbol{\Theta}|} \,, \tag{3.5.7}$$

which follows from the identity $(\hat{n}\cdot\boldsymbol{\sigma})^2 = \mathbb{1}$, where $\hat{n} = \boldsymbol{\Theta}/|\boldsymbol{\Theta}|$ is the unit vector in the direction of the rotation axis, and the power series expansion of the exponential function. To clean up we can define $\theta = |\boldsymbol{\Theta}|$ as the angle of rotation, which brings us to:

$$M_{(s=1/2)}(L) = \mathbb{1}\cos\frac{\theta}{2} - i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}{|\boldsymbol{\theta}|}\sin\frac{\theta}{2}\,. \tag{3.5.8}$$

For a general element $L \in \mathrm{SO}(1,3)$ the form of the matrices $M(L)$ is given by:

$$M(L) = \exp\left[-\frac{i}{2}\,w_{\mu\nu}J^{\mu\nu}_{(s)}\right]\,, \tag{3.5.9}$$

where $w_{\mu\nu}$ are the parameters of the Lorentz transformation that encode both rotation angles for spatial components and rapidities for boosts. $J^{\mu\nu}_{(s)}$ are now generalized generators that include both rotations and boosts, called **generalized angular momentum operators**. This means that the transformation induced by L can alternatively be obtained by the action of the infinitesimal form of the equation above. Generalized angular momentum operators obey the following relations:

$$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) = i\left(x^\mu\frac{\partial}{\partial x_\nu} - x^\nu\frac{\partial}{\partial x_\mu}\right)\,, \tag{3.5.10}$$

$$\left[J^{\mu\nu},\, J^{\rho\sigma}\right] = \cdots = i\left(g^{\mu\sigma}J^{\nu\rho} + g^{\nu\rho}J^{\mu\sigma} + g^{\mu\rho}J^{\nu\sigma} + g^{\nu\sigma}J^{\mu\rho}\right)\,, \tag{3.5.11}$$

where the second equation is the commutation relation of the angular momentum operators, which can be better understood if we split it into the commutation relation for rotations and commutation relation for boosts:

$$\hat{J}_i = \frac{1}{2}\varepsilon_{ijk}J_{jk}\,,\quad K_i = J_{0i}\,. \tag{3.5.12}$$

From this we can figure out that rotations have indices $(1,2)$, $(1,3)$, $(2,3)$ and boosts have indices $(0,1)$, $(0,2)$, $(0,3)$. $J^{\mu\nu}$ for a Dirac field with spin $s = 1/2$ are defined as:

$$J^{\mu\nu}_{(1/2)} = \frac{i}{2}\left[\gamma^\mu,\, \gamma^\nu\right] = \frac{1}{2}\sigma^{\mu\nu}\,, \tag{3.5.13}$$

where note that Peskin denotes this as $S^{\mu\nu}$. Gamma matrices obey the anti-commutation relation:

$$\left\{\gamma^\mu,\, \gamma^\nu\right\} = 2g^{\mu\nu}\mathbb{1}_{n\times n}\,. \tag{3.5.14}$$

Thus we've come to the discovery that for spin $s = 1/2$, in the chiral representation of the gamma matrices (see Equation (4.1.3)), $M_{(s=1/2)}(L)$ is block diagonal as so:

$$M_{(s=1/2)}(L) = \begin{bmatrix} \# & 0 \\ 0 & \# \end{bmatrix}\,, \tag{3.5.15}$$

where the blocks $\#$ are 2×2 matrices. The spinor field is then defined as $\psi = (\psi_L,\, \psi_R)^\mathsf{T}$, where ψ_L and ψ_R are left- and right-handed two-component Weyl/chiral spinors.

Using what we've learned above we can now figure out the form of the matrices $J^{\mu\nu}$ in the case of a vector field with spin $s = 1$:

$$(J^{\mu\nu}_{(1)})^\alpha{}_\beta = i\left(g^{\mu\alpha}\delta^\nu{}_\beta - \delta^\mu{}_\beta g^{\nu\alpha}\right)\,;\quad \alpha,\beta = 0,1,2,3\,. \tag{3.5.16}$$

These are 4×4 matrices that act on the components of a vector field V^μ.

4 Dirac Spinor Fields

To properly describe fermions in QFT we need to introduce a new type of field, the Dirac spinor field. The spinor field is defined as:

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}\,, \tag{4.0.1}$$

where ψ_L and ψ_R are left- and right-handed two-component Weyl/chiral spinors. This is a 4-component complex field that transforms under Lorentz transformations as:

$$S^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{bmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{bmatrix} \quad \text{(boost generators)}\,, \tag{4.0.2}$$

$$S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\epsilon^{ijk}\begin{bmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{bmatrix} \quad \text{(rotation generators)}\,. \tag{4.0.3}$$

Here σ^i are the Pauli matrices.

4.1 Dirac Equation and Dirac Adjoint

We have a few demands for our equations of motion:

1. The equations of motion should be Lorentz invariant.
2. Solutions of the equations of motion should have positive energy, $E = \sqrt{\boldsymbol{p}^2 + m^2}$.
3. The equations of motion should be first order in time derivatives.

While the first two demands seem pretty much self-explanatory, the third demand might need additional explanation. The reason for the third demand is that we'd like to avoid specifying boundary conditions the way we must do for second order equations (e.g. the Klein-Gordon equation). This is because the boundary conditions for second order equations also require the initial value of the first time derivative. Demanding the equation to be first order in time gives us a well-defined single-time initial condition (which means no ambiguous negative energy doubling) and hence ensures a consistent single-particle probabilistic interpretation. With that said, let's write down the Dirac equation.

Remember: Dirac Equation

The Dirac equation is given by

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\,\mathbb{1}\right)\psi(x) = 0\,, \tag{4.1.1}$$

where γ^μ are the gamma/Dirac matrices that form the representation of the Lorentz group in spinor space. In the **Dirac representation** the gamma matrices are given by:

$$\gamma^0 = \begin{bmatrix}\mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2}\end{bmatrix}\,, \quad \gamma^i = \begin{bmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{bmatrix} \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix}0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0\end{bmatrix}\,, \tag{4.1.2}$$

where σ^i are the Pauli matrices, $i = 1, 2, 3$ and γ^5 is an additional *traceless* matrix sometimes used in conjunction with the others. Take note that gamma matrices are also commonly defined in other representations, such as the Weyl (chiral) and Majorana representations. Peskin uses the chiral representation in his chapter on spinor fields, which is given by:

$$\gamma^0 = \begin{bmatrix}0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0\end{bmatrix}\,, \quad \gamma^i = \begin{bmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{bmatrix} \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix}-\mathbb{1}_{2\times2} & 0 \\ 0 & \mathbb{1}_{2\times2}\end{bmatrix}\,. \tag{4.1.3}$$

We've essentially swapped γ^0 and γ^5 compared to the Dirac representation. More information on γ^μ can be found in the appendix A.1.

Lets check if our demands have been met:

$$
\begin{aligned}
(i\gamma^\mu\partial_\mu - m)\,\psi(x) \to & \left(i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right)\Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\Lambda_{1/2}^{-1}\left(i\gamma^\mu(\lambda^{-1\nu})_\mu\partial_\nu - m\right)\Lambda_{1/2}\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2}(\Lambda^{-1\nu})_\mu\partial_\nu - m\right)\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\Lambda^\mu{}_\sigma\gamma^\sigma(\Lambda^{-1})^\nu{}_\mu\partial_\nu - m\right)\psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\left(i\gamma^\nu\partial_\nu - m\right)\psi(\Lambda^{-1}x) \\
&= 0\,.
\end{aligned}
$$

where we took into account $\tilde{x} = \Lambda^{-1}x$ and $M_{1/2} = \Lambda_{1/2}$. Lets check the second demand by acting on the Dirac equation with $(-i\gamma^\mu\partial_\mu - m)$:

$$
\begin{aligned}
0 &= (-i\gamma^\mu\partial_\mu - m)\,(i\gamma^\nu\partial_\nu - m)\,\psi \\
&= (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi \\
&= (\frac{1}{2}\{\gamma^\mu,\,\gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi \\
&= \left(\partial^2 + m^2\right)\psi\,.
\end{aligned}
$$

where we see that we came to the Klein-Gordon equation and with that the second demand is met. Finally, the Dirac equation is first order in time derivatives and thus the third demand is also met.

To be able to write down a Lagrangian for the Dirac equation we must first figure out how to multiply two Dirac spinors to form a Lorentz scalar. The obvious guess of $\psi^\dagger\psi$ does not work since under a Lorentz boost the previous becomes $\psi^\dagger\Lambda_{1/2}^\dagger\Lambda_{1/2}\psi$. If the boost matrix were unitary then $\Lambda_{1/2}^\dagger = \Lambda_{1/2}^{-1}$ and everything would be fine. However this is not the case as the generators for Lorentz transformations on spin-1/2 fields (4.0.2) are not Hermitian. We can fix this by defining the **Dirac adjoint** as follows.

Remember: Dirac Adjoint

The Dirac adjoint defines the dual operation of a Dirac spinor. It is defined as:

$$\bar{\psi} \equiv \psi^\dagger\gamma^0\,. \tag{4.1.4}$$

The Dirac adjoint transforms under Lorentz transformations as:

$$\bar{\psi}(\Lambda x) = \bar{\psi}(x)\Lambda_{1/2}^\dagger\,. \tag{4.1.5}$$

and thus the product $\bar{\psi}\psi$ is a Lorentz scalar.

4.2 Solutions of the Dirac Equation for Free Fermions

To obtain plane-wave solutions of the Dirac equation (detailed derivation in Peskin pg. 45) we can start with the plane wave ansatz:

$$\psi(x) = u(p)e^{-ip\cdot x}\,, \tag{4.2.1}$$

where, as we've said, $p^2 = m^2$. Let's concentrate on solutions with a positive frequency, meaning $p^0 > 0$. The column vector $u(p)$ must obey an additional constraint that is found by plugging it into the Dirac equation:

$$(\gamma^\mu p_\mu - m)u(p) = 0\,, \tag{4.2.2}$$

where γ^μ used here are in the **chiral representation** (4.1.3). This will naturally split the spinor components into left-handed ξ_L and right-handed two-component spinors ξ_R. Let's solve the Dirac equation in the rest frame as it's the easiest, where $p = p_0 = (m, \mathbf{0})$. The general solution can then be found using a Lorentz boost $\Lambda_{1/2}$. In the rest frame Equation (4.2.2) becomes:

$$(m\gamma^0 - m)u(p_0) = m \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u(p_0) = 0\,. \tag{4.2.3}$$

For this to hold $u(p_0)$ must be:

$$u(p_0) = \sqrt{m} \begin{bmatrix} \xi_L \\ \xi_R \end{bmatrix} = \sqrt{m} \begin{bmatrix} \xi \\ \xi \end{bmatrix}\,, \tag{4.2.4}$$

where ξ is a two-component spinor and $\xi_L = \xi_R = \xi$ in the rest frame. After a Lorentz boost, ξ_L and ξ_R are no longer identical. The boost mixes left- and right-handed components in a momentum-dependent way. We usually normalize two-component spinors as $\xi^\dagger \xi = 1$. The pre-factor of $\sqrt{m}$ has been added to satisfy the standard relativistic normalization convention for four-component spinors:

$$\overline{u}^s(p) u^r(p) = 2m\delta_{sr}\,. \tag{4.2.5}$$

ξ transforms as an ordinary two-component spinor under rotations. Which means that it determines the spin orientation of the Dirac solution in the rest frame:

$$\xi_\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad u_\uparrow(p_0) = \sqrt{m} \begin{bmatrix} \xi_\uparrow \\ \xi_\uparrow \end{bmatrix} = \sqrt{m} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{(spin up)}\,, \tag{4.2.6}$$

$$\xi_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad u_\downarrow(p_0) = \sqrt{m} \begin{bmatrix} \xi_\downarrow \\ \xi_\downarrow \end{bmatrix} = \sqrt{m} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{(spin down)}\,. \tag{4.2.7}$$

Notice how we were only able to choose two of the four components of $u(p)$. This is in accordance with the fact that we're studying spin-1/2 particles which only have two possible physical states, up and down. We can obtain $u(p)$ for any other frame by performing a Lorentz boost. A boost transforms a 4-momentum vector as:

$$\begin{bmatrix} E \\ p^3 \end{bmatrix} = \left(1 + \eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} m \\ 0 \end{bmatrix}\,, \tag{4.2.8}$$

where η is an infinitesimal parameter called the **rapidity**. It is the quantity that is additive for successive boosts. For a finite η the transformation becomes:

$$\begin{aligned} \begin{bmatrix} E \\ p^3 \end{bmatrix} &= \exp\left(\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} m \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} m \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} m\cosh\eta \\ m\sinh\eta \end{bmatrix}\,. \end{aligned} \tag{4.2.9}$$

When we apply the boost in spinor space our left- and right-handed two-component spinors mix. We can work out that the rapidities are implicitly encoded in p^0 and $|\boldsymbol{p}|$, because:

$$\cosh\eta = \frac{E}{m}\,, \tag{4.2.10}$$

$$\sinh\eta = \frac{|\boldsymbol{p}|}{m}\,. \tag{4.2.11}$$

Thus after applying a boost to $u(p_0)$ we find (with a little work) that $u(p)$ must be:

$$u^s(p) = \begin{bmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\overline{\sigma}}\,\xi^s \end{bmatrix}\,, \tag{4.2.12}$$

here we've also taken into account that we have two possible solutions for ξ thus the index $s = 1, 2$ is added. The Dirac adjoint of $u(p)$ is as expected:

$$\bar{u}^s(p) \equiv u^{s\dagger}(p)\gamma^0 = \begin{bmatrix} \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \xi^{s\dagger}\sqrt{p\cdot\sigma} \end{bmatrix} , \tag{4.2.13}$$

a row vector with flipped (from γ^0) and conjugate transposed components. If we were to convert this result into the Dirac gamma matrix representation where:

$$p\cdot\sigma \equiv p^0 \mathbb{1}_{2\times 2} + \boldsymbol{p}\cdot\boldsymbol{\sigma} , \tag{4.2.14}$$

$$p\cdot\bar{\sigma} \equiv p^0 \mathbb{1}_{2\times 2} - \boldsymbol{p}\cdot\boldsymbol{\sigma} , \tag{4.2.15}$$

we would get:

$$u^s_{\text{Dirac}}(p) = \sqrt{E+m}\begin{bmatrix} \xi^s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\xi^s \end{bmatrix} , \tag{4.2.16}$$

which some might find easier to understand as it explicitly shows the dependence on energy and mass and how they affect components. A similar procedure can be done for negative energy solutions where we take the same plane wave ansatz:

$$\psi(x) = v(p)e^{+ip\cdot x} , \tag{4.2.17}$$

where we've put a + into the exponential rather than having $p_0 < 0$, hence this ansatz looks like an opposite plane wave. Following an identical procedure as before we find that $v(p)$ has to be:

$$v^s(p) = \begin{bmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{bmatrix} \qquad \bar{v}^s(p) \equiv v^{s\dagger}(p)\gamma^0 = \begin{bmatrix} \eta^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & -\eta^{s\dagger}\sqrt{p\cdot\sigma} \end{bmatrix} , \tag{4.2.18}$$

where η is a two-component spinor and since we have two possible solutions for η we've added the index $s = 1, 2$. When working with expressions of this form it is useful to know the identity:

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2 . \tag{4.2.19}$$

σ^μ is defined as $\sigma^\mu = (1,\ \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1,\ -\boldsymbol{\sigma})$, which follows from equations (4.2.14) and (4.2.15). For completeness, in the Dirac representation $v(p)$ becomes:

$$v^s_{\text{Dirac}}(p) = \sqrt{E+m}\begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\eta^s \\ \eta^s \end{bmatrix} . \tag{4.2.20}$$

4.3 Lagrangian, Hamiltonian and Conjugated Momenta of the Dirac Equation

After our discussion, most notably on the introduction of the Dirac adjoint, we can now write down the Lagrangian for the Dirac equation. The Lagrangian is given by:

$$\mathcal{L}(\psi_a,\ \partial_\mu\psi_a,\ \bar{\psi}_a,\ \partial_\mu\bar{\psi}_a)(x) = \bar{\psi}_a\left(i\gamma^\mu\partial_\mu - m\right)_{ab}\psi_b(x) . \tag{4.3.1}$$

If we apply the Euler-Lagrange equations (2.1.8) we quickly get the Dirac equation:

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}_a} - \frac{\partial\mathcal{L}}{\partial\bar{\psi}_a} = 0 = \partial_\mu\cdot 0 - \left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)_{ab}\psi_b . \tag{4.3.2}$$

To get the conjugated moment we use the formula as we've derived in the introduction. Thus we have:

$$\Pi_b(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\psi_b} = \bar{\psi}_a i\gamma^0_{ab} = \left(\psi^\dagger\gamma^0\gamma^0\right)_b i = \psi^\dagger_b\ i . \tag{4.3.3}$$

and for the Dirac adjoint $\bar{\psi}_b$ we simply have:

$$\Pi_b(x) = \frac{\partial\mathcal{L}}{\partial\partial_0\bar{\psi}_b} = 0 . \tag{4.3.4}$$

Now that we have the two previous expressions we can write down the Hamiltonian density, again just like we've done in the introduction (2.1.15):

$$\begin{aligned}\mathcal{H} &= \sum_a \Pi_a \dot{\psi}_a - \mathcal{L} \\ &= i\psi_a^\dagger \partial_0 \psi_a - \overline{\psi}\left(i\gamma^0\partial_0 + i\boldsymbol{\gamma}\cdot\nabla - m\right)_{ab}\psi_b \\ &= \psi^\dagger\left[\gamma^0\boldsymbol{\gamma}(-i\nabla) + \gamma^0 m\right]\psi \\ &= \psi^\dagger h_D \psi\,,\end{aligned} \tag{4.3.5}$$

where $h_D\psi(x) = i\partial_0\psi(x) = -i\boldsymbol{\alpha}\cdot\nabla + m\beta$ if we mark $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, $\beta = \gamma^0$ and $\boldsymbol{p} = (-i\nabla)$. h_D is the **Dirac Hamiltonian** that we know from one-particle quantum mechanics. We normalize the linearly independent solutions of the Dirac equation as:

$$\overline{u}^r(p)u^s(p) = 2m\delta^{rs}\,, \tag{4.3.6}$$

$$\overline{v}^r(p)v^s(p) = -2m\delta^{rs}\,, \tag{4.3.7}$$

$$u^{r\dagger}(p)u^s(p) = 2E_{\boldsymbol{p}}\delta^{rs}\,, \tag{4.3.8}$$

$$v^{r\dagger}(p)v^s(p) = +2E_{\boldsymbol{p}}\delta^{rs}\,. \tag{4.3.9}$$

$$\tag{4.3.10}$$

The solutions $u(p)$ and $v(p)$ are orthogonal to each other:

$$\overline{u}^r(p)v^s(p) = \overline{v}^r(p)u^s(p) = 0\,. \tag{4.3.11}$$

however do note **that the following relations are non-zero** leading to another property:

$$u^{r\dagger}(p)v^s(p) \neq 0\,, \tag{4.3.12}$$

$$v^{r\dagger}(p)u^s(p) \neq 0\,, \tag{4.3.13}$$

$$u^{r\dagger}(\boldsymbol{p})v^s(-\boldsymbol{p}) = v^{r\dagger}(-\boldsymbol{p})u^s(\boldsymbol{p}) = 0\,. \tag{4.3.14}$$

When evaluation Feynman diagrams we will often want to sum over the polarization/spin states of a fermion. Following a derivation from Peskin pg. 48 we come to the result:

$$\sum_s u^s(p)\overline{u}^s(p) = \gamma\cdot p + m = \not{p} + m\,, \tag{4.3.15}$$

$$\sum_s v^s(p)\overline{v}^s(p) = \gamma\cdot p - m = \not{p} - m\,. \tag{4.3.16}$$

The combination $\gamma\cdot p$ occurs very often, so much so that Feynman introduced a new notation for it.

Remember: Feynman Slash Notation

Feynman's slash notation is defined as:

$$\not{A} \equiv \gamma^\mu A_\mu = \gamma^0 A_0 + \gamma^1 A_1 + \gamma^2 A_2 + \gamma^3 A_3\,, \tag{4.3.17}$$

where A_μ is a covariant vector (more generally a 1-form). The signs are positive since we only used the Einstein summation convention and not the metric. Upon using the Minkowski metric we can see that for the corresponding contravariant components A^μ we have:

$$\not{A} \equiv \gamma_\mu A^\mu = \gamma^0 A^0 - \gamma^1 A^1 - \gamma^2 A^2 - \gamma^3 A^3\,. \tag{4.3.18}$$

With that we can define p *slash*, or more formally the slashed 4-momentum, as:

$$\not{p} \equiv \gamma^0 p^0 - \gamma^i p^i\,. \tag{4.3.19}$$

4.4 Quantization of the Dirac Field

As we've done for the real scalar field we now promote the classic fields to operators however the commutation relations we postulate are different. The classic field $\psi(x)$ is given by:

$$\hat{\psi}(x) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left(A^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ipx}\, a^s_{\boldsymbol{p}} + B^s_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ipx}\, {b^s}^\dagger_{\boldsymbol{p}} \right) , \tag{4.4.1}$$

where we are free to choose $A^s_{\boldsymbol{p}}$ and $B^s_{\boldsymbol{p}}$ as we please. The operators $a^s_{\boldsymbol{p}}$ and ${b^s}^\dagger_{\boldsymbol{p}}$ are the annihilation and creation operators for particles and antiparticles respectively. We have two possible choices for the postulation of the commutation relations. The first choice is to postulate the commutation relations with **commutation relations**:

$$\left[\hat{\psi}_a(\boldsymbol{x}, t),\ \hat{\Pi}_b(\boldsymbol{y}, t)\right] = i\hbar \delta_{ab} \delta(\boldsymbol{x} - \boldsymbol{y}) , \tag{4.4.2}$$

$$\left[\hat{\psi}_a(\boldsymbol{x}, t),\ \hat{\psi}_b(\boldsymbol{y}, t)\right] = 0 , \tag{4.4.3}$$

$$\left[\hat{\Pi}_a(\boldsymbol{x}, t),\ \hat{\Pi}_b(\boldsymbol{y}, t)\right] = 0 . \tag{4.4.4}$$

This would seem like a reasonable choice, as it is how we quantized the real scalar field. However, this **does not work**, as it leads to solutions with negative energies. The correct choice is to postulate the commutation relations as **anticommutation relations**, which can be done as follows.

Remember: Commutation Relations and Field Operators of the Dirac Field

We postulate the commutation relations as **anticommutation relations**, which fixes the problem of negative energies:

$$\left\{\psi_a(\boldsymbol{x}, t),\ \psi^\dagger_b(\boldsymbol{y}, t)\right\} = \delta_{ab} \delta(\boldsymbol{x} - \boldsymbol{y}) , \tag{4.4.5}$$

$$\left\{\psi_a(\boldsymbol{x}, t),\ \psi_b(\boldsymbol{y}, t)\right\} = 0 , \tag{4.4.6}$$

$$\left\{\psi^\dagger_a(\boldsymbol{x}, t),\ \psi^\dagger_b(\boldsymbol{y}, t)\right\} = 0 , \tag{4.4.7}$$

where we've chosen $A^s_{\boldsymbol{p}} = B^s_{\boldsymbol{p}} = 1$. The field operators $\hat{\psi}(x)$ are then given by:

$$\psi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left(a^s_{\boldsymbol{p}} u^s(p) e^{-ip\cdot x} + {b^s}^\dagger_{\boldsymbol{p}} v^s(p) e^{ip\cdot x} \right) , \tag{4.4.8}$$

$$\bar{\psi}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \sum_s \left(b^s_{\boldsymbol{p}} \bar{v}^s(p) e^{-ip\cdot x} + {a^s}^\dagger_{\boldsymbol{p}} \bar{u}^s(p) e^{ip\cdot x} \right) , \tag{4.4.9}$$

where the sum goes over $s = 1, 2$ and the creation and annihilation operators satisfy the following anticommutation relation:

$$\left\{a^r_{\boldsymbol{p}},\ {a^s}^\dagger_{\boldsymbol{q}}\right\} = \left\{b^r_{\boldsymbol{p}},\ {b^s}^\dagger_{\boldsymbol{q}}\right\} = (2\pi)^3 \delta^{rs} \delta(\boldsymbol{p} - \boldsymbol{q}) , \tag{4.4.10}$$

while all other anticommutators are equal to zero. The vacuum state $|0\rangle$ is defined to be the state such that the following holds:

$$a^s_{\boldsymbol{p}}|0\rangle = b^s_{\boldsymbol{p}}|0\rangle = 0 . \tag{4.4.11}$$

With these definitions both ${a^s}^\dagger_{\boldsymbol{p}}$ and ${b^s}^\dagger_{\boldsymbol{p}}$ both create particles with positive energy and momentum $\boldsymbol{p}$. We refer to the particles created by ${a^s}^\dagger_{\boldsymbol{p}}$ as **fermions** and those created by ${b^s}^\dagger_{\boldsymbol{p}}$ as **antifermions**.

4.5 Eigenstates and Eigenvalues of Operators Applied to Fermions

Now that we've successfully quantized the Dirac spinor field we can start to study excitations of the field which represent fermions. What we're looking for:

$$\hat{\mathcal{O}} {a^s}^\dagger_{\boldsymbol{p}}|0\rangle = \lambda_{\mathcal{O}} {a^s}^\dagger_{\boldsymbol{p}}|0\rangle , \tag{4.5.1}$$

$$\hat{\mathcal{O}} {b^s}^\dagger_{\boldsymbol{p}}|0\rangle = \bar{\lambda}_{\mathcal{O}} {b^s}^\dagger_{\boldsymbol{p}}|0\rangle , \tag{4.5.2}$$

where $\lambda_{\mathcal{O}}$ and $\overline{\lambda}_{\mathcal{O}}$ are the eigenvalues of the operator $\mathcal{O}$ for fermions and antifermions respectively. One-particle states:

$$|\boldsymbol{p},\, s\rangle \equiv \sqrt{2E_{\boldsymbol{p}}}\, a^{s\,\dagger}_{\boldsymbol{p}}|0\rangle \;, \tag{4.5.3}$$

are defined so that their inner product is Lorentz invariant:

$$\langle \boldsymbol{p},\, r|\boldsymbol{q},\, s\rangle = 2E_{\boldsymbol{p}}(2\pi)^3\delta^{rs}\delta(\boldsymbol{p}-\boldsymbol{q})\,. \tag{4.5.4}$$

This implies that the operator $U(\Lambda)$ that implements Lorentz transformations on the states must be unitary, even though we know that for boosts $\Lambda_{1/2}$ is not unitary. To find the way ladder operators transform we calculate:

$$U\psi(x)U^{-1} = U\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_s\left(a^s_{\boldsymbol{p}}u^s(p)e^{-ip\cdot x} + b^{s\,\dagger}_{\boldsymbol{p}}v^s(p)e^{ip\cdot x}\right)U^{-1}\,. \tag{4.5.5}$$

Equation (4.5.3) implies that the ladder operators transform as:

$$U(\Lambda)a^s_{\boldsymbol{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\boldsymbol{p}}}{E_{\boldsymbol{p}}}}\,a^s_{\Lambda\boldsymbol{p}}\,. \tag{4.5.6}$$

4.5.1 The Hamiltoniam Operator

We've now acquired everything we need to write down the Hamiltonian operator For the Dirac field. First let's quickly discuss the Dirac Hamiltonian h_D (4.3.5) and how it works on the one-particle states (4.5.3). Applied it gives:

$$h_Du(\boldsymbol{p})e^{-ip\cdot x} = +E_{\boldsymbol{p}}u(\boldsymbol{p})e^{-ip\cdot x}\;, \tag{4.5.7}$$

$$h_Dv(\boldsymbol{p})e^{ip\cdot x} = -E_{\boldsymbol{p}}v(\boldsymbol{p})e^{ip\cdot x}\;. \tag{4.5.8}$$

Finally the Hamiltonian operator is given by:

$$\begin{aligned}\hat{H} &= \int \mathrm{d}\boldsymbol{x}\;\hat{\psi}^\dagger(\boldsymbol{x})h_D\hat{\psi}(\boldsymbol{x}) = \int \mathrm{d}\boldsymbol{x}\;\mathcal{H}(\boldsymbol{x})\\ &= \int \mathrm{d}\boldsymbol{x}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_s\left(a^{s\,\dagger}_{\boldsymbol{p}}u^{s\,\dagger}(\boldsymbol{p}) + b^s_{-\boldsymbol{p}}v^{s\,\dagger}(\boldsymbol{p})\right)e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\\ &\times\int\frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}}}}\sum_r\left(+E_{\boldsymbol{q}}a^r_{\boldsymbol{q}}u^r(\boldsymbol{q}) + (-E_{\boldsymbol{q}})b^{r\,\dagger}_{-\boldsymbol{q}}v^r(\boldsymbol{x})\right)e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\\ &= \int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\sum_s\left(a^{s\,\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}E_{\boldsymbol{p}} - b^s_{\boldsymbol{p}}b^{s\,\dagger}_{\boldsymbol{p}}E_{\boldsymbol{p}}\right)\\ &= \int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}E_{\boldsymbol{p}}\sum_s\left(a^{s\,\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}} + b^{s\,\dagger}_{\boldsymbol{p}}b^s_{\boldsymbol{p}} - (2\pi)^3\delta(0)\right)\;,\end{aligned} \tag{4.5.9}$$

The last term implies that the vacuum state has $-\infty$ energy. We sweep that under the rug by renormalizing the energy scale. We will discuss this further when we talk about the operator of electric charge. I think it deserves a box.

Remember: Hamiltonian Operator of the Dirac Field

The Hamiltonian operator for the Dirac field is given by:

$$\hat{H} = \int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}E_{\boldsymbol{p}}\sum_s\left(a^{s\,\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}} + b^{s\,\dagger}_{\boldsymbol{p}}b^s_{\boldsymbol{p}}\right)\;. \tag{4.5.10}$$

Trying out the fruits of our labor let's consider a one-particle anti-particle state:

$$\begin{aligned}\hat{H}b^{s\,\dagger}_{\boldsymbol{p}}|0\rangle &= b^{s\,\dagger}_{\boldsymbol{p}}\hat{H}|0\rangle + \left[\hat{H},\, b^{s\,\dagger}_{\boldsymbol{p}}\right]|0\rangle\\ &= E_0b^{s\,\dagger}_{\boldsymbol{p}}|0\rangle + E_{\boldsymbol{p}}b^{s\,\dagger}_{\boldsymbol{p}}|0\rangle\\ &\Rightarrow E = E_0 + E_{\boldsymbol{p}}\;,\end{aligned} \tag{4.5.11}$$

where the commutator here is calculated as:

$$
\begin{aligned}
[\hat{H},\, b^{s\dagger}_{\boldsymbol{p}}] &= \dots (a^{s'\dagger}{}_{\boldsymbol{p'}} a^{s'\dagger}{}_{\boldsymbol{p'}} b^{s\dagger}_{\boldsymbol{p}} + b^{s'\dagger}{}_{\boldsymbol{p'}} b^{s'}{}_{\boldsymbol{p}} b^{s\dagger}_{\boldsymbol{p}}) - \dots (b^{s\dagger}_{\boldsymbol{p}} a^{s'\dagger}{}_{\boldsymbol{p'}} a^{s'}{}_{\boldsymbol{p'}} + b^{s\dagger}_{\boldsymbol{p}} b^{s'\dagger}{}_{\boldsymbol{p'}} b^{s'}{}_{\boldsymbol{p'}}) \\
&= a^{s'\dagger}{}_{\boldsymbol{p'}} a^{s'\dagger}{}_{\boldsymbol{p'}} b^{s\dagger}_{\boldsymbol{p}} + b^{s'\dagger}{}_{\boldsymbol{p'}} b^{s'}{}_{\boldsymbol{p}} b^{s\dagger}_{\boldsymbol{p}} - a^{s'\dagger}{}_{\boldsymbol{p'}} a^{s'\dagger}{}_{\boldsymbol{p'}} b^{s\dagger}_{\boldsymbol{p}} - \underline{(-1) \cdot b^{s'\dagger}{}_{\boldsymbol{p'}} b^{p\dagger}_{\boldsymbol{p}}} b^{s'}{}_{\boldsymbol{p'}} \\
&= b^{s'\dagger}{}_{\boldsymbol{p'}} \left(b^{s'}{}_{\boldsymbol{p}} b^{s\dagger}_{\boldsymbol{p}} + b^{p\dagger}_{\boldsymbol{p}} b^{s'}{}_{\boldsymbol{p'}} \right) \\
&= b^{s'\dagger}{}_{\boldsymbol{p'}} \{ b^{s'}_{\boldsymbol{p'}},\, b^{s\dagger}_{\boldsymbol{p}} \} \,,
\end{aligned}
\tag{4.5.12}
$$

which is nonzero thus meaning that the additional $E_{\boldsymbol{p}}$ factor above is explained. We see that energy is well defined against the energy of the vacuum state. What about a multi-particle state of two noninteracting fermions? We have:

$$
\begin{aligned}
&a^{s_1\dagger}_{\boldsymbol{p}_1} a^{s_2\dagger}_{\boldsymbol{p}_2} |0\rangle \;:\quad E = E_0 + E_{\boldsymbol{p}_1} + E_{\boldsymbol{p}_2} \,, \\
&\text{however} \quad a^{s\dagger}_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{p}} |0\rangle = -a^{s\dagger}_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{p}} |0\rangle = 0 \,.
\end{aligned}
\tag{4.5.13}
$$

The same does not hold when it comes to combinations of particles and antiparticles, what's more they can be in the state with the same s and $\boldsymbol{p}$ meaning:

$$
a^{s\dagger}_{\boldsymbol{p}} b^{s\dagger}_{\boldsymbol{p}} |0\rangle = -b^{s\dagger}_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{p}} |0\rangle \neq 0 \,. \tag{4.5.14}
$$

4.5.2 The Momentum Operator

Derivation of the momentum operator is done using the momentum density (2.4) and in a similar fashion to what we've done for the real scalar field (2.6.1). The momentum operator for the Dirac field is given as follows.

Remember: Momentum Operator of the Dirac Field

The momentum operator for the Dirac field is given by:

$$
\hat{\boldsymbol{p}} = \int \mathrm{d}\boldsymbol{x}\; \psi^\dagger(-i\nabla)\psi = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \sum_s \boldsymbol{p} \left(a^{s\dagger}_{\boldsymbol{p}} a^s{}_{\boldsymbol{p}} + b^{s\dagger}_{\boldsymbol{p}} b^s{}_{\boldsymbol{p}} \right) \,. \tag{4.5.15}
$$

We've said this before but here we can explicitly see that $a^{s\dagger}_{\boldsymbol{p}}$ and $b^{s\dagger}_{\boldsymbol{p}}$ both create particles with positive energy and momentum $\boldsymbol{p}$. Fermions and antifermions respectively. Eigenvalues of the momentum operator are the momentum of the particles, like so:

$$
\hat{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} |0\rangle = \boldsymbol{p} a^\dagger_{\boldsymbol{p}} |0\rangle \,, \tag{4.5.16}
$$

$$
\hat{\boldsymbol{p}} b^\dagger_{\boldsymbol{p}} |0\rangle = \boldsymbol{p} b^\dagger_{\boldsymbol{p}} |0\rangle \,. \tag{4.5.17}
$$

4.5.3 The Electric Charge Operator

The electric charge operator can be found by using the Noether current for the $U(1)$ symmetry of the Dirac field. In practice this can be done by considering a global phase transformation of the Dirac field and applying Noether's theorem (2.3). The transformation in question is:

$$
\psi(x) \to e^{i\alpha}\psi(x) \text{ for } \alpha = \text{const.}\,, \bar{\psi}(x) \to e^{-i\alpha}\bar{\psi}(x) \,. \tag{4.5.18}
$$

The Noether current is given by:

$$
j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \Delta\psi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} \Delta\bar{\psi} - J^\mu \,, \tag{4.5.19}
$$

where we have an additional term due to our field being complex. $J^\mu = 0$ since the Lagrangian is invariant under global phase transformations. Remember our Lagrangian is:

$$
\mathcal{L} = \bar{\psi}_a (i\gamma^\mu \partial_\mu - m)_{ab} \psi_b \,, \tag{4.5.20}
$$

from which we get that $\Delta\bar{\psi} = -i\bar{\psi}$. In the first order of α we can express the transformation as:

$$\psi \quad \rightarrow \quad \psi' = (1 + i\alpha)\psi \,. \tag{4.5.21}$$

Looking back to the Noether current (4.5.19) and our Lagrangian we can see that the second term is also 0 as we have no derivatives $\partial_\mu\bar{\psi}$. Thus all that is left is to calculate the first term:

$$\frac{\partial\mathcal{L}}{\partial(\partial_a\psi_b)} = \bar{\psi}_b i(\gamma^\mu)_{ab} \,. \tag{4.5.22}$$

From which we can simply put together the Noether current:

$$j^\mu = \bar{\psi}_a i(\gamma^\mu)_{ab}(i\psi_b) \,. \tag{4.5.23}$$

Due to the fact that this current is conserved we can multiply it by an arbitrary constant. From this we get the **continuity equation for electrical charge**:

$$j^\mu_{\text{EM}} = e\bar{\psi}\gamma^\mu\psi \,, \tag{4.5.24}$$
$$\partial_\mu j^\mu_{\text{EM}} = 0 \,, \tag{4.5.25}$$

where e is the electric charge of the fermion and $j^\mu_{\text{EM}} = (\rho_{\text{el.}}, \boldsymbol{j}_{\text{EM}})$. The electric charge of the fermion is equal to $-e_0$ in the case of the electron field, but can also be $2/3\, e_0$ or $-1/3\, e_0$ for the quark fields. Anyways the electric charge can be found from the 0th component of the current. As per Noether's theorem:

$$\begin{aligned} Q_{\text{el.}} &= \int \mathrm{d}^3x e\bar{\psi}\gamma^0\psi = \int \mathrm{d}^3\rho_{\text{el.}}(x) \qquad \text{but with operators also} \\ &= \cdots = \\ &= e\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\sum_s({a^s}^\dagger_{\boldsymbol{p}}{a^s}_{\boldsymbol{p}} + {b^s}_{\boldsymbol{p}}{b^s}^\dagger_{\boldsymbol{p}}) \,. \end{aligned} \tag{4.5.26}$$

We've come to an important result which arguably deserves its own box.

Remember: Electric Charge Operator of the Dirac Field

The electric charge operator for the Dirac field is given by:

$$Q_{\text{el.}} = e\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\sum_s({a^s}^\dagger_{\boldsymbol{p}}{a^s}_{\boldsymbol{p}} - {b^s}^\dagger_{\boldsymbol{p}}{b^s}_{\boldsymbol{p}} + (2\pi)^3\delta(0)) \,. \tag{4.5.27}$$

The continuity equation for electrical charge is given by the conserved Noether current:

$$j^\mu_{\text{EM}} = e\bar{\psi}\gamma^\mu\psi \,, \tag{4.5.28}$$
$$\partial_\mu j^\mu_{\text{EM}} = 0 \,. \tag{4.5.29}$$

Eigenvalues of the electric charge operator are the electric charges of the particles, like so:

$$\hat{Q}_{\text{el.}}a^\dagger_{\boldsymbol{p}}|0\rangle = \left[\hat{Q}_{\text{el.}}, a^\dagger_{\boldsymbol{p}}\right]|0\rangle = ea^\dagger_{\boldsymbol{p}}|0\rangle \,, \tag{4.5.30}$$
$$\hat{Q}_{\text{el.}}b^\dagger_{\boldsymbol{p}}|0\rangle = \left[\hat{Q}_{\text{el.}}, b^\dagger_{\boldsymbol{p}}\right]|0\rangle = -eb^\dagger_{\boldsymbol{p}}|0\rangle \,, \tag{4.5.31}$$

where we've ignored the charge of the vacuum state. More on this right below.

Charge of the Vacuum State: The vacuum state has a charge of ∞ which does not really make sense. This is also a problem that we sweep under the rug later on. We can see that the vacuum state has a charge of ∞ by using the anticommutation relations of the creation and annihilation operators for antifermions to get them into their *normal* ordered form:

$$\ldots({a^s}^\dagger_{\boldsymbol{p}}{a^s}_{\boldsymbol{p}} + {b^s}_{\boldsymbol{p}}{b^s}^\dagger_{\boldsymbol{p}}) = \ldots({a^s}^\dagger_{\boldsymbol{p}}{a^s}_{\boldsymbol{p}} - {b^s}^\dagger_{\boldsymbol{p}}{b^s}_{\boldsymbol{p}} + (2\pi)^3\delta(0)) \,. \tag{4.5.32}$$

The implication that the vacuum state has a charge of ∞ is made by the third term where we have the delta function $\delta(0)$ which is infinite.

What about Majorana fermions? We've yet to discuss Majorana fermions. Their special property is that they are their own antiparticles. This means that the creation and annihilation operators are the same for Majorana particles and antiparticles. If we remember the field operators for the Dirac field (4.4), they state something along the lines of:

$$\hat{\psi}_M(\boldsymbol{x},t) = \int \dots (a^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ip\cdot x} + a^{s\dagger}_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ip\cdot x}) \,, \tag{4.5.33}$$

where we've used the same creation and annihilation operators for particles and antiparticles. We're we to calculate the value of the electric charge operator for Majorana fermions we would get:

$$\hat{Q}_{\text{el.}} = e \int \hat{\psi}^\dagger_M \hat{\psi}_M = 0 \,, \tag{4.5.34}$$

which is an interesting property of Majorana fermions. They have no electric charge.

4.5.4 Interpretation of Vacuum Charge and Energy for the Electron Field

As we've seen in our journey through the quantization of the Dirac field, the vacuum state has a charge of $-\infty$ and an energy of $-\infty$, where we've taken into account that for electrons $e = -e_0$. These are the results of the anticommutation relations of the creation and annihilation operators for antifermions. Dirac interpreted this result as what is known as the **Dirac sea**. His idea was that the vacuum state is the sea of electrons with negative energy. This sea would naturally have $E_0 = -\infty$ and $Q_{\text{el.}} = -\infty$.

It is interesting to note that $E_0^{\text{fermion}} = -E_0^{\text{scalar}}$ would hold if the masses of the fermions and scalars were equal. We know in nature however that they are not. This gives rise to **supersymmetric theories** where every fermion has a scalar supersymmetric partner and vice versa. The electron e^- has a scalar partner the selectron $\tilde{e}^-$ and the photon γ has a fermionic partner the photino $\tilde{\gamma}$. These are currently only theoretical constructs though and have yet to be observed experimentally.

4.5.5 The Angular Momentum Operator

In the same fashion as we've derived the operator of the electrical charge we can derive the operator of the angular momentum. This time the transformation we're going to consider is a rotation of the Dirac field. We will get operator of the angular momentum from the Noether current of the rotation symmetry. The transformation in question is:

$$\psi \quad \to \quad \psi' = \Lambda_{1/2}\psi(\Lambda^{-1}x) = (1 - i\theta J^{12}_{(1/2)})\psi(x+\theta y, y - \theta x, z) = \psi + \theta\Delta\psi \,, \tag{4.5.35}$$

where $\Lambda_{1/2}$ is the representation of the Lorentz group for spin 1/2 particles, which we've discussed before (3). The matrix $J^{12}_{(1/2)}$ is the generator of rotations in the $x-y$ plane, given as such:

$$J^{12}_{(1/2)} = \frac{1}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix} = \frac{1}{2}\Sigma^3 \,, \tag{4.5.36}$$

meaning that $\Lambda_{1/2}$ can be written as:

$$\Lambda_{1/2} \approx 1 - \omega_{\mu\nu} J^{\mu\nu}_{(1/2)} = 1 - \frac{i}{2}\theta\Sigma^3 \,, \tag{4.5.37}$$

since $\omega_{12} = -\omega_{21} = \theta$ and all other off diagonal components of $\omega_{\mu\nu}$ are 0. $\Lambda^{-1}x$ gives the rotation of the coordinates:

$$\Lambda^{-1}x = \begin{bmatrix} 1 & \theta & 0 \\ -\theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+\theta y \\ y - \theta x \\ z \end{bmatrix} \,. \tag{4.5.38}$$

Using a Taylor expansion we can find $\Delta\psi$ from the transformation (4.5.35):

$$\psi + \theta\Delta\psi = \psi + \theta\left[-\frac{i}{2}\Sigma_3\psi + \partial_x\psi\cdot y - \partial_y\psi\cdot x\right] \tag{4.5.39}$$

We've already evaluated the Noether current for spinors in Equation ((4.5.23)) is given by:

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\Delta\psi = \bar{\psi}i\gamma^\mu\Delta\psi\ . \tag{4.5.40}$$

And as expected the operator of the angular momentum is given by the first component of the Noether current:

$$\begin{aligned} Q &= \int \mathrm{d}^3x\ j^0(\boldsymbol{x}) \\ &= \int \mathrm{d}^3x\ \psi^\dagger(\boldsymbol{x})\left[\frac{1}{2}\Sigma^3 + (\boldsymbol{x}\times(-i\nabla))^3\right]\psi(\boldsymbol{x}) \\ \Rightarrow\quad \hat{J}_z &= \int \mathrm{d}^3x\ \hat{\psi}^\dagger(\boldsymbol{x})\left[\frac{1}{2}\Sigma^3 + (\boldsymbol{x}\times(-i\nabla))^3\right]\hat{\psi}(\boldsymbol{x})\ , \end{aligned} \tag{4.5.41}$$

where note that the power 3 at the cross product actually denotes the z component of the cross product. We've again come to an important result which deserves its own box. Let's write down the operator for any direction of the angular momentum.

Remember: Angular Momentum Operator of the Dirac Field

The total angular momentum operator for the Dirac field is given by:

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{S}} + \hat{\boldsymbol{L}} = \int \mathrm{d}^3x\ \hat{\psi}^\dagger\left[\frac{1}{2}\Sigma + (\boldsymbol{x}\times(-i\nabla))\right]\hat{\psi}\ , \tag{4.5.42}$$

where the matrix vector $\boldsymbol{\Sigma}$ is given by:

$$\Sigma^i = \begin{bmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{bmatrix}\ . \tag{4.5.43}$$

In the case of the z component of the angular momentum we can write states with well-defined angular like so:

$$\hat{J}_z {a^{s=1}}^\dagger_{\mathbf{0}}|0\rangle = +\frac{1}{2}{a^{s=1}}^\dagger_{\mathbf{0}}|0\rangle\ , \tag{4.5.44}$$

$$\hat{J}_z {a^{s=2}}^\dagger_{\mathbf{0}}|0\rangle = -\frac{1}{2}{a^{s=2}}^\dagger_{\mathbf{0}}|0\rangle\ , \tag{4.5.45}$$

$$\hat{J}_z {b^{s=1}}^\dagger_{\mathbf{0}}|0\rangle = -\frac{1}{2}{b^{s=1}}^\dagger_{\mathbf{0}}|0\rangle\ , \tag{4.5.46}$$

$$\hat{J}_z {b^{s=2}}^\dagger_{\mathbf{0}}|0\rangle = +\frac{1}{2}{b^{s=2}}^\dagger_{\mathbf{0}}|0\rangle\ . \tag{4.5.47}$$

From here we see that fermions really do correspond to particles with spin 1/2. The operator of the total angular momentum for any direction is given by the sum of the spin operator and the orbital angular momentum operator. The value of $\hat{S}$ is related to $\hat{J}_z$ in the particles rest frame.

4.5.6 z Component of Spin and Helicity for Particles with Momentum

In the event that we have a particle with $\boldsymbol{p} \neq 0$ the state of the particle is not an eigenstate of the z component of the angular momentum, meaning:

$$\hat{S}_z {a^{s=1}}^\dagger_{p}|0\rangle \not\propto {a^{s=1}}^\dagger_{p}|0\rangle\ , \tag{4.5.48}$$

since $\Sigma_3 u^s(p) \neq u^s(p)$. As we've said in the previous section, spin is related to the z component of the angular momentum in the particles rest frame. If we consider commutators between these various operators and the Hamiltonian we find that the Spin operator or the Orbital angular momentum operator do not commute with the Hamiltonian, meaning they are not good quantum numbers.

$$[\hat{\boldsymbol{S}},\, \hat{H}] \neq 0 \neq [\hat{\boldsymbol{L}},\, \hat{H}]\,. \tag{4.5.49}$$

However combining them together to get the total angular momentum operator we find that it does commute with the Hamiltonian and is thus a good quantum number. We've already done so in the box above, without explicitly stating the reason for doing so. Once again for posterity:

$$\hat{\boldsymbol{J}} = \hat{\boldsymbol{S}} + \hat{\boldsymbol{L}}\,, \tag{4.5.50}$$
$$[\hat{\boldsymbol{J}},\, \hat{H}] = 0\,. \tag{4.5.51}$$

We can however find another good quantum number if we consider the projection of spin onto the direction of momentum. This is known as **Helicity**. Truth be told it is a often neglected operator and thus deserves a box of its own.

Remember: Helicity Operator for Particles with Momentum

The helicity operator for particles with momentum is given by:

$$\hat{h} = \frac{\hat{\boldsymbol{S}} \cdot \boldsymbol{p}}{|\boldsymbol{p}|}\,. \tag{4.5.52}$$

The eigenvalues of the helicity operator are the helicities of the particles, like so:

$$\hat{h} a^{s=1\dagger}_{\boldsymbol{p}} |0\rangle = +\frac{1}{2} a^{s=1\dagger}_{\boldsymbol{p}} |0\rangle\,, \quad \hat{h} a^{s=2\dagger}_{\boldsymbol{p}} |0\rangle = -\frac{1}{2} a^{s=2\dagger}_{\boldsymbol{p}} |0\rangle\,. \tag{4.5.53}$$

The helicity operator commutes with the Hamiltonian:

$$\left[\hat{h},\, \hat{H}\right] = 0\,, \tag{4.5.54}$$

thus making it a good quantum number. The helicity of a particle is **right-handed** if spin is aligned with the direction of momentum, thus making helicity positive and **left-handed** if it is opposite to the direction of momentum, thus making it negative. Do note that the helicity operator is **not Lorentz invariant**. It can change sign under Lorentz boosts.

4.5.7 Causality for Operators of the Dirac Field

This is here just as a short reminder as it is in general pretty important. Anyways the point is that for two space-like seperated points of spacetime the commutator of the various operators of the Dirac field we've listed here must be zero. Operators at space-like seperated points must commute. This is a consequence of the theory being Lorentz invariant. For example for a short excercise we can consider the following:

$$\begin{gathered} \left[\hat{\mathcal{O}}_1(x),\, \hat{\mathcal{O}}_2\right] = 0 \text{ for } (x-y)^2 < 0\,, \\ \text{where } \hat{\mathcal{O}}_{1,2}(x) = \psi^\dagger_a(x) \Gamma^{1,2}_{ab} \psi_b(x)\,. \end{gathered} \tag{4.5.55}$$

Our operators here have a pair of fields. We can expand the commutator to get:

$$\begin{aligned} \left[\hat{\mathcal{O}}_1(\boldsymbol{x},0)\, \hat{\mathcal{O}}_2(\boldsymbol{y},0)\right] &= \psi^\dagger_a(x)\gamma^1_{ab}\psi_b(x)\psi^\dagger_c(y)\gamma^2_{cd}\psi_d(y) - \psi^\dagger_c(y)\gamma^2_{cd}\psi_d(y)\psi^\dagger_a(x)\gamma^1_{ab}\psi_b(x) \\ &= \psi^\dagger_c(y)\psi^\dagger_a(x)\gamma^1_{ab}\psi_b(x)\gamma^2_{cd}\psi_d(y) - 2.\text{ term in similar fashion} \\ &= \psi^\dagger_c\psi_d\psi^\dagger_a\gamma^1_{ab}\psi_b\gamma^2_{cd} - 2.\text{ term} \\ &= 0\,. \end{aligned} \tag{4.5.56}$$

4.6 Fermionic Propagator

Calculating propagation amplitudes for the Dirac field should now be a piece of cake. We're interested in the fermionic propagator which is given by:

$$\Delta_F(x-y)_{ab} = \langle 0|\mathcal{T}\{\psi_a(x)\bar{\psi}_b(y)\}|0\rangle = \begin{cases} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle & \text{for } x^0 > y^0 \,, \\ -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle & \text{for } x^0 < y^0 \,, \end{cases} \tag{4.6.1}$$

where as we've now seen a couple of times $\mathcal{T}$ is the time ordering operator. We can quickly evaluate the two cases as follows:

$$\begin{aligned} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_s u_a^s(\boldsymbol{p})\bar{u}_b^s(\boldsymbol{p})e^{-ip\cdot(x-y)} \\ &= (i\partial\!\!\!/_x + m)_{ab}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{-ip\cdot(x-y)} \end{aligned} \,, \tag{4.6.2}$$

$$\begin{aligned} \langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_s v_a^s(\boldsymbol{p})\bar{v}_b^s(\boldsymbol{p})e^{-ip\cdot(x-y)} \\ &= -(i\partial\!\!\!/_x - m)_{ab}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{-ip\cdot(x-y)} \end{aligned} \,, \tag{4.6.3}$$

where $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. Just as we've done for the scalar field we can now construct the propagator as the **retarded** (trust me it's in Peskin pg. 62) Green's function of the Dirac equation, while taking into account the boundary conditions:

$$\Delta_F^{ab}(x-y) \equiv \theta(x^0-y^0)\langle 0|\{\psi_a(x),\, \bar{\psi}_b(y)\}|0\rangle \,, \tag{4.6.4}$$

where we've insulted the Green's function by calling it retarded since it contains the Heaviside step function $\theta(x^0-y^0)$. If we act on the previous function with the Dirac equation we quickly verify that it is indeed the Green's function:

$$(i\partial\!\!\!/_x - m)\Delta_F(x-y) = i\delta^{(4)}(x-y)\cdot \mathbb{1}_{4\times 4} \,. \tag{4.6.5}$$

The Green's function of the Dirac equation can also be found via a Fourier transform as:

$$i\delta^{(4)}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}(p\!\!\!/ - m)e^{-ip\cdot(x-y)}\tilde{\Delta}_F(p) \,, \tag{4.6.6}$$

where $\tilde{\Delta}_F(p)$ is:

$$\tilde{\Delta}_F(p) = \frac{i}{p\!\!\!/ - m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2} \,. \tag{4.6.7}$$

Yet again we've come to important knowledge which deserves a box.

Remember: Fermionic Propagator

The fermionic propagator for the Dirac field is given by the Green's function of the Dirac equation. Taking into account Feynman boundary conditions we get:

$$\Delta_F(x-y) \equiv \langle 0|\mathcal{T}\{\psi(x)\bar{\psi}(y)\}|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon}e^{-ip\cdot(x-y)} \,. \tag{4.6.8}$$

where we've used the **Feynman prescription** where the poles of the propagator are slighly displaced above and below the real axis $p^0 = \pm(E_{\boldsymbol{p}} - i\varepsilon)$. We've used the same prescription for the propagator of the scalar field (2.11.3). When drawing Feynman diagrams we draw the fermionic propagator as a straight line with an arrow pointing in the direction of the flow of positive charge, like so:

(4.6.9)

4.7 Majorana Fermions in Contrast to Dirac Fermions

We've already mentioned Majorana fermions when discussing the electric charge operator of the Dirac field. Majorana fermions are their own antiparticle. As we've seen (4.5.34) they cannot hold charge since their particle, antiparticle ladder operators are the same. This means that exications of Majorana fields are always neutral particles. The Majorana field operator is given by:

$$\hat{\psi}_M(\boldsymbol{x}, t) = \int \ldots (a^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{-ip\cdot x} + a^{s\dagger}_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{ip\cdot x}) \,. \tag{4.7.1}$$

For example it is unknown whether the neutrino is a Dirac or a Majorana fermion. We could identify this through observing beta decays. A process that is known to occur is the double beta decay where two neutrons decay into two protons, two electrons and two electron antineutrinos. This process can occur for both Dirac and Majorana neutrinos, however there exists a theoretical process known as the neutrino-less double beta decay where the two antineutrinos annihilate each other which would only be possible if the neutrino is a Majorana fermion. This process has yet to be observed experimentally.

4.8 Lepton Number Conservation

Another conserved quantity that applies to leptons is the **lepton number**, defined as:

$$L = L_e + L_\mu + L_\tau \,. \tag{4.8.1}$$

For example the lepton number of the electron and electron neutrino is $L_e = 1$, while the lepton number of the positron and electron antineutrino is $L_e = -1$ and so on for the other leptons. The lepton number is conserved in all interactions. We know for example that L_e is not conserved due to neutrino oscillations, however the total lepton number is conserved. This does not hold true if neutrinos are Majorana fermions. Another reason why the neutrino-less double beta decay has yet to be observed.

4.8.1 Neutrino Mass

The discovery of neutrino oscillations has shown that neutrinos have mass. This is a problem for the Standard Model where neutrinos are massless. This can be corrected by introducing right-handed neutrinos, which we do not know how to get directly from interactions. In theory it is possible to create a right-handed neutrino essentially by transposition:

$$\hat{\psi}_{L,M} = \frac{1}{2}(1-\gamma_5)\hat{\psi} \,, \tag{4.8.2}$$

$$\hat{\psi}_{R,M} = \frac{1}{2}(1+\gamma_5)\hat{\psi} \,, \hat{\psi}_{R,M} = \ldots \hat{\psi}^\dagger_{L,M} \,. \tag{4.8.3}$$

But this can only be done if the neutrino is a Majorana fermion.

5 Complex Scalar Fields

Complex scalar fields describe spinless charged scalar particles. The main difference between complex scalar fields and real scalar fields is that for complex scalar fields $\hat{\phi} \neq \hat{\phi}^\dagger$. Examples of charged spinless particles are mesons such as pions, which if we're more specific are pseudoscalars as they change sign under parity transformations, while real scalars do not.

Remember: Complex Scalar Fields

The Lagrangian for a complex scalar field is:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger - m^2 \phi \phi^\dagger \,. \tag{5.0.1}$$

The complex scalar field operator is defined as:

$$\hat{\phi}(\boldsymbol{x}, t) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + b^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right) \neq \hat{\phi}^\dagger(\boldsymbol{x}, t) \,. \tag{5.0.2}$$

The commutation relations are:

$$\left[a_{\boldsymbol{p}},\, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{q}) \,, \tag{5.0.3}$$

$$\left[\hat{\phi}(\boldsymbol{x}, t),\, \hat{\Pi}(\boldsymbol{y}, t)\right] = i\delta(\boldsymbol{x} - \boldsymbol{y}) \,. \tag{5.0.4}$$

With these we can derive the various operators.

If we have a quick look at the causality for complex scalar fields, considering two operators $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$ in two space-like separated points, we have:

$$\left[\hat{\mathcal{O}}_1(\boldsymbol{x}_1, 0),\, \hat{\mathcal{O}}_2(\boldsymbol{x}_2, 0)\right] = 0 \quad \text{for} \quad \boldsymbol{x} \neq \boldsymbol{y} \,. \tag{5.0.5}$$

We've already seen this in the commutation relations we postulated for complex scalars (for $\boldsymbol{x} \neq \boldsymbol{y}$, $\left[\hat{\phi}(\boldsymbol{x}, t),\, \hat{\Pi}(\boldsymbol{y}, t)\right] = 0$) but how about the commutation relations for the fields themselves? We have:

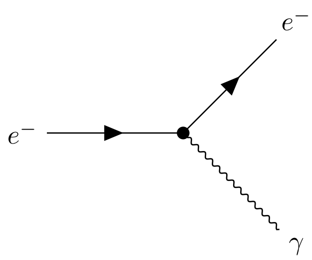
$$\begin{aligned} \left[\hat{\phi}(\boldsymbol{x}, t),\, \hat{\phi}(\boldsymbol{y}, t)\right] &= \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3} \int \frac{\mathrm{d}\boldsymbol{p'}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \frac{1}{\sqrt{2E_{\boldsymbol{p'}}}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x} - i\boldsymbol{p'}\cdot\boldsymbol{y}} \left(\left[a_{\boldsymbol{p}},\, a^\dagger_{-\boldsymbol{p'}}\right] + \left[b^\dagger_{\boldsymbol{p}},\, b_{-\boldsymbol{p'}}\right] \right) \\ &= 0 \\ &\text{however} \\ &= \langle 0 | \phi(\boldsymbol{x}, 0) \phi^\dagger(\boldsymbol{y}, 0) | 0 \rangle - \langle 0 | \phi^\dagger(\boldsymbol{y}, 0) \phi(\boldsymbol{x}, 0) | 0 \rangle \\ &= \boldsymbol{x} \xrightarrow{\text{particle}} \boldsymbol{y} \;+\; \boldsymbol{x} \xleftarrow{\text{antiparticle}} \boldsymbol{y} \;= 0 \,. \end{aligned} \tag{5.0.6}$$

We can see that even the final line is zero if interpret it in terms of Feynman diagrams as the two propagators cancel each other out. Thus for this commutator to be zero and for causality to be preserved antiparticles must also exist. This must hold for every $\boldsymbol{x} \neq \boldsymbol{y}$. In the case of real scalar fields these demands are trivially satisfied via the postulated commutation relations.

6 Interactions Between Fields & Feynman Diagrams

We'd like to understand how fields interact with each other. We'll introduce that fields interact in the same point in spacetime $(\boldsymbol{x}, t)$. We'll also introduce the concept of a Feynman diagram. The figure (6.0.1) shows a process that we'd like to understand:

$$\mathcal{H}_{\text{int}} = e\bar{\psi}(x)\gamma^\mu \psi(x) A_\mu(x) \,, \tag{6.0.1}$$

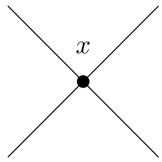


6.1 Principles of Introducing Interactions to QFT

6.1.1 Principle 1: Locality and Causality

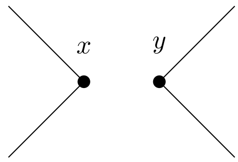
Interactions are local in spacetime. This means that the interaction Hamiltonian is a function of the fields at the same point in spacetime. So the following process would be **allowed**:

$$\mathcal{L}_{\text{int}}(x) \propto \phi^4(x) \,, \tag{6.1.1}$$



But the following process would be **forbidden**:

$$\mathcal{L}_{\text{int}}(x,y) \propto \phi^2(x)\phi^2(y) \,, \tag{6.1.2}$$



6.1.2 Principle 2: Commutation Relations

The evolution of a field without interactions is relatively easy to calculate:

$$\begin{aligned} \hat{\phi}_0(\boldsymbol{x},t) &= e^{i\mathcal{H}_0 t}\phi(\boldsymbol{x},0)e^{-i\mathcal{H}_0 t} \\ &= \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}(a_{\boldsymbol{p}}e^{-ip\cdot x} + a^\dagger_{\boldsymbol{p}}e^{ip\cdot x}) \,, \end{aligned} \tag{6.1.3}$$

In the case of interactions, the evolution of the field is more complicated:

$$\begin{aligned} \hat{\phi}(\boldsymbol{x},t) &= e^{i\mathcal{H}t}\phi(\boldsymbol{x},0)e^{-i\mathcal{H}t} \\ &= e^{i\mathcal{H}_{\text{int}}t}e^{-i\mathcal{H}_0 t}\phi_{\text{int}}(\boldsymbol{x},0)e^{i\mathcal{H}_0 t}e^{-i\mathcal{H}_{\text{int}}t} \\ &= U^{-1}(t,0)\phi_{\text{int}}(\boldsymbol{x},0)U(t,0) \,, \end{aligned} \tag{6.1.4}$$

where $U(t,0)$ is the time evolution operator from $t=0$ to t. Just in case, we should verify that our postulated commutation relations hold even for fields evolved in time:

$$\begin{aligned} \big[\hat{\phi}(\boldsymbol{x},t),\, \hat{\Pi}(\boldsymbol{y},t)\big] &= \big[U^{-1}\hat{\phi}_{\text{int}}(\boldsymbol{x},0)U,\, U^{-1}\hat{\Pi}_{\text{int}}(\boldsymbol{y},0)U\big] \\ &= U^{-1}\big[\hat{\phi}_{\text{int}}(\boldsymbol{x},0),\, \hat{\Pi}_{\text{int}}(\boldsymbol{y},0)\big]U \\ &= U^{-1}(i\delta^3(\boldsymbol{x}-\boldsymbol{y}))U \\ &= 0 \,. \end{aligned} \tag{6.1.5}$$

6.1.3 Principle 3: Lorentz Invariance

This principle is essentially the basis of a lot of our work so far. We've been working with Lorentz invariant Lagrangians and Hamiltonians and we'll demand that our interactions are Lorentz invariant as well.

$$\mathcal{L}(x) \underset{\text{L. T.}}{\rightarrow} \mathcal{L}(\Lambda^{-1}x) \,. \tag{6.1.6}$$

6.1.4 Principle 4: Renormalizability

Later on, we'll discuss renormalization in more detail. For now, we'll just say that we want our theory to be renormalizable. This means that we want to be able to absorb divergences into a redefinition of the parameters of the theory. For example, Quantum Electrodynamics (QED) has only two parameters m and e. In general our theory is not renormalizable if the units of the coupling constant are raised to a negative power, like so:

$$[g] = \text{eV}^N \,, \quad N < 0 \,, \quad \text{non-renormalizable} \,. \tag{6.1.7}$$

If $N \geq 0$, then the theory might be renormalizable. For completness:

$$[g] = \text{eV}^N \,, \quad N \geq 0 \,, \quad \text{possibly renormalizable} \,. \tag{6.1.8}$$

6.1.5 Principle 5: Symmetries

The previous four principles have already severly constrained our theory, but even these constraints leave too many possibilities. This is why we introduce symmetries as additional constraints on our theories. Our theories can have global symmetries or local/gauge symmetries. We'll discuss this in more detail later. For example the full standard model has only 3 symmetries added:

$$\text{Standard Model Symmetries} \rightarrow \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \,. \tag{6.1.9}$$

Fun Fact: Translations of the term *Gauge*

In our native language Slovenian, the term *gauge* is translated to *umeritev*, meaning *gauge symmetries* are translated to *umeritvene simetrije*. Our professor Saša did mention that when attending a conference in Croatia the term used in Croatian was *baždarna teorija*. This has been verified with Croatian colleagues.

Examples of $\mathcal{H}_{\text{int}}$ and $\mathcal{L}_{\text{int}}$ among ϕ, ψ, A_μ: Our Lagrangian could be made up of a free part and an interaction part:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \,. \tag{6.1.10}$$

Calculating the Hamiltonian, we find:

$$\begin{aligned} &\mathcal{H} = \Pi\dot{\phi} - \mathcal{L}_{\text{int}} - \mathcal{L}_0 = (\Pi\dot{\phi} - \mathcal{L}_0) - \mathcal{L}_{\text{int}} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \,, \\ &\Rightarrow \mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} \,. \end{aligned} \tag{6.1.11}$$

Mass Dimensions of Fields: As an exercise we can try to calculate the mass dimension of the fields we've been working with. So for our 4-momentum $p^\mu = (E, \boldsymbol{p})$, we have:

$$[p] = [E] = \mathrm{eV} \,. \tag{6.1.12}$$

We know that for our Hamiltonian and Hamiltonian Density, we have:

$$H = \int \mathrm{d}^3\boldsymbol{x}\mathcal{H} \Rightarrow [H] = \mathrm{eV} \,, \tag{6.1.13}$$

$$[\mathcal{H}] = \frac{\mathrm{eV}}{\mathrm{eV}^{-3}} = \mathrm{eV}^4 \quad \Rightarrow \quad [\mathcal{L}] = \mathrm{eV}^4 \,. \tag{6.1.14}$$

So for onto our fields, for **scalar fields**:

$$\begin{aligned} \mathcal{L} &= \dots \frac{\partial}{\partial x^\mu}\phi \frac{\partial}{\partial x_\mu}\phi + \dots \,, \\ [\mathcal{L}] &= \mathrm{eV}^4 = [\mathrm{eV}][\phi][\mathrm{eV}][\phi] \,, \\ &\Rightarrow [\phi] = \mathrm{eV} \,. \end{aligned} \tag{6.1.15}$$

For **spinor fields**:

$$\begin{aligned} \mathcal{L} &= \bar{\psi} i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi + \dots \,, \\ [\mathcal{L}] &= \mathrm{eV}^4 = [\bar{\psi}] i\gamma^\mu [\mathrm{eV}][\psi] \,, \\ &\Rightarrow [\psi] = \mathrm{eV}^{3/2} \,. \end{aligned} \tag{6.1.16}$$

And for **vector fields**:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \dots \,, \\ [\mathcal{L}] &= \mathrm{eV}^4 = \left([\mathrm{eV}][A_\nu] - [\mathrm{eV}][A_\mu]\right)^2 \,, \\ &\Rightarrow [A_\mu] = \mathrm{eV} \,. \end{aligned} \tag{6.1.17}$$

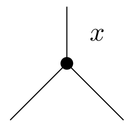
6.2 Interactions of Scalar Fields

Our interaction Lagrangian for scalar fields has the form:

$$\mathcal{L}_{\text{int}} = A_n \phi^n(x) \,. \tag{6.2.1}$$

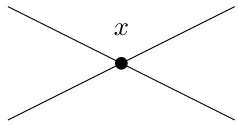
For our theory to be renormalizable, we need $[A_n] = \text{eV}^{4-n}$. This means that $n = 3, 4$ are allowed. Taking $\phi = \phi^\dagger$, we have the following Feynman diagram for $n = 3$:

$$\mathcal{L}_{\text{int}} \propto \phi^3(x) \,, \tag{6.2.2}$$



which is generally not present if additional symmetries are imposed (like $\phi \to -\phi$). For $n = 4$, we have the following Feynman diagram:

$$\mathcal{L}_{\text{int}} \propto \phi^4(x) \,, \tag{6.2.3}$$



which represents the interaction of the Higgs field in nature. In general this interaction would have a Hamiltonian of the form:

$$\mathcal{H}_{\text{int}}(x) = \frac{\lambda}{4!}\phi^4(x) \,, \tag{6.2.4}$$

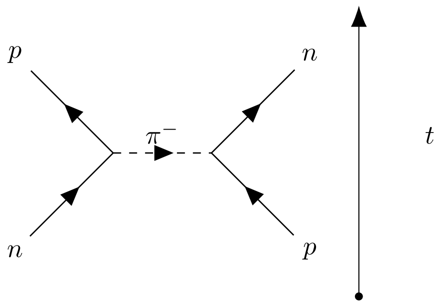
where we've chosen the factor of 4! for convenience.

6.3 Interactions of Fermions and Scalars

We can now construct the interaction Lagrangian between fermions and scalars. Notice that the interaction Lagrangian has **two fermions and one scalar**. It has the form:

$$\mathcal{L}_{\text{int}} = g\bar{\psi}(x)\psi(x)\phi(x) \,. \tag{6.3.1}$$

Taking a look at the coupling constant g, we have $[\text{eV}]^4 = [g][\text{eV}]^{3/2}[\text{eV}]^{3/2}[\text{eV}] \Rightarrow [g] = \text{eV}$. There is no other possible option for the coupling constant. The Feynman diagram for this interaction is:



(6.3.2)

6.3.1 Example: Interaction between Higgs Field and Fermions

The interaction Lagrangian between the Higgs field and fermions is:

$$\mathcal{L}_{\text{int}} = -g_f \cdot v\bar{\psi}_f\psi_f - g_f\bar{\psi}_f\psi_f H \,, \tag{6.3.3}$$

where the coupling is $m_f = g_f \cdot v$ and $v = 246\,\text{GeV}$ is the non-zero vacuum expectation value of the Higgs field that spans the entire universe. This coupling is responsible for the mass of the fermions. We see that the coupling constant g_f is proportional to the mass of the fermion m_f.

6.4 Interactions of Fermions and Vector Fields

When considering interactions between fermions and vector fields, we must consider that we have an additional requirement which is known as **local gauge invariance**. These are arbitrary local transformations of the fields which can vary from point to point, for example a phase transformation in the case of fermionic fields or an addition of a gradient to the EM vector potential which leaves the fields $\boldsymbol{B}$ and $\boldsymbol{E}$ unchanged. The two transformations we mentioned can be mathematically expressed as:

$$\psi(x) \quad \to \quad \psi'(x) = e^{i\alpha(x)}\psi(x) \, , \tag{6.4.1}$$

$$A_\mu(x) \quad \to \quad A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \, , \tag{6.4.2}$$

where the transformations are global if $\alpha = \text{const.}$ and local if $\alpha = \alpha(x)$. The Lagrangian for the interaction between fermions and vector fields is:

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = e\bar{\psi}\gamma^\mu \psi A_\mu = j^\mu_{\text{EM}} A_\mu \, , \tag{6.4.3}$$

where one can quickly see that the current j^μ_{EM} is Lorentz invariant:

$$j^\mu_{\text{EM}} \xrightarrow[\text{L.T.}]{} \Lambda^\mu{}_\nu j^\nu_{\text{EM}} \, . \tag{6.4.4}$$

Taking a look at the units of the coupling constant e we see we have:

$$\begin{aligned} &[\mathcal{H}_{\text{int}}] = [\text{eV}]^4 = [e][\text{eV}]^{3/2}[\text{eV}]^{3/2}[\text{eV}] \\ &\Rightarrow \ [e] = \text{eV}^0 \, , \end{aligned} \tag{6.4.5}$$

which holds true if we remember the fine structure constant $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$.

6.4.1 Example: Quantum Electrodynamics (QED)

The gauge transformations we've mentioned above are actually the ones postulated by QED. We will go into the details of QED later on. For completeness, lets list the transformations again:

$$\psi(x) \quad \to \quad \psi'(x) = e^{i\alpha(x)}\psi(x) \, , \tag{6.4.6}$$

$$A_\mu(x) \quad \to \quad A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \, , \tag{6.4.7}$$

$$\text{where} \quad \boldsymbol{E} \to \boldsymbol{E} \, , \quad \boldsymbol{B} \to \boldsymbol{B} \, .$$

As a fun exercise we can calculate the QED Lagrangian from this. We have:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \, , \tag{6.4.8}$$

$$D_\mu = \partial_\mu + ieA_\mu \, , \tag{6.4.9}$$

where D_μ is known as the **covariant derivative**. The covariant derivative is introduced to ensure that the Lagrangian is invariant under the gauge transformations. Now we must prove that $D_\mu\psi \to e^{i\alpha(x)}D_\mu\psi$ under the gauge transformation. We have:

$$\begin{aligned} D_\mu\psi &\to (\partial_\mu + ieA_\mu - i(\partial_\mu\alpha(x)))(e^{i\alpha(x)}\psi(x)) \\ &= \psi(x)e^{i\alpha(x)}i\partial_\mu\alpha(x) + e^{i\alpha(x)}\partial_\mu\psi(x) - ieA_\mu e^{i\alpha(x)}\psi(x) \\ &= e^{i\alpha(x)}(\partial_\mu + ieA_\mu)\psi(x) \\ &= e^{i\alpha(x)}D_\mu\psi(x) \, . \end{aligned} \tag{6.4.10}$$

We must also check the gauge invariance under the second transformation:

$$\begin{aligned} \mathcal{L} &\to \psi^\dagger e^{i\alpha(x)}\gamma^0(i\gamma^\mu e^{i\alpha(x)}D_\mu\psi - me^{i\alpha(x)}\psi) \\ &= \bar{\psi}(iD_\mu\psi - m\psi) \\ &= \bar{\psi}(iD_\mu - m)\psi \, , \end{aligned} \tag{6.4.11}$$

and thus we come to the QED Lagrangian (some things here are hat pulled though, more later):

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{6.4.12}$$

Using similar logic we could find the Quantum Chromodynamics (QCD) Lagrangian if we also took additional symmetries due to color charge into account.

6.5 Interactions of Scalar and Vector Fields

The interaction Lagrangian between scalar and vector fields is again obtained by considering additional gauge symmetries. The symmetries are the same as in the case of fermions and vector fields, where we have a local phase transformation for the scalar field and a local gauge transformation for the vector field, mathematically expressed as:

$$V^\mu(x) \quad \to \quad V'^\mu(x) = V^\mu(x) - \frac{1}{\tilde{g}}\partial^\mu\alpha(x)\,, \tag{6.5.1}$$

$$\phi(x) \quad \to \quad \phi'(x) = e^{i\alpha(x)}\phi(x)\,, \tag{6.5.2}$$

where I'd like to remind the reader that a general vector field can be marked as $V^\mu(x)$. The desired invariance is achieved by the following replacement:

$$\partial_\mu\phi \quad \to \quad D_\mu\phi = (\partial_\mu + i\tilde{g}V_\mu) \tag{6.5.3}$$

This means that our free Lagrangian for the scalar field changes to:

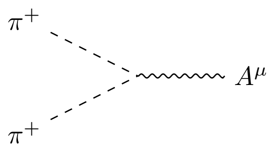
$$\mathcal{L}_0 \propto \partial_\mu\phi\partial^\mu\phi^\dagger \quad \to \quad D_\mu\phi D^\mu\phi^\dagger = (\partial_\mu\phi + i\tilde{g}V_\mu\phi)(\partial^\mu\phi^\dagger - i\tilde{g}V^{\mu\dagger}\phi^\dagger)\,, \tag{6.5.4}$$

otherwise the Lagrangian would not be invariant under the gauge transformations. This makes our free Lagrangian:

$$\mathcal{L}_0 = (\partial_\mu\phi + i\tilde{g}V_\mu\phi)(\partial^\mu\phi^\dagger - i\tilde{g}V^{\mu\dagger}\phi^\dagger) - m^2\phi^\dagger\phi\,. \tag{6.5.5}$$

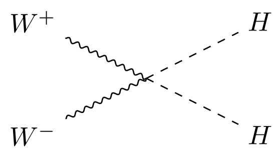
Again studying the units of the coupling constant $\tilde{g}$ to give our theory a chance of being renormalizable, we have two options, where in both cases $[\tilde{g}] = \text{eV}^0$:

$$\mathcal{L}_{\text{int}} = \tilde{g}V_\mu\phi\partial^\mu\phi^\dagger\,, \tag{6.5.6}$$



which represents for example, the interaction of the π^+ meson with the electromagnetic field. The second option is:

$$\mathcal{L}_{\text{int}} = \tilde{g}V_\mu V^{\mu\dagger}\phi\phi^\dagger\,, \tag{6.5.7}$$



which for example represents the interaction of the W^+ and W^- bosons with the Higgs field.

Note: There are derivatives in both the above interaction Lagrangians which means that commutation relations are ensured by other means.

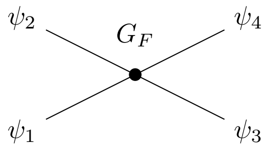
6.6 Summary of Interactions

What we've seen so far is essentially the **Standard Model** of interactions between fundamental fields. All interactions between them are listed above. It makes sense to emphasize that in regular Quantum Mechanics the choice of potential $V(\boldsymbol{x})$ which is used to describe the interaction between particles is practically arbitrary. In Quantum Field Theory interactions are constrained by the various principles we've discussed in the previous sections (6.1). Unfortunately none of the interactions in QFT are analytically solvable in more than 2D spacetime.

6.6.1 Example: Is the Four-Fermion Fermi Interaction Renormalizable?

Let's consider the following interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = G_F \bar{\psi}_1 \gamma^\mu (1 - \gamma_5) \psi_2 \bar{\psi}_3 \gamma_\mu (1 - \gamma_5) \psi_4 \ , \tag{6.6.1}$$

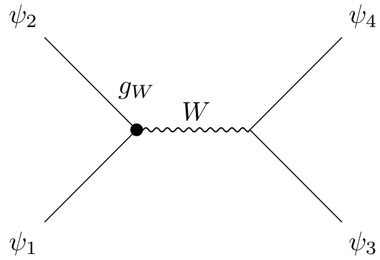


where G_F is the Fermi coupling constant and I'd like to remind the reader that we've defined the γ matrices when discussing the Dirac equation (4.1.2). If we check the units of the coupling constant G_F we have:

$$\begin{aligned} &[\mathcal{L}_{\text{int}}] = [G_F]([\text{eV}]^{3/2})^4 \\ &\Rightarrow [G_F] = [\text{eV}]^{4-6} = \text{eV}^{-2} \ . \end{aligned} \tag{6.6.2}$$

Since the coupling constant has units raised to a negative power, the theory is non-renormalizable. Indeed this process does not correspond to any fundamental interactions, however we can think of it as a low-energy effective theory where the momentum of incoming particles is much smaller than $m^2_{W,Z}$. The correct interaction is mediated by the W and Z vector bosons and has an interaction Lagrangian of the form:

$$\mathcal{L}_{\text{int}} \propto g_W \bar{\psi}_1 \gamma^\mu (1 - \gamma_5) \psi_2 W_\mu \ , \tag{6.6.3}$$



where the interaction Lagrangian stated above is the contribution of a single vertex (noted by the dot). The process illustrated by the Feynman diagram above is already a full diagram, meaning we'd need to take into account all the vertices, propagators etc. We'll get to this a little later. Taking a look at the units of the coupling constant g_W we now have:

$$\begin{aligned} &[\mathcal{L}_{\text{int}}] = [g_W]([\text{eV}]^{3/2})^2[\text{eV}] \\ &\Rightarrow [g_W] = [\text{eV}]^{4-4} = \text{eV}^0 \ . \end{aligned} \tag{6.6.4}$$

This means that the theory is renormalizable.

Note: The notation we used so far for the interaction Lagrangians and Hamiltonians is the corrected version of what we used in lectures. Prof. Saša originally mentioned that what was once denoted by $\mathcal{H}_i$ is now denoted by $\mathcal{L}_{\text{int}}$ and the same for Lagrangians. This is the notation we've used so far, however I wanted to add a note on this if anyone were to look at any handwritten notes from our lectures. Moving forward from now $\mathcal{H}_i$ and $\mathcal{L}_i$ will denote something related but slightly different.

7 Correlation Functions C

7.1 Motivation for Computing Correlation Functions

The motivation for computing correlation functions will come naturally once we see that these functions are where all the information about the physics of the theory is encoded. Here are two examples of correlation functions:

$$C_2 = \langle \Omega | \phi(x_1) \phi(x_2) | \Omega \rangle \ , \tag{7.1.1}$$
$$C_4 = \langle \Omega | \bar{\psi}(x_1) \Gamma \psi(x_2) \bar{\psi}(x_3) \Gamma \psi(x_4) | \Omega \rangle \ , \tag{7.1.2}$$

where we've denoted $|\Omega\rangle$ as the **true vacuum of the full Hamiltonian**. Analogous to the vacuum state $|0\rangle$ for a free Hamiltonian. Both represent ground states. Friendly reminder that our fields now evolve with the full Hamiltonian, for example:

$$\phi(\boldsymbol{x}, t) = e^{+iH(t-t_0)}\phi(\boldsymbol{x}, t_0)e^{-iH(t-t_0)} , \tag{7.1.3}$$

where it makes sense to mind that $\phi(x)$ is now a complicated object that evolves with a Hamiltonian that includes interactions $H = H_0 + H_{\text{int}}$. Let's have a quick look at what is possible to compute with correlation functions.

7.1.1 Calculating Eigenenergies E_n from C

Consider the following correlation function:

$$C(t) = \langle\Omega|\hat{\mathcal{O}}(\mathbf{0}, t)\hat{\mathcal{O}}^\dagger(\mathbf{0}, 0)|\Omega\rangle , \tag{7.1.4}$$

where $\hat{\mathcal{O}}$ is some operator that annihilates a system with the desired quantum numbers. For example:

- $\hat{\mathcal{O}} = \phi(\boldsymbol{x})$ annihilates a scalar particle.
- $\hat{\mathcal{O}} = \bar{\psi}_e(\boldsymbol{x})\Gamma^\mu\psi_e(\boldsymbol{x})$ annihilates positronium e^+e^- with $J^p = 1^-$.
- $\hat{\mathcal{O}} = \bar{\psi}_u\gamma_5\psi_d$ annihilates a pion π.
- $\hat{\mathcal{O}} = \psi_u\left[\psi_u\Gamma\psi_d\right]$ annihilates a proton p.

All the previous examples annihilate what is written next to them but also other states with the same quantum numbers. We can expand the correlation function by evolving the first operator in time and then writing it in terms of eigenstates of the Hamiltonian:

$$\begin{aligned} C(t) &= \langle\Omega|e^{+iHt}\hat{\mathcal{O}}(\mathbf{0}, 0)e^{-iHt}\sum_n |n\rangle\langle n|\hat{\mathcal{O}}^\dagger(\mathbf{0}, 0)|\Omega\rangle \\ &= \sum_n \langle\Omega|\hat{\mathcal{O}}(\mathbf{0}, 0)|n\rangle e^{-iE_nt}\langle n|\hat{\mathcal{O}}^\dagger(\mathbf{0}, 0)|\Omega\rangle \\ &= \sum_n A_n e^{-iE_nt} , \end{aligned} \tag{7.1.5}$$

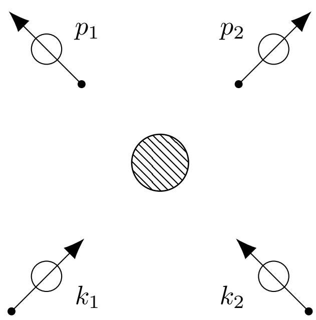
where $\hat{H}|n\rangle = E_n|n\rangle$ are the eigenstates and eigenenergies and that $\hat{H}|\Omega\rangle = 0$. From this we see that if the correlation function is computed as a function of time, one can determine the eigenenergies E_n.

In general C allows the computation of correlations of various observables at different locations or times.

7.1.2 Calculating Cross Sections from C

We can calculate the scattering matrix S, also known as the S-matrix by computing a function of the form:

$$\langle p_1, p_2|e^{-2iTH}|k_1, k_2\rangle \tag{7.1.6}$$



where $|k_1, k_2\rangle$ are the initial states at time $t \to -\infty = -T$ and $|p_1, p_2\rangle$ are the final states at time $t \to \infty = T$. The function above is correlated to C after being expressed in terms of fields. Later we will see a rigorous relation between the S-matrix and correlation functions when we discuss the LSZ reduction formula.

7.1.3 Reminder: Interaction Picture

At this point I think it makes sense to remind the reader that we want to express correlation functions in terms of fields in the interaction picture. This picture is where both the fields and operators evolve with the free Hamiltonian H_0. So just as an example:

$$\phi_I(t) = e^{+iH_0(t-t_0)}\phi_I(t_0)e^{-iH_0(t-t_0)} \, , \tag{7.1.7}$$

$$\begin{aligned} |\psi_I(t)\rangle &= e^{iH_0(t-t_0)}|\psi_S(t)\rangle \\ &= e^{iH_0(t-t_0)}e^{-iH(t-t_0)}|\psi_I(t_0)\rangle \\ &= U(t,t_0)|\psi_I(t_0)\rangle \, , \end{aligned} \tag{7.1.8}$$

where $|\psi_S(t)\rangle$ is the state in the Schrödinger picture, $|\psi_I(t)\rangle$ is the state in the interaction picture and $U(t,t_0)$ is the time evolution operator in the interaction picture. For further reading I direct the reader to the introduction (1.6.3).

7.2 Perturbation Expansion of Correlation Functions

For the purpose of studying interacting fields let us rederive a time-dependent perturbation theory in a form that is convenient for our purposes. We'll start by trying to calculate the **two-point correlation function**, also known as the **two-point Green's function** in ϕ^4 theory. The two-point correlation function is defined as:

$$\langle\Omega|\mathcal{T}\{\phi(x)\phi(y)\}|\Omega\rangle \, , \tag{7.2.1}$$

The correlation function can be interpreted physically as the amplitude for propagation of a particle or excitation between the two points x and y. In free field theory it is simply the Feynman propagator:

$$\langle\Omega|\mathcal{T}\{\phi(x)\phi(y)\}|\Omega\rangle_{\text{free}} = D_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\varepsilon} \, . \tag{7.2.2}$$

We'd like to know how this expression changes in the interacting theory. We can write the full Hamiltonian of the ϕ^4 theory as:

$$H = H_0 + H_{\text{int}} = H_{\text{Klein-Gordon}} + \int \mathrm{d}^3x \frac{\lambda}{4!}\phi^4(\boldsymbol{x}) \, . \tag{7.2.3}$$

We want to express the two-point correlation function as a power series in λ, which will allow us to do perturbation theory. As a start the interaction Hamiltonian enters in two places. First in the definition of the **Heisenberg field**:

$$\phi(x) = e^{+iHt}\phi(\boldsymbol{x})e^{-iHt} \, , \tag{7.2.4}$$

and second in the definition of the new ground state $|\Omega\rangle$. We'd like to express both of these in terms of free fields and the free theory vacuum, since these are quantities that we know how to manipulate. Starting of with the Heisenberg field, for any fixed time t_0 we can expand the field in terms of ladder operators:

$$\phi(\boldsymbol{x},t_0) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \, , \tag{7.2.5}$$

Then to obtain $\phi(\boldsymbol{x},t)$ for $t \neq t_0$ we we switch to the Heisenberg picture by evolving the field with the full Hamiltonian:

$$\phi(\boldsymbol{x},t) = e^{+iH(t-t_0)}\phi(\boldsymbol{x},t_0)e^{-iH(t-t_0)} \, . \tag{7.2.6}$$

In the special case that $\lambda = 0$ the previous expression simplifies to:

$$\phi(\boldsymbol{x},t)\Big|_{\lambda=0} = e^{+iH_0(t-t_0)}\phi(\boldsymbol{x},t_0)e^{-iH_0(t-t_0)} \equiv \phi_I(\boldsymbol{x},t) \, . \tag{7.2.7}$$

When λ is small the previous expression will still give us the main contribution to the field, thus it makes sense to give this expression a name, the **interaction picture field** $\phi_I(\boldsymbol{x},t)$. Explicitly stated:

$$\phi_I(\boldsymbol{x},t) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\Big|_{x^0=t-t_0} \, . \tag{7.2.8}$$

Now to express the full field in the Heisenberg picture we use the time evolution operator $U(t,t_0)$, as was hinted in the previous section:

$$U(t,t_0) = e^{+iH_0(t-t_0)}e^{-iH(t-t_0)}\,, \tag{7.2.9}$$

$$\phi(\boldsymbol{x},t) = U^\dagger(t,t_0)\phi_I(\boldsymbol{x},t)U(t,t_0)\,. \tag{7.2.10}$$

Peskin mentions that the operator $U(t,t_0)$ is sometimes known as the interaction picture propagator. We'd like to express this operator in terms of just the interaction picture field. This is done by solving the Schrödinger equation for $U(t,t_0)$, where the initial conditions are $U(t_0,t_0)=1$:

$$\begin{aligned} i\frac{\partial}{\partial t}U(t,t_0) &= e^{+iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} \\ &= e^{+iH_0(t-t_0)}(H_{\text{int}})e^{-iH(t-t_0)} \\ &= e^{+iH_0(t-t_0)}(H_{\text{int}})e^{-iH(t-t_0)}e^{+iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= H_I(t)U(t,t_0)\,, \end{aligned} \tag{7.2.11}$$

where the interaction Hamiltonian written in the interaction picture is:

$$H_I(t) = e^{+iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)} = \int \mathrm{d}^3x\frac{\lambda}{4!}\phi_I^4(\boldsymbol{x},t)\,. \tag{7.2.12}$$

The solution to the previous equation can be expressed as a power series in λ:

$$\begin{aligned} U(t,t_0) = 1 &+ (-i)\int_{t_0}^t \mathrm{d}t_1 H_I(t_1) + (-i)^2\int_{t_0}^t \mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2 H_I(t_1)H_I(t_2) + \\ &+ (-i)^3\int_{t_0}^t \mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\int_{t_0}^{t_2}\mathrm{d}t_3 H_I(t_1)H_I(t_2)H_I(t_3) + \dots\,. \end{aligned} \tag{7.2.13}$$

Note that the various factors of H_I stand in **time order**. Using the time ordering operator $\mathcal{T}$ we can simplify the expression. For a general term in the series we have:

$$\int_{t_0}^t \mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\dots\int_{t_0}^{t_{n-1}}\mathrm{d}t_n H_I(t_1)H_I(t_2)\dots H_I(t_n) = \frac{1}{n!}\int_{t_0}^t \mathrm{d}t_1\dots\mathrm{d}t_n\mathcal{T}\big\{H_I(t_1)H_I(t_2)\dots H_I(t_n)\big\}\,. \tag{7.2.14}$$

If we write the first couple of terms we will see that what forms is actually the Taylor expansion of the exponential function:

$$\begin{aligned} U(t,t_0) &= 1 + (-i)\int_{t_0}^t \mathrm{d}t_1 H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^t \mathrm{d}t_1\int_{t_0}^t \mathrm{d}t_2\mathcal{T}\big\{H_I(t_1)H_I(t_2)\big\} + \dots \\ &= \mathcal{T}\left\{\exp\left[-i\int_{t_0}^t \mathrm{d}t' H_I(t')\right]\right\}\,. \end{aligned} \tag{7.2.15}$$

Before moving onto the ground state lets generalize the definition of U to allow it a second value other than t_0. I like Peskin's comment that the correct definition is now *quite natural*:

$$U(t,t') \equiv \mathcal{T}\left\{\exp\left[-i\int_{t'}^t \mathrm{d}t'' H_I(t'')\right]\right\}\,. \tag{7.2.16}$$

It is good to know that U follows the following identities for $t_1 \le t_2 \le t_3$:

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)\,, \tag{7.2.17}$$

$$U(t_1,t_3)\left[U(t_2,t_3)\right]^\dagger = U(t_1,t_2)\,. \tag{7.2.18}$$

One can check by using the Schrödinger equation that the this definition of the time evolution operator satisfies our initial condition and that the operator really is **unitary**. Now we move onto the ground state. We will start with $|0\rangle$ and evolve it through time:

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle\,, \tag{7.2.19}$$

In the next step we must assume that $|\Omega\rangle$ has some overlap with the vacuum state $|0\rangle$, meaning $\langle\Omega|0\rangle \neq 0$. Were this not true, perturbation theory would not be possible. We can write the previous as:

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle \, . \tag{7.2.20}$$

for $E_0 \equiv \langle\Omega|H|\Omega\rangle$. Since $E_n \geq E_0$ we can get rid of all the terms in the sum by sending T to ∞ but in a slightly imaginary direction $T \to \infty(1-i\varepsilon)$. The exponential factor at $n = 0$ falls the slowest, thus we're left with:

$$\begin{aligned}
|\Omega\rangle &= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle \\
&= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))}e^{-iH_0(-T-t_0)}|0\rangle \\
&= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle \, ,
\end{aligned} \tag{7.2.21}$$

where we've used the fact that $H_0|0\rangle = 0$. The expression we got tells us that we can get the ground state by evolving the vacuum state from time $-T$ to t_0 with the time evolution operator $U(t_0, -T)$. In a similar fashion we can express the state $\langle\Omega|$ as:

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \, . \tag{7.2.22}$$

Putting everything together now, if we imagine that $x^0 > y^0 > t_0$ we can express the two-point correlation function as:

$$\begin{aligned}
\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \langle 0|U(T,t_0) \\
&\quad \times \left[U(x^0,t_0)\right]^\dagger \phi_I(x)U(x^0,t_0)\left[U(y^0,t_0)\right]^\dagger \phi_I(y)U(y^0,t_0) \\
&\quad \times U(t_0,-T)|0\rangle \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} \\
&= \lim_{T\to\infty(1-i\varepsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \\
&\quad \times \langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle \, .
\end{aligned} \tag{7.2.23}$$

This kind of looks like a mess but we can engage in a bit of math trickery to simplify it. We will be dividing the expression by 1, which we're going to express as the following:

$$1 = \langle\Omega|\Omega\rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \langle 0|U(T,t_0)U(t_0,-T)|0\rangle \, . \tag{7.2.24}$$

After that simplification we get the following:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \, , \tag{7.2.25}$$

where this expression holds true for $x^0 > y^0$. However the expression would still be correct if $x^0 < y^0$ if the fields are in proper time order. Thus we arrive at the main result of this section:

$$\langle\Omega|\mathcal{T}\big\{\phi(x)\phi(y)\big\}|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\mathcal{T}\big\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}\mathrm{d}tH_I(t)\right]\big\}|0\rangle}{\langle 0|\mathcal{T}\big\{\exp\left[-i\int_{-T}^{T}\mathrm{d}tH_I(t)\right]\big\}|0\rangle} \, . \tag{7.2.26}$$

We generalize this to an n-point correlation function which of course deserves its own box.

Remember: n-point Correlation Function

The n-point correlation function in the ϕ^4 interaction theory is given by:

$$\langle\Omega|\mathcal{T}\big\{\phi(x_1)\dots\phi(x_n)\big\}|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|\mathcal{T}\big\{\phi_I(x_1)\dots\phi_I(x_n)\exp\left[-i\int_{-T}^{T}\int \mathrm{d}^4z\mathcal{H}_{\text{int}}(z)\right]\big\}|0\rangle}{\langle 0|\mathcal{T}\big\{\exp\left[-i\int_{-T}^{T}\int \mathrm{d}^4z\mathcal{H}_{\text{int}}\right]\big\}|0\rangle} . \tag{7.2.27}$$

The above can also be written as:

$$C_n(x_1,\dots,x_n) = \frac{\sum_N \tilde{C}_n^{(N)}(x_1,\dots,x_n)}{\sum_N \tilde{C}_0^{(N)}} , \tag{7.2.28}$$

where N indicates the order of $\mathcal{O}(\lambda^N)$ in the Taylor expansion in λ. For example:

$$\tilde{C}_{n=2}^{(N=1)} = \langle 0|\mathcal{T}\big\{\phi_I(x)\phi_I(y)\left(-i\frac{\lambda}{4!}\right)\int \mathrm{d}^4z\phi_I(z)\phi_I(z)\phi_I(z)\phi_I(z)\big\}|0\rangle . \tag{7.2.29}$$

7.3 Wick's Theorem for Calculating Correlation Functions

We've now come to the realization that calculation correlation functions means evaluating expressions of the form:

$$\langle 0|\mathcal{T}\big\{\phi_I(x_1)\phi_I(x_2)\dots\phi_I(x_n)\big\}|0\rangle , \tag{7.3.1}$$

that is vacuum expectation values of time-ordered products of a finite number of free field operators. We've said before that for two fields this is just the Feynman propagator. For more fields we could try and evaluate the expression by brute force, but in this section we will see how to simplify the calculation by using Wick's theorem, which helps tremendously.

Let's consider again the case of two fields. Since the result is known we can use that to our advantage to rewrite the expression in a form that is easy to evaluate and easy to generalize. We start of by decomposing the interaction picture field into positive and negative frequency parts:

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x) , \tag{7.3.2}$$

where the positive and negative frequency parts are defined as:

$$\phi_I^+(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a_{\boldsymbol{p}}e^{-ip\cdot x} , \quad \phi_I^-(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}a_{\boldsymbol{p}}^\dagger e^{ip\cdot x} . \tag{7.3.3}$$

This decomposition is very handy since it can be done for any free field and it give us the following:

$$\phi_I^+(x)|0\rangle = 0 , \quad \langle 0|\phi_I^-(x) = 0 . \tag{7.3.4}$$

We can use this in the expression for the time-ordered product of two fields:

$$\begin{aligned}\mathcal{T}\big\{\phi_I(x)\phi_I(y)\big\} &\overset{x^0\geq y^0}{=} \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y)\\ &= \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \left[\phi_I^+(x),\, \phi_I^-(y)\right] .\end{aligned} \tag{7.3.5}$$

In the previous expression all terms except the commutator have a vanishing vacuum expectation value. This is due to them being in the *normal order* (e.g., $a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{k}} a_{\boldsymbol{l}}$). For convenience we can define the **normal ordering symbol** to put all operators in normal order:

$$\mathcal{N}(a_{\boldsymbol{p}}a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{q}}) \equiv a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{p}} a_{\boldsymbol{q}} . \tag{7.3.6}$$

It is worth noting that if we considered the case $x^0 < y^0$ above we would have gotten the same expression but with the commutator term having switched points $\left[\phi_I^+(y),\, \phi_I^-(x)\right]$. We now introduce an important definition, the **contraction** of two fields:

$$\wick{\c\phi(x)\c\phi(y)} \equiv \begin{cases} \left[\phi^{+}(x),\ \phi^{-}(y)\right] & \text{if } x^0 > y^0 \ , \\ \left[\phi^{+}(y),\ \phi^{-}(x)\right] & \text{if } x^0 < y^0 \ . \end{cases} \tag{7.3.7}$$

Note: From this point forward I, as well as many other texts, often drop the I subscript from the fields. This is because we're working in the interaction picture and it is understood that all fields are in the interaction picture.

The contraction between two fields is exactly the Feynman propagator:

$$\wick{\c\phi(x)\c\phi(y)} = D_F(x-y) \ . \tag{7.3.8}$$

The relation between the time-ordering and normal-ordering operators is now simple to express as:

$$\mathcal{T}\left\{\phi(x)\phi(y)\right\} = \mathcal{N}\left\{\phi(x)\phi(y) + \wick{\c\phi(x)\c\phi(y)}\right\} \ , \tag{7.3.9}$$

and this can easily be generalized for n fields, thus earning its own box.

Theorem: Wick's Theorem for m Fields

Wick's theorem states that the time-ordered product of m fields can be expressed fields in the *normal order* plus the sum of all possible contractions:

$$\mathcal{T}\left\{\phi(x_1)\phi(x_2)\dots\phi(x_m)\right\} = \mathcal{N}\left\{\phi(x_1)\phi(x_2)\dots\phi(x_m) + \text{all possible contractions}\right\} \ , \tag{7.3.10}$$

where the contraction between two fields is defined as:

$$\wick{\c\phi(x)\c\phi(y)} \equiv \begin{cases} \left[\phi^{+}(x),\ \phi^{-}(y)\right] & \text{if } x^0 > y^0 \ , \\ \left[\phi^{+}(y),\ \phi^{-}(x)\right] & \text{if } x^0 < y^0 \ . \end{cases} \tag{7.3.11}$$

Example: For $m = 4$ fields we have the following:

$$\begin{aligned} \mathcal{T}\left\{1\,2\,3\,4\right\} = \mathcal{N}\{1\,2\,3\,4 &+ \wick{\c1 \c2 3\,4} + \wick{\c1 2 \c3 4} + \wick{\c1 2\,3 \c4} \\ &+ \wick{1 \c2 \c3 4} + \wick{1 \c2 3 \c4} + \wick{1\,2 \c3 \c4} \\ &+ \wick{\c1 1 \c1 2 \c2 3 \c2 4} + \wick{\c1 1 \c2 2 \c1 3 \c2 4} + \wick{\c2 1 \c1 2 \c1 3 \c2 4}\} \ , \end{aligned} \tag{7.3.12}$$

where we've marked ϕ_1 with 1 etc. for brevity and clarity.

7.3.1 Proof of Wick's Theorem for $m = 3$ Fields

Let's prove Wick's theorem for $m = 3$ fields. We start by writing the time-ordered product of three fields:

$$\mathcal{T}\left\{\phi_1\phi_2\phi_3\right\} = \mathcal{N}\left(\phi_1\phi_2\phi_3 + \wick{\c\phi_1\c\phi_2}\phi_3 + \wick{\c\phi_1\phi_2\c\phi_3} + \phi_1\wick{\c\phi_2\c\phi_3}\right) \ . \tag{7.3.13}$$

This is what Wick's theorem tells us. Lets start with the LHS and derive it to RHS in the case where $x_1^0 > x_{2,3}^0$:

$$\begin{aligned} \mathcal{T}\left\{\phi_1\phi_2\phi_3\right\} &= (\phi_1^+ + \phi_1^-)\mathcal{T}\left\{\phi_2\phi_3\right\} \\ &= (\phi_1^+ + \phi_1^-)\mathcal{N}(\phi_2\phi_3 + \wick{\c\phi_2\c\phi_3}) \\ &= \mathcal{N}(\phi_1^+\phi_2\phi_3 + \phi_1^+\wick{\c\phi_2\c\phi_3}) + \mathcal{N}(\phi_2\phi_3)\phi_1^- + \left[\phi_1^-,\ \mathcal{N}(\phi_2\phi_3)\right] + \wick{\c\phi_2\c\phi_3}\phi_1^- \\ &= \mathcal{N}(\phi_1\phi_2\phi_3) + \mathcal{N}(\phi_1\wick{\c\phi_2\c\phi_3}) + \mathcal{N}(\wick{\c\phi_1\c\phi_2}\phi_3 + \wick{\c\phi_1\phi_2\c\phi_3}) \ , \end{aligned} \tag{7.3.14}$$

where we've used quite a few tricks including using Wick's theorem for two fields in the second line, as well as putting the field ϕ_1^- into normal order from the second to the third line. The commutator we've used above can be evaluated as:

$$
\begin{aligned}
\left[\phi_1^-,\, \mathcal{N}(\phi_2\phi_3)\right] &= \left[\phi_1^-,\, \phi_2^+\phi_3^+\right] + \left[\phi_1^-,\, \phi_2^+\phi_3^-\right] + \left[\phi_1^-,\, \phi_3^+\phi_2^-\right] + \left[\phi_1^-,\, \phi_2^-\phi_3^-\right] \\
&\qquad \text{where} \quad
\begin{aligned}
&\left[\phi_1^-,\, \phi_2^+\phi_3^+\right] = \left[\phi_1^-,\, \phi_2^+\right]\phi_3^+ + \phi_2^+\left[\phi_1^-,\, \phi_3^+\right] = \wick{\c{\phi}_1\c{\phi}_2}\phi_3^+ + \phi_2^+\wick{\c{\phi}_1\c{\phi}_3}\,, \\
&\left[\phi_1^-,\, \phi_2^+\phi_3^-\right] = \left[\phi_1^-,\, \phi_2^+\right]\phi_3^- = \wick{\c{\phi}_1\c{\phi}_2}\phi_3^-\,, \\
&\left[\phi_1^-,\, \phi_3^+\phi_2^-\right] = \phi_2^-\left[\phi_1^-,\, \phi_3^+\right] = \phi_2^-\wick{\c{\phi}_1\c{\phi}_3}\,, \\
&\left[\phi_1^-,\, \phi_2^-\phi_3^-\right] = 0\,,
\end{aligned} \\
&= \mathcal{N}(\wick{\c{\phi}_1\c{\phi}_2}\phi_3 + \wick{\c{\phi}_1\phi_2\c{\phi}_3})\,,
\end{aligned}
\tag{7.3.15}
$$

where we've used the commutation identity:

$$
\left[A,\, BC\right] = ABC - BCA = \left[A,\, B\right]C - B\left[C,\, A\right] = ABC - BAC - BCA + BAC\,. \tag{7.3.16}
$$

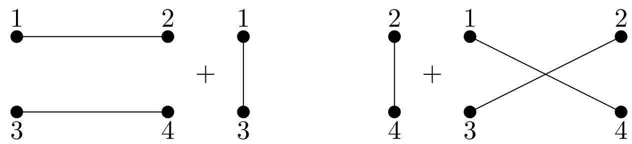
The previous commutator indicates that A gets contracted with all remaining operators. We've now shown that Wick's theorem holds for $m = 3$ fields.

8 Feynman Diagrams

Wick's theorem is brilliant since it allows us to turn any expression of the form:

$$
\langle 0|\mathcal{T}\left\{\phi_1(x_1)\phi_2(x_2)\ldots\phi_n(x_n)\right\}|0\rangle\,, \tag{8.0.1}
$$

into a sum of products of Feynman propagators/two-point correlation functions. **Feynman diagrams** are a diagrammatic interpretation of such expressions. Consider for example four fields the expression for which we expanded in the previous section (7.3.12). We can represent each point in the expression as a dot and each propagator as a line connecting two dots. The process can be represented as a sum of three Feynman diagrams:

$\langle 0|\mathcal{T}\left\{\phi_1\phi_2\phi_3\phi_4\right\}|0\rangle =$  (8.0.2)

This isn't a measurable quantity but it does give hint at an interpretation of the expression. Particles are created at two spacetime points, then each particle propagates to another spacetime point and then they annihilate. Since this can happen in three ways we have three diagrams. The total amplitude is the sum of the three diagrams.

8.1 Example: ϕ^4 Interaction to the 2nd Order

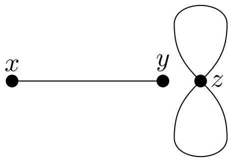
Interesting things start to happen when we have two or more fields in the same spacetime point. Lets consider the ϕ^4 interaction to the second order. We must evaluate the following:

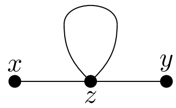
$$
\tilde{C}_2(x,y) = \langle 0|\mathcal{T}\left\{\phi(x)\phi(y)\left(1 - i\frac{\lambda}{4!}\int \mathrm{d}^4z\phi^4(z)\right)\right\}|0\rangle = \tilde{C}_2^{(0)} + \tilde{C}_2^{(1)}\,. \tag{8.1.1}
$$

We've seen before that the first term is just the Feynman propagator $\Delta_F(x-y)$. The second term is where things start to get interesting. We can expand the expression using Wick's theorem and evaluate:

$$
\begin{aligned}
\tilde{C}_2^{(1)} &= -i\frac{\lambda}{4!}\int \mathrm{d}^4z\, \langle 0|\mathcal{T}\left\{\phi(x)\phi(y)\phi^4(z)\right\}|0\rangle \\
&= 3\cdot\left(\frac{-i\lambda}{4!}\right)\Delta_F(x-y)\int \mathrm{d}^4z\, \Delta_F(z-z)\Delta_F(z-z) \\
&\quad + 12\cdot\left(\frac{-i\lambda}{4!}\right)\int \mathrm{d}^4z\, \Delta_F(x-z)\Delta_F(y-z)\Delta_F(z-z)\,.
\end{aligned}
\tag{8.1.2}
$$

We can better understand this expression if we draw each term as a Feynman diagram. Here we must distinguish between internal and external points.x and y are external points of $\tilde{C}_2$. The internal points z can be *anywhere* and are integrated over. Thus we can draw the following diagrams for the previous expression:

$\tilde{C}_2^{(1)} =$ 

$+$ 

(8.1.3)

We refere to the lines in these diagrams as **propagators**, since they represent the propagation amplitude Δ_F. Internal points where lines meet are called **vertices**. Wick's theorem represents $\tilde{C}_2$ as a sum of products of free propagators between points. The scalar particle is created at y and the propagated over to x where it is annihilated.

8.2 Feynman Rules in Coordinate Space

We can summarize a few rules for calculating the numerator of what is stated by Wick's theorem (7.3) which is:

$$\langle 0|\mathcal{T}\{\phi_I(x)\phi_I(y)\exp\left[-i\int \mathrm{d}tH_I(t)\right]\}|0\rangle = \left(\begin{smallmatrix}\text{sum of all possible diagrams}\\ \text{with two external points}\end{smallmatrix}\right) . \tag{8.2.1}$$

We build each diagram out of propagators, vertices and external points. The rules for associating analytic expressions with pieces of diagrams are called the *Feynman rules*. The rules for the ϕ^4 interaction are as follows.

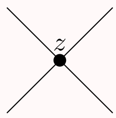
Remember: Feynman Rules for ϕ^4 Interaction

The *position-space* Feynman rules for the ϕ^4 interaction are as follows:

1. For each **propagator**:

$= \Delta_F(x-y)$, (8.2.2)

2. For each **vertex**:



$= -i\lambda \int \mathrm{d}^4 z$, (8.2.3)

3. For each **external point**:

$= 1$, (8.2.4)

4. Divide by the symmetry factor, given as:

$$\frac{1}{S} = \frac{1}{N!}\left(\frac{1}{4!}\right)^N f , \tag{8.2.5}$$

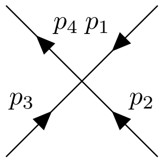
where f is the number of Wick contractions that render the same value of contribution to $\tilde{C}_n^N$.

8.3 Feynman Rules in Momentum Space

We've seen the Feynman rules for the ϕ^4 interaction in coordinate space, however it is more convenient to express the Feynman rules in terms of momenta. We can do this by using the Fourier transform of the propagator:

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} . \tag{8.3.1}$$

We can represent this in the diagram by assigning a 4-momentum p to each propagator and since $\Delta_F(x-y)=\Delta_F(y-x)$ the direction of the arrow in the diagram is arbitrary. We get the following:



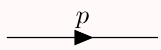
$$\longleftrightarrow \lim_{T\to\infty(1-i\varepsilon)} \int \mathrm{d}z^0\mathrm{d}^3z e^{-ip_1\cdot z}e^{-ip_2\cdot z}e^{-ip_3\cdot z}e^{+ip_4\cdot z} = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3-p_4)\,, \tag{8.3.2}$$

where the reader can ignore the limit as $T\to\infty(1-i\varepsilon)$ for now. We will discuss this later (8.3.1). In other words what we got is the fact that momentum is conserved at each vertex. We can use these delta functions from the vertices to perform some of the momentum integrals in propagators. That leaves us with the following Feynman rules in momentum space.

Remember: Feynman Rules for ϕ^4 Interaction in Momentum Space

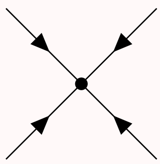
The *momentum-space* Feynman rules for the ϕ^4 interaction are as follows:

1. For each **propagator**:



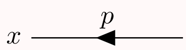
$$= \frac{i}{p^2-m^2+i\epsilon}\,, \tag{8.3.3}$$

2. For each **vertex**:

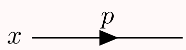


$$= -i\lambda\,, \tag{8.3.4}$$

3. For each **external point**:



$$= e^{-ip\cdot x}\,, \tag{8.3.5}$$



$$= e^{ip\cdot x}\,, \tag{8.3.6}$$

4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined momentum:

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,, \tag{8.3.7}$$

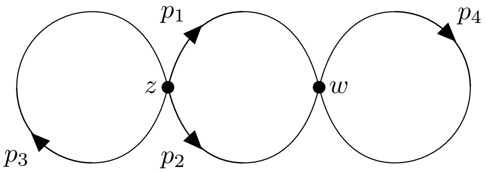
6. Divide by the symmetry factor, given as:

$$\frac{1}{S} = \frac{1}{N!}\left(\frac{1}{4!}\right)^N f\,, \tag{8.3.8}$$

where f is the number of Wick contractions that render the same value of contribution to $\tilde{C}_n^N$.

8.3.1 What happens to T the slightly imaginary infinite time?

We've ignored the limit as $T\to\infty(1-i\varepsilon)$ up until now. Time to lift up the rug and clean the mess we've made underneath. Consider the expression for momentum conservation at each vertex (8.3.2). The exponential blows up as $z^0\to\infty$ or $z^0\to-\infty$ unless it's argument is purely imaginary. We can achieve this by taking each p^0 to have a small imaginary part $p^0\propto(1+i\varepsilon)$. This sometimes called the **Feynman $i\varepsilon$ prescription**. It's precisely what we do when following the Feynman boundary conditions for computing Δ_F. We integrate along a contour that is slightly rotated away from the real axis, from where we get $p^0 \propto (1+i\varepsilon)$. The explicit dependence on T seems to disappear when we take the limit $T \to \infty$, but consider the diagram:



(8.3.9)

The delta function for the left-hand vertex is $(2\pi)^4\delta^{(4)}(p_1+p_2)$, so momentum conservation at the right-hand vertex is automatically satisfied and we get $(2\pi)^4\delta^{(4)}(0)$. We can better understand where things go wrong if we move back to the position space. Writing the right-hand vertex contribution we get:

$$\int \mathrm{d}^4 w(\text{const.}) \propto (2T)\cdot(\text{volume of space})\,, \tag{8.3.10}$$

which tells us that the spacetime process shown in the diagram above (8.3.9) can happen at any place in space and at any time between $-T$ and T. Every **disconnected** piece of a diagram contributes a factor of $(2\pi)^4\delta(0) = 2T\cdot V$. A disconnected diagram is one that is not connected to an external point.

We can understand this even further if we consider, what Peskin calls, *a very pretty identity*, called **the exponentiation of the disconnected diagrams**. Lets label the various possible disconnected pieces by a set V_i. The elements of this set are connected internally but disconnected from external points. Let's assume that some Feynman diagram has n_i pieces of the form V_i in addition to its one piece that is already connected to x and y. Let's let the value V_i also denote the value of the disconnected piece V_i. We can then write the value of a diagram as follows:

$$(\text{value of connected piece})\cdot\prod_i \frac{1}{n_i!}(V_i)^{n_i}\,, \tag{8.3.11}$$

where the $1/n_i!$ factor is the symmetry factor that comes from interchanging the n_i copies of V_i. The sum of all diagrams, representing the numerator in the formula for the two-point correlation function is then:

$$\sum_{(\text{all connected pieces})}\;\sum_{(\text{all }\{n_i\})}\left(\substack{\text{value of}\\ \text{connected piece}}\right)\times\left(\prod_i \frac{1}{n!}(V_i)^{n_i}\right)\,, \tag{8.3.12}$$

where all $\{n_i\}$ denotes all ordered sets of $\{n_1, n_2, \ldots\}$ such. For convenience we will mark the summation terms as $(\sum(\textit{connected}))$. The sum of all the connected pieces factors out just this expression:

$$=\left(\sum \text{connected}\right)\times\sum_{(\text{all }\{n_i\})}\left(\prod_i\frac{1}{n_i!}(V_i)^{n_i}\right)\,. \tag{8.3.13}$$

We can factor the rest of the expression in a similar fashion:

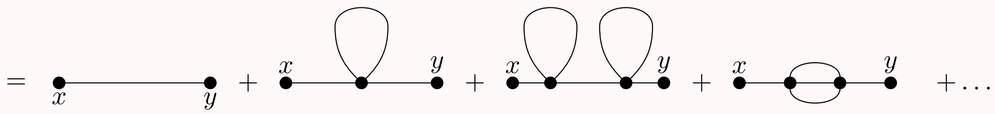
$$\begin{aligned}
&=\left(\sum\text{connected}\right)\times\left(\sum_{n_1}\frac{1}{n_1!}(V_1)^{n_1}\right)\left(\sum_{n_2}\frac{1}{n_2!}(V_2)^{n_2}\right)\cdots\\
&=\left(\sum\text{connected}\right)\times\prod_i\left(\sum_{n_i}\frac{1}{n_i!}V_i^{n_i}\right)\\
&=\left(\sum\text{connected}\right)\times\prod_i\exp(V_i)\\
&=\left(\sum\text{connected}\right)\times\exp\left(\sum_i V_i\right)\,.
\end{aligned} \tag{8.3.14}$$

We've just shown that the sum of all diagrams is equal to the sum of all *connected* diagrams, times the exponential of the sum of all *disconnected* diagrams. Now consider the denominator of the two-point correlation function expression. By the same argument as above is the exponential of the disconnected diagrams, like in the numerator. Thus we reach the final simplification of the two-point correlation function formula, which is the sum of all connected diagrams.

Remember: Simplified Two-Point/Generalized Correlation Function Formula

The two-point correlation function formula can be simplified using the above arguments to:

$$\begin{aligned}\langle\Omega|\mathcal{T}\{\phi(x)\phi(y)\}|\Omega\rangle &= \\ &= \text{sum of all connected diagrams with two external points}\end{aligned}$$



(8.3.15)

Likewise we can generalize this to higher order correlation functions, where we simply get:

$$\langle\Omega|\mathcal{T}\{\phi(x_1)\dots\phi(x_n)\}|\Omega\rangle = \left(\begin{smallmatrix}\text{sum of all connected diagrams}\\ \text{with } n \text{ external points}\end{smallmatrix}\right) . \tag{8.3.16}$$

Using the same argument as before we can conclude that disconnected diagrams exponentiate, factor and cancel out. Friendly reminder here that not all external points need to be connected with each other in the case of a connected diagram. We use the term **disconneceted** diagram to describe a diagram that is completely disconnected from all external points.

(a) An example of a *connected* graph. (b) An example of a *disconnected* graph.

Figure 8.1: Graphs for demonstrative purposes.

9 Cross Sections and the Scattering Matrix

We now have a very handy formula (8.3.16) with which we can calculate n-point correlation functions. However, the concept of a correlation function is quite abstract and not very physical. We'd now like to find a way to relate this to physical quantities like cross sections and decay rates, that can actually be measured in experiments. We will approach this task by relating correlation functions to an even more primitive quantity called the S-matrix.

9.1 The S-matrix

If we want to discuss the probability of an event happening, we first have a few conditions we need to satisfy. Most notably, our initial states must be normalized. If we try to do this with something like planar waves, we quickly run into trouble. Thus we take a different approach where we use **Gaussian Wavepackets** as our initial states. A wavepacket representing a state $|\phi\rangle$ is given by:

$$|\phi(x)\rangle = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\phi(k)|k\rangle , \tag{9.1.1}$$

where $\phi(k)$ is the Fourier transform of the spatial wavefunction $\phi(x)$ and $|k\rangle$ is a one-particle state of momentum k in the interacting theory. In the case of a free theory this would simply be:

$$|k\rangle = \sqrt{2E_k}a_k^\dagger|0\rangle . \tag{9.1.2}$$

The factor $\sqrt{2E_k}$ is there to ensure that our relativistic normalization of $|k\rangle$ is consistent with conventional normalization (in which all probabilities add up to 1). Mathematically this is expressed as:

$$\langle\phi|\phi\rangle = 1 \qquad \text{if} \qquad \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left|\phi(k)\right|^2 = 1\,. \tag{9.1.3}$$

This gives us the probability:

$$\mathcal{P} = \left|\langle\phi_1\phi_2\ldots|\phi_A\phi_B\ldots\rangle\right|^2\ , \tag{9.1.4}$$

where $|\phi_A\phi_B\rangle$ is the *in* state of two wavepackets constructed in the far past and $\langle\phi_1\phi_2\ldots|$ is a state of several wavepackets (more precisely one per final-state particle) constructed in the far future. Peskin reminds us here that we are using the Heisenberg picture here, which means that states are time-independent. However the name we give a state depends on the eigenvalues of time-dependent operators. This means that states with the same name created at different times have non-trivial overlap, which is a result of the time-dependent operators.

For states $|\phi_A\phi_B\rangle$ we can take the limit in which the wavepackets $\phi_i(k_i)$ become concentrated about definite momenta p_i. With this we are able to define an *in* state $|p_Ap_B\rangle$ with definite initial momenta. Another technicality to take into account is the fact that the wavepacket ϕ_B is transversly displaced relative to ϕ_A in position space. Here we adopt the convention that out reference momentum-space wavefunctions are collinear (meaning, the impact parameter is $b=0$) and write $\phi_B(k_B)$ with an explicit factor of $\exp(-ib\cdot k_B)$ to account for the translation in physical space. Now since ϕ_A and ϕ_B are constructed independently at different locations, we can write the initial state as:

$$|\phi_A\phi_B\rangle_{\text{in}} = \int\frac{\mathrm{d}^3 k_A}{(2\pi)^3}\int\frac{\mathrm{d}^3 k_B}{(2\pi)^3}\frac{\phi_A(k_A)\phi_B(k_B)e^{-ib\cdot k_B}}{\sqrt{(2E_A)(2E_B)}}\,|k_Ak_B\rangle_{\text{in}}\,. \tag{9.1.5}$$

It is possible to simmilarly expand $\langle\phi_1\phi_2\ldots|$ in terms of *out* states of definite momentum which are formed in the far future

$$\mathrm{d}\mathcal{P}_{\text{AB}\to 12} = \left\{\prod_{f=1,2}\frac{\mathrm{d}^3 p_f}{(2\pi)^3 2E_f}\right\}\left|{}_{\text{out}}\langle p_1p_2|\phi_A\phi_B\rangle_{\text{in}}\right|^2\ . \tag{9.1.6}$$

However, for our *out* states it is much simpler to use the *out* states of definite momentum as as the final states in the probability amplitude and to multiply by the various normalization factors after squaring the amplitude. This is sound as long as the detectors of final-state particles mainly measure momentum instead of resolving positions (at the level of de Broglie wavelengths). We can now relate the probability of scattering measured in a real experiment to an idealized set of transition amplitudes between the asymptotically defined *in* and *out* states of definite momentum. It is probably a good idea to remind ourselves that these states are related by time translation (this is evident if we go back to the Schrodinger picture):

$${}_{\text{out}}\langle p_1p_2\ldots|k_Ak_B\rangle_{\text{in}} \overset{\to\text{Schrodinger}}{=} \lim_{T\to\infty}\langle p_1p_2\ldots|e^{-iH(2T)}|k_Ak_B\rangle\,. \tag{9.1.7}$$

Where we've defined the states in Schrodinger's picture at any common reference time. From this we can see that *in* and *out* states are truly related by a limit of a sequence of unitary operators. This limiting unitary operator is called the S**-matrix**:

$${}_{\text{out}}\langle p_1p_2\ldots|k_Ak_B\rangle_{\text{in}} \equiv {}_{\text{out}}\langle p_1p_2\ldots|\,S\,|k_Ak_B\rangle_{\text{in}}\,. \tag{9.1.8}$$

For particles that do not interact at all, the S-matrix is simply the identity operator. Even if the theory contains interactions there is some nonzero chance of particles missing each other. We can isolate the interesting part of the S-matrix, the one that describes the interactions, by defining the T-matrix as:

$$S = \mathbb{1} + iT\,. \tag{9.1.9}$$

The **optical theorem**

$$i(\hat{T}-\hat{T}^\dagger) = -\hat{T}\hat{T}^\dagger\ , \tag{9.1.10}$$

holds true for the T-matrix which we can see from:

$$\hat{S}\hat{S}^\dagger = \mathbb{1} = (\mathbb{1} + i\hat{T})(\mathbb{1} - i\hat{T}^\dagger) = \mathbb{1} + i(\hat{T} - \hat{T}^\dagger) + \hat{T}\hat{T}^\dagger \,. \tag{9.1.11}$$

We can extract the *invariant matrix element* $\mathcal{M}$ by considering the fact that elements of the S-matrix must conserve 4-momentum. Thus S or T must always contain a factor of $\delta^{(4)}(k_A + k_B - \sum p_f)$ which guarantees such. We define the invariant matrix element $\mathcal{M}$ by extracting this factor

$$\langle p_1 p_2 \ldots | iT | k_A k_B \rangle = (2\pi)^4 \delta^{(4)}(k_A + k_B - \sum p_f) \cdot i\mathcal{M}(k_A, k_B \to p_f) \,. \tag{9.1.12}$$

A reminder here that all 4-momenta are on mass-shell, thus $p^0 = E_p$, $k^0 = E_k$. Additionally this entire derivation only holds true for the specific case where we have only two particles in the initial state. For more particles we can analogously define such constructions. Peskin adds to this, that such complicated experiments are left for another book. The invariant matrix element $\mathcal{M}$ represents the scattering amplitude f that we know from one-particle quantum mechanics.

9.2 Calculating Cross Sections and Decay Rates

To relate $\mathcal{M}$ to the scattering cross section σ lets first calculate the probability for the initial state $|\phi_A \phi_B\rangle$ to scatter and become a final state of n particles whose momenta lie in a small region $\mathrm{d}^3 p_1 \ldots \mathrm{d}^3 p_n$. This is given as

$$\mathcal{P}(\mathcal{AB} \to 1\,2 \ldots\, n) = \left(\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\, {}_{\text{out}}\langle p_1 \ldots p_n | \phi_A \phi_B \rangle_{\text{in}} |^2 \,. \tag{9.2.1}$$

For a single target particle $\mathcal{A}$ and many incident particles $\mathcal{B}$ with different parameters b the number of scattering events is

$$N = \sum_{\substack{\text{all incident} \\ \text{particles } i}} \mathcal{P}_i = \int \mathrm{d}^2 b \; n_B \mathcal{P}(b) \,, \tag{9.2.2}$$

where n_B represents the number density of incident particles $\mathcal{B}$. Since we are assuming that this number density is constant over the range of the interaction, we can take n_B out of the integral giving us the cross section σ as

$$\sigma = \frac{N}{n_B N_A} = \frac{N}{n_B \cdot 1} \int \mathrm{d}^2 b \; \mathcal{P}(b) \,. \tag{9.2.3}$$

With this we can derive an expression for $\mathrm{d}\sigma$ in terms of $\mathcal{M}$ by combining Equations (9.2.3), (9.2.1) and (9.1.5):

$$\begin{aligned} \mathrm{d}\sigma = \left(\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \right) \int \mathrm{d}^2 b \left(\prod_{i=\mathcal{A},\mathcal{B}} \int \frac{\mathrm{d}^3 k_i}{(2\pi)^3} \frac{\phi_i(k_i)}{\sqrt{2E_i}} \int \frac{\mathrm{d}^3 \bar{k}_i}{(2\pi)^3} \frac{\phi_i^*(\bar{k}_i)}{\sqrt{2\bar{E}_i}} \right) \\ \times e^{ib\cdot(\bar{k}_\mathcal{B} - k_\mathcal{B})} \left({}_{\text{out}}\langle \{p_f\} \,|\, \{k_i\} \rangle_{\text{in}} \right) \left({}_{\text{out}}\langle \{p_f\} \,|\, \{\bar{k}_i\} \rangle_{\text{in}} \right)^* \,, \end{aligned} \tag{9.2.4}$$

where $\bar{k}_\mathcal{A}, \bar{k}_\mathcal{B}$ are dummy integration variables for the second half of the squared amplitude. We can perform the integral over $\mathrm{d}^2 b$ from which we get a factor of $(2\pi)^2 \delta^{(2)}(k_\mathcal{B}^\perp - \bar{k}_\mathcal{B}^\perp)$. We can conjure up more delta functions by writing the final two factors from Equation (9.2.4) in terms of the invariant matrix element $\mathcal{M}$:

$$\left({}_{\text{out}}\langle \{p_f\} \,|\, \{\bar{k}_i\} \rangle_{\text{in}} \right) = i\mathcal{M}(\{k_i\} \to \{p_f\}) (2\pi)^4 \delta^{(4)} \left(\sum k_i - \sum p_f \right) \,, \tag{9.2.5}$$

$$\left({}_{\text{out}}\langle \{p_f\} \,|\, \{\bar{k}_i\} \rangle_{\text{in}} \right)^* = -i\mathcal{M}^*(\{\bar{k}_i\} \to \{p_f\}) (2\pi)^4 \delta^{(4)} \left(\sum \bar{k}_i - \sum p_f \right) \,. \tag{9.2.6}$$

Using all these delta functions together we can perform all six integrals over $\bar{k}$ in (9.2.4). Of those six integrals, two ($\bar{k}^z_{\mathcal{A}}$ and $\bar{k}^z_{\mathcal{B}}$) need some some extra work. Let's take a look:

$$\begin{aligned}
&\int \mathrm{d}\bar{k}^z_{\mathcal{A}} \mathrm{d}\bar{k}^z_{\mathcal{B}}\, \delta(\bar{k}^z_{\mathcal{A}} + \bar{k}^z_{\mathcal{B}} - \sum p^z_f)\, \delta(\bar{E}_{\mathcal{A}} + \bar{E}_{\mathcal{B}} - \sum E_f) \\
&= \int \mathrm{d}\bar{k}^z_{\mathcal{A}}\, \delta\left(\sqrt{\bar{k}^2_{\mathcal{A}} + m^2_{\mathcal{A}}} + \sqrt{\bar{k}^2_{\mathcal{B}} - m^2_{\mathcal{B}}} - \sum E_f\right)\Bigg|_{\bar{k}^z_{\mathcal{B}} = \sum p^z_f - \bar{k}^z_{\mathcal{A}}} \\
&= \frac{1}{\left|\dfrac{\bar{k}^z_{\mathcal{A}}}{\bar{E}_{\mathcal{A}}} - \dfrac{\bar{k}^z_{\mathcal{B}}}{\bar{E}_{\mathcal{B}}}\right|} \equiv \frac{1}{|v_{\mathcal{A}} - v_{\mathcal{B}}|}\,.
\end{aligned} \tag{9.2.7}$$

This ensures that momenta and energies are conserved in the z-direction. The remaining 4 integrals enforce the constraints $\bar{k}^{\perp}_{\mathcal{A}} = k^{\perp}_{\mathcal{A}}$ and $\bar{k}^{\perp}_{\mathcal{B}} = k^{\perp}_{\mathcal{B}}$. The difference $|v_{\mathcal{A}} - v_{\mathcal{B}}|$ is the relative velocity of the two beams as viewed from the laboratory frame. The initial wavepackets we created are localized in momentum space, centered on $p_{\mathcal{A}}$ and $p_{\mathcal{B}}$. This means that we can evaluate all factors that are smooth functions of $k_{\mathcal{A}}$ and $k_{\mathcal{B}}$ at $p_{\mathcal{A}}$ and $p_{\mathcal{B}}$ and pull them out of the integrals. These factors include $E_{\mathcal{A}}, E_{\mathcal{B}}$, $|v_{\mathcal{A}} - v_{\mathcal{B}}|$, and everything $\mathcal{M}$ related. All that remains is the following delta function:

$$\begin{aligned}
\mathrm{d}\sigma = &\left(\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\right) \frac{|\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\})|}{2E_{\mathcal{A}} 2E_{\mathcal{B}}\, |v_{\mathcal{A}} - v_{\mathcal{B}}|} \int \frac{\mathrm{d}^3 k_{\mathcal{A}}}{(2\pi)^3} \int \frac{\mathrm{d}^3 k_{\mathcal{B}}}{(2\pi)^3} \\
&\times |\phi_{\mathcal{A}}(k_{\mathcal{A}})|^2\, |\phi_{\mathcal{B}}(k_{\mathcal{B}})|^2\, (2\pi)^4 \delta^{(4)}\left(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\right)\,.
\end{aligned} \tag{9.2.8}$$

To further simplify this expression we need to consider the fact that real particle detectors project mainly onto eigenstates of momentum, but since they have finite resolution, they sum incoherently over momentum bites of finite size. This means that the measurement of the final state momentum is not such high quality that we could resolve the small variations in this momentum that results from the momentum spread of the initial wavepackets. In that case we can treat the momentum vector in the delta function $k_{\mathcal{A}} + k_{\mathcal{B}}$ as being well approximated by its center value $p_{\mathcal{A}} + p_{\mathcal{B}}$. We can then perform integrals over $k_{\mathcal{A}}$ and $k_{\mathcal{B}}$, while still taking into account the fact that we'd like conventional normalization of the wavepackets in which the sum of all probabilities is 1. This gives us the relation between the S-matrix and the cross sections, which deserves its own little box (9.2). Notice how all dependence on the shapes of the initial wavepackets has disappeared from the final expression.

$$\mathrm{d}\sigma = \frac{|\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\})|^2}{2E_{\mathcal{A}} E_{\mathcal{B}}\, |v_{\mathcal{A}} - v_{\mathcal{B}}|} \left(\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3 2E_f}\right) (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f\right)\,. \tag{9.2.9}$$

If we were to integrate over all the final-state momenta p_f in Equation (9.2.9) we get an expression which has the structure:

$$\int \mathrm{d}\Pi_n = \left(\prod_f \int \frac{\mathrm{d}^3 p_f}{(2\pi)^3} (2\pi)^4 \delta^{(4)}\left(P - \sum p_f\right)\right)\,, \tag{9.2.10}$$

where $P = p_{\mathcal{A}} + p_{\mathcal{B}}$ is the total initial momentum. This integral is manifestly Lorentz invariant, since it is built from invariant 3-momentum integrals constrained by a 4-momentum delta function. It is known as the **relativistically invariant n-body phase space**. Since the matrix element $\mathcal{M}$ is also Lorentz invariant the only transformation property for the cross section comes from the relative velocity pre-factor:

$$\frac{1}{E_{\mathcal{A}} E_{\mathcal{B}}\, |v_{\mathcal{A}} - v_{\mathcal{B}}|} = \frac{1}{E_{\mathcal{B}} p^z_{\mathcal{A}} - E_{\mathcal{A}} p^z_{\mathcal{B}}} = \frac{1}{\varepsilon_{\mu x y \nu} p^{\mu}_{\mathcal{A}} p^{\nu}_{\mathcal{B}}}\,, \tag{9.2.11}$$

which is **not** completely Lorentz invariant. It is only invariant under boosts along the z-axis. This expression has the exact properties we'd expect from a cross-sectional area. We can simplify the Equation (9.2.9) further in the case of two particles in the final state by evaluating the phase-space integrals in the center-of-mass (CMS) frame. We'll label the outgoing momenta as p_1 and p_2. We can integrate all three components of p_2 over the delta functions enforcing 3-momentum conservation, which sets $p_2 = -p_1$. The two-body phase space integral is then of the form:

$$\int \mathrm{d}\Pi_2 = \int \frac{\mathrm{d}p_1\, p_1^2 \mathrm{d}\Omega}{(2\pi)^3 2E_1 2E_2}(2\pi)\delta(E_{\text{CMS}} - E_1 - E_2)\,, \tag{9.2.12}$$

where E_{cms} is the total initial energy and $E_i = \sqrt{p_i^2 + m_i^2}$ is the energy of the final-state particles. Evaluating the previous integral yields:

$$\begin{aligned}\int \mathrm{d}\Pi_2 &= \int \mathrm{d}\Omega \frac{p_1^2}{16\pi^2 E_1 E_2}\left(\frac{p_1}{E_1} + \frac{p_1}{E_2}\right)^{-1} \\ &= \int \frac{1}{16\pi^2}\frac{|p_1|}{E_{\text{CMS}}}\,. \end{aligned} \tag{9.2.13}$$

For reactions symmetric around the collision axis, the two-body phase space integral can be simply evaluated as an integral over the polar angle in the center-of-mass frame $\mathrm{d}(\cos\theta)$. Using this result we can rewrite the differential cross section from Equation (9.2.9) in the center-of-mass frame as:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CMS}} = \frac{|\mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to p_1, p_2)|^2}{2E_{\mathcal{A}}E_{\mathcal{B}}\,|v_{\mathcal{A}} - v_{\mathcal{B}}|}\frac{|p_1|}{(2\pi)^2 E_{\text{CMS}}}\,. \tag{9.2.14}$$

In the special case where all four particles have the same mass (including the limit $m \to 0$) this can be even further simplified to:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CMS}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{CMS}}^2}\,. \tag{9.2.15}$$

We are also interested in the differential decay rate $\mathrm{d}\Gamma$ in terms of $\mathcal{M}$ which we get by slightly modifying the expression in Equation (9.2.9). All we do is remove the factors that do not make sense for a initial state of a single particle. Noteworthy is also the fact that by definition the decaying particle is at rest, thus we can set the normalization factor $(2E_{\mathcal{A}})^{-1} = (2m_{\mathcal{A}})^{-1}$. With that we can now make a box for all of our relations where the differential decay rate $\mathrm{d}\Gamma$ is given in Equation (9.2.19).

Remember: Relation Between S-matrix, Cross Section and Decay Rate

The differential cross section $\mathrm{d}\sigma$ is given by:

$$\mathrm{d}\sigma = \frac{|\mathcal{M}(p_\mathcal{A}, p_\mathcal{B} \to \{p_f\})|^2}{2E_\mathcal{A}E_\mathcal{B}\,|v_\mathcal{A} - v_\mathcal{B}|} \left(\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3 2E_f}\right) (2\pi)^4 \delta^{(4)}\left(p_\mathcal{A} + p_\mathcal{B} - \sum p_f\right) , \tag{9.2.16}$$

which can be further simplified in the case of two outgoing particles to:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CMS}} = \frac{|\mathcal{M}(p_\mathcal{A}, p_\mathcal{B} \to p_1, p_2)|^2}{2E_\mathcal{A}E_\mathcal{B}\,|v_\mathcal{A} - v_\mathcal{B}|} \frac{|p_1|}{(2\pi)^2 E_{\text{CMS}}} . \tag{9.2.17}$$

This can be even further simplified in the case where all four particles have the same mass (including the limit $m \to 0$) to:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CMS}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{CMS}}^2} \qquad \text{all masses identical} . \tag{9.2.18}$$

If we consider the decay of a single particle $\mathcal{A}$ into n particles with momenta $p_1, p_2, \ldots, p_n$ we can write the differential decay rate as:

$$\mathrm{d}\Gamma = \frac{|\mathcal{M}(m_\mathcal{A} \to \{p_f\})|^2}{2m_\mathcal{A}} \left(\prod_f \frac{\mathrm{d}^3 p_f}{(2\pi)^3 2E_f}\right) (2\pi)^4 \delta^{(4)}\left(p_\mathcal{A} - \sum p_f\right) , \tag{9.2.19}$$

where the decay rate considers the decaying particle to be at rest ($E_\mathcal{A} = m_\mathcal{A}$).
When computing any total cross section σ or decay rate Γ we must take care to avoid multiple counting of the same final state if the outgoing particles are indistinguishable. This is done by either restricting integration to only inequivalent configurations or by dividing by the factorial of the number of indistinguishable particles in the final state after integrating over all momenta.

9.3 The LSZ Reduction Formula

9.3.1 Motivation

We want express the S-matrix in terms of correlation functions. Let us try to show that:

$$\sqrt{Z} = \langle \Omega | \hat{\phi}(0) | S(\boldsymbol{p} = 0) \rangle , \tag{9.3.1}$$

where $|S(\boldsymbol{p} = 0)\rangle$ is a one-particle state at rest and Z the field-strength renormalization constant (or literally the residue of the single-particle pole). Consider a two-point correlation function:

$$C^{(2)}(x, y) = \langle \Omega | \hat{\phi}(x) \hat{\phi}(y) | \Omega \rangle . \tag{9.3.2}$$

Now we insert a complete set of momentum eigenstates between the two field operators:

$$\mathbb{1} = \sum_\lambda \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{q}}} |\lambda_{\boldsymbol{q}}\rangle\langle\lambda_{\boldsymbol{q}}| , \tag{9.3.3}$$

where λ labels all possible states with momentum $\boldsymbol{q}$ from some Fock space basis, which in general means, one-particle states, multi-particle states, excitations. Higher-particle states do contribute extra terms to the two-point function but since they lack poles at the single-particle mass they are irrelevant when we will take the on-shell limit $p^2 \to m^2$. Thus we only need to consider the one-particle states. We can get states with $\boldsymbol{q} \neq 0$ by boosting the state $|S(\boldsymbol{p} = 0)\rangle$:

$$C^{(2)}(x, y) \overset{\text{single-particle only}}{\approx} \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{q}}} \langle \Omega | \hat{\phi}(x) | S(\boldsymbol{q}) \rangle \langle S(\boldsymbol{q}) | \hat{\phi}(y) | \Omega \rangle , \tag{9.3.4}$$

where $E_{\boldsymbol{q}} = \sqrt{\boldsymbol{q}^2 + m_\lambda^2}$. Next we make use of translational invariance to rewrite the field operator as:

$$\hat{\phi}(x) = e^{i\hat{p}\cdot x}\hat{\phi}(0)e^{-i\hat{p}\cdot x} , \tag{9.3.5}$$

where $\hat{p}^\mu$ is the four-momentum operator. This gives us:

$$\langle\Omega|\hat{\phi}(x)|S(\boldsymbol{q})\rangle = e^{-iq\cdot x}\langle\Omega|\hat{\phi}(0)|S(\boldsymbol{q})\rangle . \tag{9.3.6}$$

Now using another Lorentz transformation back to the rest frame of the particle we get:

$$e^{-iq\cdot x}\langle\Omega|\hat{\phi}(0)|S(\boldsymbol{q})\rangle = e^{-iq\cdot x}\langle\Omega|U^{-1}U\hat{\phi}(0)U^{-1}U|S(\boldsymbol{p})\rangle = \langle\Omega|\hat{\phi}(0)|S(\boldsymbol{p}=0)\rangle , \tag{9.3.7}$$

where U is the unitary operator that performs the Lorentz boost. Since $\hat{\phi}$ is a scalar field it is invariant under Lorentz transformations. Our vacuum state is manifestly Lorentz invariant from which it follows that this term is momentum independent, which is a very useful property which allows us to define a single constant $\sqrt{Z}$ for the overlap of the field operator with a one-particle state, regardless of its momentum.

With that said we can rewrite Equation (9.3.4) as:

$$C^{(2)}(x,y) \approx \int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}}}e^{-i\boldsymbol{q}\cdot(x-y)}|\langle\Omega|\hat{\phi}(0)|S(\boldsymbol{p}=0)\rangle|^2 . \tag{9.3.8}$$

We can transform the two-point correlation function from Equation (9.3.8) into momentum space via Fourier transform:

$$\begin{aligned} C^{(2)}(p) &= \int \mathrm{d}^4x\; e^{ip\cdot x}C^{(2)}(x,0) \\ &= \int \mathrm{d}^4x\; e^{ip\cdot x}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2 - m_\lambda^2 + i\varepsilon}e^{-iq\cdot x}|\langle\Omega|\hat{\phi}(0)|S(\boldsymbol{p}=0)\rangle|^2 , \end{aligned} \tag{9.3.9}$$

where we've imposed the on-shell condition $p^2 = m_\lambda^2$ for the single-particle state, such that we can swap the Lorentz-invariant phase space measure by the four-dimensional integral of the propagator:

$$\int \frac{\mathrm{d}\boldsymbol{q}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}}} \sim \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2 - m_\lambda^2 + i\varepsilon}\delta^{(4)}(q^2 - m_\lambda^2) . \tag{9.3.10}$$

Now we perform the integral over d^4x in the above expression:

$$\begin{aligned} C^{(2)}(p) &= \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2 - m_\lambda^2 + i\varepsilon}(2\pi)^4\delta^{(4)}(p-q)|\langle\Omega|\hat{\phi}(0)|S(\boldsymbol{p}=0)\rangle|^2 \\ &= \frac{i}{p^2 - m_\lambda^2 + i\varepsilon}|\langle\Omega|\hat{\phi}(0)|S(\boldsymbol{p}=0)\rangle|^2 . \end{aligned} \tag{9.3.11}$$

From here we can read off that the residue of the single-particle pole is given by:

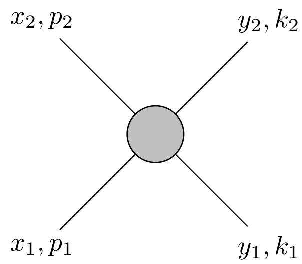
$$\mathrm{Res}(C^{(2)}(p)) = Z = |\langle\Omega|\hat{\phi}(0)|S(\boldsymbol{p}=0)\rangle|^2 , \tag{9.3.12}$$

which was what we wanted to show. This way the two-point correlation function has given us a way to relate the field-strength renormalization constant Z and particle mass m_λ. The LSZ uses exactly this same idea of extracting residues of poles from correlation functions for each external particle. Once we know the pole structure, we can systematically amputate the external legs of the correlation functions and take the on-shell limit to turn Green's functions into S-matrix elements.

9.3.2 The LSZ Theorem

The LSZ theorem named after Lehmann, Symanzik and Zimmermann provides a way to relate S-matrix elements to time-ordered correlation functions. Let us derive the LSZ theorem for an interaction between

4 scalar fields ϕ. Diagramatically this means we have:



(9.3.13)

where the gray blob represents all possible interactions between the four fields. We want to amputate the external legs of this diagram. Mathematically the diagram corresponds to the 4-point correlation function:

$$C^{(4)}(x_1, x_2, y_1, y_2) = \langle\Omega|\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}|\Omega\rangle \,, \tag{9.3.14}$$

which we can transform into momentum space via Fourier transform:

$$\tilde{C}^{(4)}(p_1, p_2, k_1, k_2) = \prod_{\substack{i=1,2 \\ j=1,2}} \int \mathrm{d}^4 x_i e^{ip_i \cdot x_i} \int \mathrm{d}^4 y_j e^{-ik_j \cdot y_j} \langle\Omega|\mathcal{T}\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}|\Omega\rangle \,. \tag{9.3.15}$$

As we've seen in the previous section (see Equation (9.3.11)), each external leg gives us a contribution from the single-particle pole:

$$\int \mathrm{d}^4 x_i e^{ip_i \cdot x_i} \phi(x_i) \sim \frac{i\sqrt{Z}}{p_i^2 - m_R^2 + i\varepsilon} \,. \tag{9.3.16}$$

Thus for all four external legs we have:

$$\frac{i\sqrt{Z}}{p_1^2 - m_R^2 + i\varepsilon} \frac{i\sqrt{Z}}{p_2^2 - m_R^2 + i\varepsilon} \frac{i\sqrt{Z}}{k_1^2 - m_R^2 + i\varepsilon} \frac{i\sqrt{Z}}{k_2^2 - m_R^2 + i\varepsilon} \,. \tag{9.3.17}$$

After amputation, the piece that is left corresponds to the fully interacting amplitude $\langle \boldsymbol{p_1}, \boldsymbol{p_2}|\hat{S}|\boldsymbol{k_1}, \boldsymbol{k_2}\rangle$ which is exactly the S-matrix element between asymptotic states $\boldsymbol{p}_1, \boldsymbol{p_2}$ and $\boldsymbol{k}_1, \boldsymbol{k}_2$. This is because it holds:

$$\overset{\text{Heisenberg Picture}}{{}_{\text{out}}\langle \boldsymbol{p}_1, \boldsymbol{p}_2|\boldsymbol{k}_1, \boldsymbol{k}_2\rangle_{\text{in}}} = \overset{\text{Schrödinger Picture}}{\langle \boldsymbol{p}_1, \boldsymbol{p}_2|\hat{S}|\boldsymbol{k}_1, \boldsymbol{k}_2\rangle} \,, \tag{9.3.18}$$

where $\hat{S} = \lim_{T\to\infty} \exp[-iH(2T)]$ is the S-matrix operator. Remember, the S-matrix is defined as the time-evolution operator in the interaction picture, which evolves states from the past to the future. So fully combining all these pieces we arrive at:

$$\tilde{C}^{(4)}(p_1, p_2, k_1, k_2) \simeq \prod_{i=1,2} \frac{i\sqrt{Z}}{p_i^2 - m_R^2 + i\varepsilon} \prod_{j=1,2} \frac{i\sqrt{Z}}{k_j^2 - m_R^2 + i\varepsilon} \langle \boldsymbol{p_1}, \boldsymbol{p_2}|\hat{S}|\boldsymbol{k_1}, \boldsymbol{k_2}\rangle \,, \tag{9.3.19}$$

where m_R is the renormalized mass of the particle. This is exactly the LSZ theorem for $2 \to 2$ particle scattering. So we've found the LSZ procedure for scalar fields, which deserves its own box.

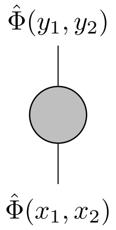
Remember: LSZ Reduction Procedure for Scalar Fields

For any n-point correlation function of scalar fields we can obtain the S-matrix element by:

1. Take the n-point correlation function $C^{(n)}(x_1, x_2, \ldots, y_1, y_2)$.
2. Insert complete sets of momentum eigenstates between the field operators.
3. Transform it into momentum space $\tilde{C}^{(n)}(p_1, p_2, \ldots, k_1, k_2)$.
4. Amputate the external legs by dividing by the single-particle propagators to isolate the S-matrix element.

9.3.3 Example: LSZ Reduction for Two-Particle States

Think back to Equation (9.3.3) where we inserted a complete set of states from the Fock space basis. Instead of only considering one-particle states we can also use multi-particle states. The result is more or less the same, but bookkeeping changes a bit so it might be worth it to go through as an example. So consider the 4-point correlation function where each leg now represents a composite operator that creates a two-particle state:



$$= C^{(4)}(x_1, x_2; y_1, y_2) = \langle \Omega | \mathcal{T}\{\hat{\Phi}(x_1, x_2)\hat{\Phi}^\dagger(y_1, y_2)\} | \Omega \rangle \,, \tag{9.3.20}$$

where $\hat{\Phi}(x_1, x_2) = \hat{\phi}(x_1)\hat{\phi}(x_2)$ is a composite operator that creates two particles at the spacetime points x_1 and x_2. This still represents a 4-point correlation function as each composite operator contains two fields. The Fourier transform of this correlation function is:

$$\tilde{C}^{(4)}(p_1, p_2; k_1, k_2) = \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4y_1\, \mathrm{d}^4y_2\; e^{i(p_1\cdot x_1 + p_2\cdot x_2) - i(k_1\cdot y_1 + k_2\cdot y_2)} C^{(4)}(x_1, x_2; y_1, y_2) \,. \tag{9.3.21}$$

Now we insert a complete set of two-particle states between the field operators:

$$\mathbb{1} = \int \frac{\mathrm{d}\boldsymbol{q}_1}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}\boldsymbol{q}_2}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}_2}} |S(\boldsymbol{q}_1)S(\boldsymbol{q}_2)\rangle\langle S(\boldsymbol{q}_1)S(\boldsymbol{q}_2)| \,, \tag{9.3.22}$$

where $|S(\boldsymbol{q}_1)S(\boldsymbol{q}_2)\rangle$ is a two-particle state with momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. Additionally we make use of translational invariance to rewrite the field operators as:

$$\langle \Omega | \hat{\phi}(x_1)\hat{\phi}(x_2) | S(q_1)S(q_2)\rangle = e^{-i(q_1\cdot x_1 + q_2\cdot x_2)} \langle \Omega | \hat{\phi}(0)\hat{\phi}(0) | S(q_1)S(q_2)\rangle \,. \tag{9.3.23}$$

All combined this gives us:

$$\begin{aligned}
\tilde{C}^{(4)}(p_1, p_2; k_1, k_2) = &\int \mathrm{d}^4x_1 e^{ip_1\cdot x_1} \int \mathrm{d}^4x_2 e^{ip_2\cdot x_2} \int \mathrm{d}^4y_1 e^{-ik_1\cdot y_1} \int \mathrm{d}^4y_2 e^{-ik_2\cdot y_2} \\
&\times \int \frac{\mathrm{d}\boldsymbol{q}_1}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}_1}} \int \frac{\mathrm{d}\boldsymbol{q}_2}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}_2}} e^{-i(q_1\cdot x_1 + q_2\cdot x_2)} \\
&\times \int \frac{\mathrm{d}\boldsymbol{q}'_1}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}'_1}} \int \frac{\mathrm{d}\boldsymbol{q}'_2}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{q}'_2}} e^{i(q'_1\cdot y_1 + q'_2\cdot y_2)} \\
&\times \langle \Omega | \hat{\phi}(0)\hat{\phi}(0) | S(q_1)S(q_2)\rangle\langle S(q_1)S(q_2) | \hat{\phi}(0)\hat{\phi}(0) | \Omega\rangle \,.
\end{aligned} \tag{9.3.24}$$

What is left is to evaluate the coordinate integrals which will impose the on-shell condition $p_i^2 = m_R^2$ for each external leg (see Equation (9.3.10)):

$$\begin{aligned}
= &\int \frac{\mathrm{d}^4q_1}{(2\pi)^4}\frac{i}{q_1^2 - m_R^2 + i\varepsilon}(2\pi)^4\delta^{(4)}(p_1 - q_1) \int \frac{\mathrm{d}^4q_2}{(2\pi)^4}\frac{i}{q_2^2 - m_R^2 + i\varepsilon}(2\pi)^4\delta^{(4)}(p_2 - q_2) \\
&\times \int \frac{\mathrm{d}^4q'_1}{(2\pi)^4}\frac{i}{{q'_1}^2 - m_R^2 + i\varepsilon}(2\pi)^4\delta^{(4)}(k_1 - q'_1) \int \frac{\mathrm{d}^4q'_2}{(2\pi)^4}\frac{i}{{q'_2}^2 - m_R^2 + i\varepsilon}(2\pi)^4\delta^{(4)}(k_2 - q'_2) \\
&\times \langle \Omega | \hat{\phi}(0)\hat{\phi}(0) | S(p_1)S(p_2)\rangle\langle S(p_1)S(p_2) | \hat{\phi}(0)\hat{\phi}(0) | \Omega\rangle \\
= &\frac{i\sqrt{Z}}{p_1^2 - m_R^2 + i\varepsilon}\frac{i\sqrt{Z}}{p_2^2 - m_R^2 + i\varepsilon}\frac{i\sqrt{Z}}{k_1^2 - m_R^2 + i\varepsilon}\frac{i\sqrt{Z}}{k_2^2 - m_R^2 + i\varepsilon} \\
&\times \langle \Omega | \hat{\phi}(0)\hat{\phi}(0) | S(p_1)S(p_2)\rangle\langle S(p_1)S(p_2) | \hat{\phi}(0)\hat{\phi}(0) | \Omega\rangle \,.
\end{aligned} \tag{9.3.25}$$

Near the simultaneous single-particle poles the two-field matrix element factorizes as:

$$\langle \Omega | \hat{\phi}(0)\hat{\phi}(0) | S(p_1)S(p_2)\rangle \simeq \langle \Omega | \hat{\phi}(0) | S(p_1)\rangle\langle \Omega | \hat{\phi}(0) | S(p_2)\rangle + (\text{other pieces}) \simeq \sqrt{Z}\sqrt{Z} \,, \tag{9.3.26}$$

We only want to keep the simultaneous single-particle poles and the other pieces, which leaves us with:

$$\begin{aligned}&\langle\Omega|\hat{\phi}(0)\hat{\phi}(0)|S(p_1)S(p_2)\rangle\langle S(p_1)S(p_2)|\hat{\phi}(0)\hat{\phi}(0)|\Omega\rangle\\&\simeq Z^2(2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)i\mathcal{M}(p_1,p_2\to k_1,k_2)\,,\end{aligned}\tag{9.3.27}$$

where the overall momentum-conserving delta function comes from translation invariance of the full correlator. What remains is the invariant amplitude with which we can now write the S-matrix element:

$$\tilde{C}^{(4)}(p_1,p_2;k_1,k_2)=\frac{i\sqrt{Z}}{p_1^2-m_R^2+i\varepsilon}\frac{i\sqrt{Z}}{p_2^2-m_R^2+i\varepsilon}\langle\boldsymbol{p}_1,\boldsymbol{p}_2|\hat{S}|\boldsymbol{k}_1,\boldsymbol{k}_2\rangle\frac{i\sqrt{Z}}{k_1^2-m_R^2+i\varepsilon}\frac{i\sqrt{Z}}{k_2^2-m_R^2+i\varepsilon}\,.\tag{9.3.28}$$

For identical particles we must sum over all possible permutations of the external legs; we absorb that factor into the invariant amplitude $\mathcal{M}$. That was a hell of a lot of work to reproduce the same result as before, but it is important to see that the LSZ reduction procedure works for any number of particles, not just one-particle states.

Remember: General LSZ-Reduced Correlator

Following the same procedure as in Section 9.3.3 we can write down the general LSZ-reduced correlator for n incoming and m outgoing particles:

$$\begin{aligned}&\tilde{C}^{(n+m)}(p_1,\ldots,p_n;k_1,\ldots,k_m)=\\&=\left(\prod_{i=1}^{n}\frac{i\sqrt{Z}}{p_i^2-m_R^2+i\varepsilon}\right)\langle\boldsymbol{p}_1,\ldots,\boldsymbol{p}_n|\hat{S}|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_m\rangle\left(\prod_{j=1}^{m}\frac{i\sqrt{Z}}{k_j^2-m_R^2+i\varepsilon}\right)\,.\end{aligned}\tag{9.3.29}$$

For identical particles we must sum over all possible permutations of the external legs; we absorb that factor into the invariant amplitude $\mathcal{M}$.

9.4 Feynman Rules for Fermions

Thus far we've only discussed Feynman rules for the ϕ^4 theory. So before we go on to adding fermions to our theory we need to generalize what we learned in Sections (7), (8) and thus far in Section (9) to also apply to fermions. Our perturbative expansion of the n-point correlation function in terms of the interaction Hamiltonian is simple to generalize since Lorentz invariance requires that the interaction Hamiltonian is made of a product of an even number of spinor fields which means we have no difficulty in defining the time-ordered exponential of the interaction Hamiltonian. However to apply Wick's theorem (7.3) we need to generalize the definition of the time-ordering and normal-ordering operators to also include fermions. To stay consistent with how we defined the fermionic propagator (4.6) earlier we define the time-ordering operator as:

$$\mathcal{T}\{\psi(x)\bar{\psi}(y)\}\equiv\begin{cases}\psi(x)\bar{\psi}(y) & \text{if } x^0>y^0\,,\\ -\bar{\psi}(y)\psi(x) & \text{if } x^0<y^0\,.\end{cases}\tag{9.4.1}$$

For products of more than two spinor fields we generalize this definition in the natural way where the time-ordered product picks up a minus sign for each interchange of operators that is necessary to put the fields in time order. Same for the definition of the normal-ordering operator $\mathcal{N}$. Due to the anticommuting nature of fermionic fields it is possible to write a normal-ordered product in several ways, which are essentially equivalent:

$$\mathcal{N}\{a_pa_qa_r^\dagger\}=(-1)^2a_r^\dagger a_pa_q=(-1)^3a_r^\dagger a_qa_p\,.\tag{9.4.2}$$

With that we can now generalize the Wick's theorem to fermions which deserves its own box.

Theorem: Wick's Theorem for m Spinor Fields

Wick's theorem states that the time-ordered product of m spinor fields can be expressed as the fields in the *normal order* plus the sum of all possible contractions:

$$\mathcal{T}\{\psi_1\bar{\psi}_2\psi_3\dots\} = \mathcal{N}\{\psi_1\bar{\psi}_2\psi_3\dots + \text{all possible contractions}\}\,, \tag{9.4.3}$$

where the contraction between two spinor fields is defined as:

$$\wick{\c1\psi(x)\c1{\bar{\psi}}(y)} \equiv \begin{cases} \{\psi^+(x),\, \psi^-(y)\} & \text{if } x^0 > y^0\,, \\ -\{\psi^+(y),\, \psi^-(x)\} & \text{if } x^0 < y^0\,. \end{cases} \tag{9.4.4}$$

$$\wick{\c1\psi(x)\c1\psi(y)} = \wick{\c1{\bar{\psi}}(x)\c1{\bar{\psi}}(y)} \equiv 0\,. \tag{9.4.5}$$

Example: For $m = 4$ spinor fields we have the following:

$$\begin{aligned} \mathcal{T}\{1\,2\,\bar{3}\,\bar{4}\} = \mathcal{N}\{&1\,2\,3\,4 + \wick{\c1 1\,2\,\c1{\bar{3}}\,\bar{4}} + \wick{\c1 1\,2\,\bar{3}\,\c1{\bar{4}}} + \wick{1\,\c1 2\,\c1{\bar{3}}\,\bar{4}} \\ &+ \wick{1\,\c1 2\,\bar{3}\,\c1{\bar{4}}} + \wick{\c1 1\,\c2 2\,\c1{\bar{3}}\,\c2{\bar{4}}} + \wick{\c1 1\,\c2 2\,\c2{\bar{3}}\,\c1{\bar{4}}}\}\,, \end{aligned} \tag{9.4.6}$$

where we've marked ψ_1 with 1 etc. for brevity and clarity. We can expand a contraction to see more clearly what it works. Say for example:

$$\mathcal{N}\{\wick{\c1\psi_1(x_1)\psi_2(x_2)\c1{\bar{\psi}}_3(x_3)}\bar{\psi}_4(x_4)\} = -S_F(x_1 - x_3)^{(1,\,3)}\mathcal{N}\{\psi_2(x_2)\bar{\psi}_4(x_4)\}\,. \tag{9.4.7}$$

9.5 Example: Yukawa Theory/Interactions Between Fermions and Scalars

Now that we have the framework needed to work with fermions and scalars it is quite simple to join the two in a single theory. The simplest way to do this is to add a **Yukawa interaction** term to the Hamiltonian density:

$$\mathcal{H} = \mathcal{H}_{\text{Dirac}} + \mathcal{H}_{\text{Klein-Gordon}} + g\bar{\psi}\psi\phi\,, \tag{9.5.1}$$

for a coupling constant g. This is a simplified way of QED which we will discuss in more detail in the next chapter. So lets work out the rules of calulation in this theory so that we can guess the rules later for QED.

9.5.1 Two Fermion to Two Fermion Scattering

Let's consder a two-particle scattering procees which we can abstract as:

$$\text{fermion}(p,\, s) + \text{fermion}(k,\, r) \to \text{fermion}(p',\, s') + \text{fermion}(k',\, r')\,, \tag{9.5.2}$$

for initial momenta p and k, final momenta p' and k' and initial and final spin indices (polarizations) s, r and s', r'. We want to evaluate the transition amplitude (T-matrix element) for this process as before:

$$\langle f|iT|i\rangle = {}_0\langle|\mathcal{T}\{\exp\left[-ig\int \mathrm{d}^4z\bar{\psi}(z)\psi(z)\phi(z)\right]\}|i\rangle_0\,. \tag{9.5.3}$$

The tree-order Feynman diagram for this process is the 2nd order diagram in the interaction Hamiltonian, which means we have to expand the time-ordered exponential to second order. We do not need to worry about the identity term in the expansion since we're computing the T-matrix element, where the identity term (ie. no interaction) does not contribute like in the S-matrix element. The 1st order expansion term is also not relevant since it does not relate to a physical process, ie. $\phi \to \bar{\psi}\psi$ is not something that can happen. Thus we need to evaluate the second order term in the expansion:

$$
\begin{aligned}
&= f^{(1)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y \ {}_0\langle f|\mathcal{T}\{\overline{\psi}(x)\psi(x)\phi(x)\overline{\psi}(y)\psi(y)\phi(y)\}|i\rangle_0 \\
&= f^{(1)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y \ {}_0\langle f(k',\, r')|\langle f(p',\, s')|\mathcal{N}\{\overline{\psi}_x\psi_x\phi_x\overline{\psi}_y\psi_y\phi_y\}|f(p,\, s)\rangle|f(k,\, r)\rangle_0 \,, \qquad (9.5.4)
\end{aligned}
$$

where we have no extermal scalar particles, which means we need to contract the scalar field with itself. The constant $f^{(1)} = 2$ since we have two indistinguishable particles and we could feasibly do $x \leftrightarrow y$ and get the same Feynman diagram. We'd need to calculate the contractions on the initial and final states, for example:

$$
\begin{aligned}
\psi|p,\, s\rangle \equiv \psi^-|p,\, s\rangle &= \int \frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'} a_{p'}^{s'}u^{s'}(p')e^{-ip'\cdot x}\sqrt{2E_p}a_p^{s\dagger}|0\rangle \,, \\
&= \int \frac{\mathrm{d}^3p'}{(2\pi)^3\sqrt{2E_{p'}}}\sum_{s'} u^{s'}(p')(2\pi)^3\delta^{(3)}(p'-p)\delta_{s,s'}e^{-ip'\cdot x}\sqrt{2E_p}|0\rangle \,, \\
&= e^{-ip'\cdot x}u^s(p)|0\rangle \,, \qquad (9.5.5)
\end{aligned}
$$

where ψ^- represents the **positive** frequency part (ie. where the annihilation operator is) of the spinor field ψ. What we got is percisely the external leg factor in the Feynman rules for an incoming fermion. Likewise we could evaluate a simmilar contraction and get the following for the external leg factor of an outgoing fermion:

$$
\langle p,\, s|\overline{\psi} \equiv \langle p,\, s|\overline{\psi}^+ = \cdots = e^{ip\cdot x}\overline{u}^s(p) \,. \qquad (9.5.6)
$$

With this knowledge we can evaluate the contractions in Equation (9.5.4) along with the definition of the scalar propagator (2.11.3) to get the following expression:

$$
\begin{aligned}
\langle f|iT|i\rangle = \frac{p_W^{(1)}f^{(1)}(-ig)^2}{2!}&\int \mathrm{d}^4x \int \mathrm{d}^4y \ \overline{u}_x^{r'}(k')e^{ik'\cdot x}u_x^r(k)e^{-ik\cdot x}\overline{u}_y^{s'}(p')e^{up'\cdot y}u_y^s(p)e^{-ip\cdot y} \\
&\times \int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{i}{q^2-m_\phi^2+i\varepsilon}e^{-iq\cdot(x-y)} \,. \qquad (9.5.7)
\end{aligned}
$$

We can easily perform the integrals over x, y and the undetermined momentum q as:

$$
\begin{aligned}
&\int \frac{\mathrm{d}^4q}{(2\pi)^4}\int \mathrm{d}^4x \ e^{-ix\cdot(k-k'+q)}\int \mathrm{d}^4y \ e^{-iy\cdot(p-p'-q)} \\
&= \int \frac{\mathrm{d}^4q}{(2\pi)^4} \ (2\pi)^4\delta^{(4)}(k-k'+q)(2\pi)^4\delta^{(4)}(p-p'+q) \\
&= (2\pi)^4\delta^{(4)}(k-k'+p-p') \,, \qquad (9.5.8)
\end{aligned}
$$

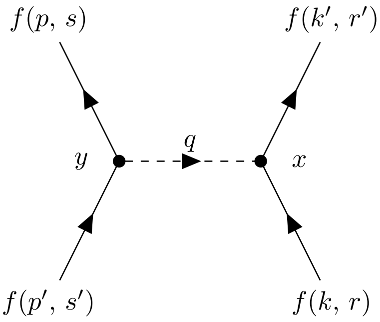
which we see gives us the momentum conservation delta function. We need to determine the factor $p_W^{(1)}$ which gives us the sign which we get from the anticommuting nature of the spinor fields while doing Wick contractions. We need to get the operators into normal order before we can do the contractions. Each swap of two spinor fields gives us a minus sign. Swaps for scalar fields do not change anything since scalars commute. We could rewrite the expression in Equation (9.5.4) as:

$$
\begin{aligned}
\langle f|iT|i\rangle &= f^{(1)}p_W^{(1)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y \ \langle 0|a_{k'}^{r'}b_{p'}^{s'}\mathcal{N}\{\overline{\psi}_x\psi_x\phi_x\overline{\psi}_y\psi_y\phi_y\}b_p^{s\dagger}a_k^{r\dagger}|0\rangle \,, \\
&= f^{(1)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y \ (-1)(-1)(-1)^2\langle 0|a_{k'}^{r'}\overline{\psi}_x b_{p'}^{s'}\overline{\psi}_y\phi_x\phi_y\psi_y b_p^{s\dagger}\psi_x a_k^{r\dagger}|0\rangle \,. \qquad (9.5.9)
\end{aligned}
$$

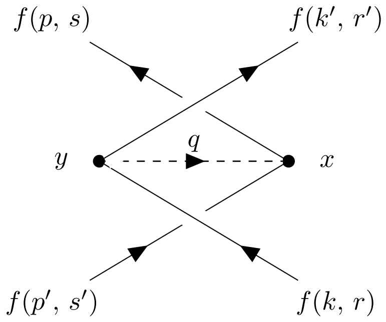
We can see that we needed to perform an even number of swaps to get the fields into normal order, which gives us a factor of $p_W^{(1)} = (-1)(-1)(-1)^2 = 1$. From this we can finally read the scattering amplitude $i\mathcal{M}$ from the T-matrix element as we defined it in Equation (9.1.12):

$$
i\mathcal{M}^{(1)} = \frac{p_W^{(1)}}{s^{(1)}}\overline{u}^{r'}(k')(-ig\,\mathbb{1})u^r(k)\overline{u}^{s'}(p')(-ig\,\mathbb{1})u^s(p)\frac{i}{(p-p')^2-m_\phi^2} \,, \qquad (9.5.10)
$$

where $1/s^{(1)} = f^{(1)}/n!$ is the symmetry factor for the process, which in our case is 1 since $f^{(1)} = 2$ as we have fields that cannot substitute for each other in contractions and $n! = 2! = 2$, as we have two particles in the final state. Thus the amplitude (9.5.10) represents the following two Feynman diagrams:



or if $x \leftrightarrow y$



. (9.5.11)

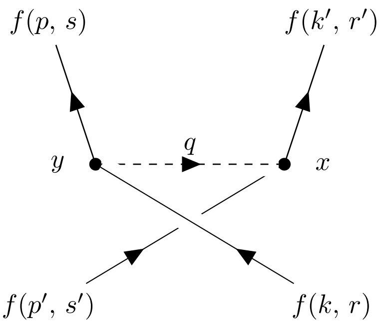
We're not done yet however as there is another possible way we can do contractions in the expansion from Equation (9.5.4):

$$
\begin{aligned}
&= f^{(2)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y \;{}_0\langle f|\mathcal{T}\{\bar{\psi}(x)\psi(x)\phi(x)\bar{\psi}(y)\psi(y)\phi(y)\}|i\rangle_0 \\
&= f^{(2)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y \;{}_0\langle f(k',\, r')|\langle f(p',\, s')|\mathcal{N}\{\bar{\psi}_x\psi_x\phi_x\bar{\psi}_y\psi_y\phi_y\}|f(p,\, s)\rangle|f(k,\, r)\rangle_0 \\
&= f^{(2)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y \;(-1)(-1)(-1)\langle 0|a^{r'}_{k'}\bar{\psi}_x b^{s'}_{p'}\bar{\psi}_y\phi_x\phi_y\psi_x b^{s\,\dagger}_p\psi_y a^{r\,\dagger}_k|0\rangle\,,
\end{aligned}
\tag{9.5.12}
$$

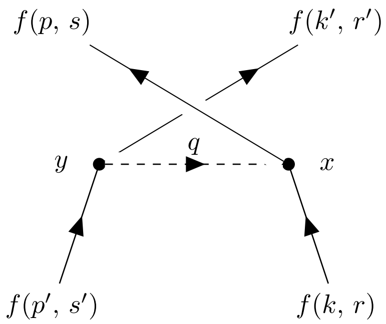
where we can see that in this case we had to perform an odd number of swaps to get the fields into normal order, which gives us a factor of $p_W^{(2)} = (-1)(-1)(-1) = -1$. Which we could continue to evaluate this expression as analagously as before to get the other scattering amplitude:

$$
i\mathcal{M}^{(2)} = \frac{p_W^{(2)} f^{(2)}}{2!}\bar{u}^{r'}(k')(-ig\,\mathbb{1})u^s(p)\bar{u}^{s'}(p')(-ig\,\mathbb{1})u^r(k)\frac{i}{(p'-k)^2 - m_\phi^2}\,, \tag{9.5.13}
$$

where $1/s^{(2)} = 1$ again with the same arguments as before. This amplitude represents the following two Feynman diagrams:



or if $x \leftrightarrow y$



. (9.5.14)

The total scattering amplitude is then the sum of the two amplitudes. The set of diagrams it represents depends on which order for x and y we choose.

9.5.2 Fermion, Anti-fermion Annihilation to Pair of Bosons

As an exercise we can also consider a 2nd order process where a pair of fermion and anti-fermion annihilate to produce a pair of bosons. We can abstract this process as:

$$
\text{fermion}(p,\, s) + \overline{\text{fermion}}(p',\, s') \to \text{boson}(k) + \text{boson}(k)\,, \tag{9.5.15}
$$

where we have initial momenta p and p' and final momenta k and k'. Notice that bosons of course have no spin indices. As before we want to evaluate the transition amplitude (T-matrix element) like in Equation

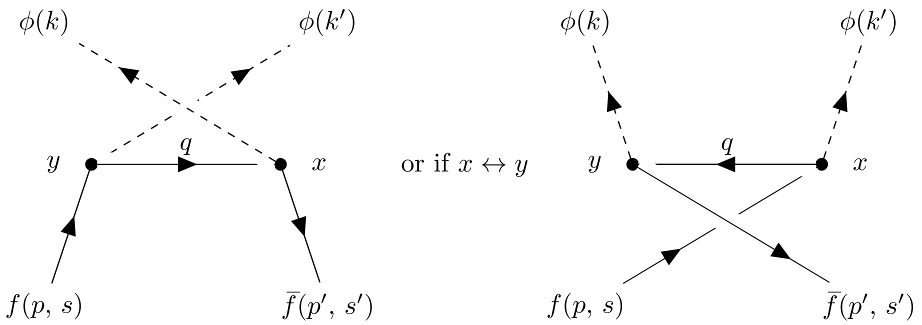
(9.5.3). Similarly to the previous example we need to expand the time-ordered exponential to second order as such:

$$\begin{aligned}\langle f|iT|i\rangle &= f^{(0)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x\int \mathrm{d}^4y\ {}_0\langle \phi(k)\phi(k')|\mathcal{T}\{\bar{\psi}_x\psi_x\phi_x\bar{\psi}_y\psi_y\phi_y\}|f(p,\,s)\bar{f}(p',\,s')\rangle_0 \\ &= f^{(0)}\frac{(-ig)^2}{2!}\int \mathrm{d}^4x\int \mathrm{d}^4y\ {}_0\langle \phi(k)\phi(k')|\mathcal{N}\{\bar{\psi}_x\psi_x\phi_x\bar{\psi}_y\psi_y\phi_y\}|f(p,\,s)\bar{f}(p',\,s')\rangle_0\end{aligned} \tag{9.5.16}$$

where we have fermions in the out state which means that we need to contract the fermion fields amongst themselves. We could now evaluate the contractions as we did before but we can also read the contractions from the Feynman rules (see Appendix **??**), which gives us the following:

$$i\mathcal{M} = \frac{p_W^{(0)}}{s^{(0)}}\bar{v}^{s'}(p')(-ig\,\mathbb{1})\frac{i(\not{q}+m_\psi)}{q^2-m_\psi^2+i\varepsilon}u^s(p)\,, \tag{9.5.17}$$

where we could figure out that $p_W^{(0)} = 1$ and $s^{(0)} = 2/2! = 1$ since we have two fermions in the initial state and they cannot be interchanged and we can swap $x \leftrightarrow y$. Remember here that external leg contractions for scalar fields simply yield 1. This amplitude represents the following Feynman diagrams:



. (9.5.18)

10 Quantum Electrodynamics

10.1 From Yukawa Theory to QED

To transition from Yukawa theory to Quantume Electrodynamics (QED), we need to replace the scalar particle ϕ with a vector particle A_μ and replace the Yukawa interaction Hamiltonian with the electromagnetic interaction Hamiltonian:

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = -e\bar{\psi}_\alpha\gamma^\mu_{\alpha\beta}\psi_\beta A_\mu\,, \tag{10.1.1}$$

where $e = -e_0$ is the electron charge in the case of the electron field. This term will appear naturally in the Lagrangian of QED as we will see.

Remember: Spinor Indices

Don't forget that the spinor indices α, β are contracted with the Dirac matrices $\gamma^\mu_{\alpha\beta}$. $\bar{\psi}_\alpha$ is mathematically a vector and as such has additional indices. Since this is more or less evident, I will be dropping the spinor indices in the rest of this section, but please keep in mind that they are there otherwise contractions would not make sense.

We've introduced the 4-potential of the electromagnetic field, A_μ, which is a vector field. It and the electromagnetic field strength tensor $F_{\mu\nu}$ are defined as:

$$A = (\phi, -\boldsymbol{A})\,, \tag{10.1.2}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{10.1.3}$$
$$F^{0i} = E^i\,, \tag{10.1.4}$$
$$F^{ij} = -\frac{1}{2}\varepsilon^{ijk}B_k\,, \tag{10.1.5}$$
$$\partial_\mu F^{\mu\nu} = 0\,, \tag{10.1.6}$$

where the last equation is the source-free Maxwell equations ie. the equations of motion for the electromagnetic field. Since QED is a gauge theory, we need to introduce the **covariant derivative**:

$$D_\mu = \partial_\mu + ieA_\mu(x) \,, \tag{10.1.7}$$

as a way to ensure gauge invariance under local $U(1)$ transformations. Where of course, the transformations occur as:

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \,, \tag{10.1.8}$$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \,. \tag{10.1.9}$$

From that, we can write the Lagrangian of QED as:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,, \tag{10.1.10}$$

where if we expand the covariant derivative, we get:

$$\mathcal{L}_{\text{QED}} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - e\bar{\psi}\gamma^\mu A_\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,, \tag{10.1.11}$$

where we can explicitly see in the third term the interaction between the fermionic field ψ and the vector field A_μ.

10.2 Gauge Fixing and Photon Polarization in QED

As we've just discussed, QED is a gauge theory, which means we have too many degrees of freedom in our vector field A_μ. That is to say that not all components of A_μ are physical, some are redundant exactly due to gauge invariance. To account for this redundancy we need to fix the gauge with a gauge fixing term, which imposes a condition on the vector field A_μ.

10.2.1 Lorenz Gauge

A very common gauge fixing condition is the Lorenz gauge, which is defined as:

$$\partial_\mu A^\mu = 0 \,. \tag{10.2.1}$$

Lets consider just the EM field part of the Lagrangian for now, ignoring all matter fields:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,. \tag{10.2.2}$$

By imposing the Lorenz gauge condition on the equations of motion for the EM field (10.1.6) we get the following new equation of motion:

$$\partial_\mu F^{\mu\nu} = \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu \overset{\partial_\mu A^\mu = 0}{=} \partial_\mu\partial^\mu A^\nu = \Box A^\nu = 0 \,, \tag{10.2.3}$$

which we can do since partial derivatives commute (at least in flat spacetime, but that is beyond the scope of this discussion) and $\Box$ is the d'Alembert operator. Lets try and solve this wave equation by assuming a plane wave solution ansatz as:

$$A^\mu(x) = \varepsilon^\mu(k)e^{-ik\cdot x} \,, \tag{10.2.4}$$

where $\varepsilon^\mu(k)$ is the polarization vector of the wave. With this choice we see that $k^2 = 0$, so this represents a massless field. The Lorenz gauge condition (10.2.1) then gives us the constraint on the polarization vector:

$$\partial^\mu A_\mu = 0 \Rightarrow k_\mu\varepsilon^\mu(k) = 0 \,, \tag{10.2.5}$$

which essentially means that the polarization vector must be orthogonal to the wave vector k^μ/momentum 4-vector of the wave. Since we imposed a single constraint on a 4-vector, we have 3 degrees of freedom left, but we still have degree of gauge freedom even with the Lorenz gauge condition. To see this concretely, let's imagine that we have a photon traveling in the z direction. The wave vector of such a photon is then:

$$k^\mu = (|\boldsymbol{k}|,\, 0,\, 0,\, |\boldsymbol{k}|)\ , \tag{10.2.6}$$

and since the polarization vector must be orthogonal to the wave vector, we get the condition:

$$k^0\varepsilon^0 - k^3\varepsilon^3 = 0 \quad \Rightarrow \quad \varepsilon^0 = \varepsilon^3\,. \tag{10.2.7}$$

This gives us a general polatization vector of the form:

$$\varepsilon^\mu = \left(\varepsilon^0,\, \varepsilon^1,\, \varepsilon^2,\, \varepsilon^0\right)\ , \tag{10.2.8}$$

which we can easily see has 3 degrees of freedom. $\varepsilon^1, \varepsilon^2$ are the transverse polarizations, while $\varepsilon^0 = \varepsilon^3$ is a combination of the time-like and longitudinal polarizations. If we wanted to fix this explicitly we can also get rid of the gauge freedom that we still have under the Lorenz gauge condition by using a gauge transformation of the form:

$$A^\mu \to A^\mu - \frac{1}{e}\partial_\mu\alpha(x)\,, \tag{10.2.9}$$

$$\varepsilon_\mu e^{-ik\cdot x} \to \varepsilon_\mu e^{-ik\cdot x} - \frac{1}{e}C(-ik_\mu)e^{-ik\cdot x}\,, \tag{10.2.10}$$

where C is a constant. We will label $D = \frac{-i}{e}C$ such that the transformation resembles the form $\varepsilon_\mu \to \varepsilon_\mu + Dk_\mu$. This essentially allows us to shift the polatization vector by a constant along the direction of the wave vector, so we can choose such a D that $\varepsilon^0 = \varepsilon^3 = 0$, which gives us the transverse polarizations only:

$$\varepsilon^\mu \to \varepsilon^\mu + D(\omega,\, 0,\, 0,\, \omega) \quad \Rightarrow \quad \varepsilon^\mu = \left(0,\, \varepsilon^1,\, \varepsilon^2,\, 0\right)\ . \tag{10.2.11}$$

10.2.2 Coulomb Gauge

Another common gauge fixing condition is the Coulomb gauge, which is defined as:

$$\nabla\cdot\boldsymbol{A} = 0\,. \tag{10.2.12}$$

This would lead us to essentially the same results where we eliminate all longitudinal polarizations and are left with only transverse polarizations, with the added caveat that the Coulomb gauge condition is not Lorentz invariant, so it is not a good choice for relativistic theories.

10.2.3 Naive Canonical Quantization

With the gauge fixing condition in place, we can now promote our fields to operators and quantize the theory. We can naively try to canonically quantize the theory. Let's define the **conjugate momenta** of the fields as:

$$\Pi_\nu \equiv \frac{\partial\mathcal{L}}{\partial_\nu(\partial_0 A^\nu)}\,. \tag{10.2.13}$$

With such a definition and with the Lorenz gauge condition (10.2.1) (and transformation (10.2.10)) we can quickly see that:

$$\Pi_i = \frac{\partial\mathcal{L}}{\partial(\dot{A}_i)} \neq 0\,, \tag{10.2.14}$$

$$\Pi_0 = \frac{\partial\mathcal{L}}{\partial(\dot{A}_0)} = 0\,, \tag{10.2.15}$$

which means that with our choices of gauge fixing condition and transformation, A_0 has no dynamics and is not a true degree of freedom but rather a constraint. This means that canonical quantization does not work for gauge theories like QED, since we cannot use such a constraint. One can find rigorous ways to quantize the EM field for example in Chapter 9.4 of Peskin but we will not go into that here. For those interested, the subject *Gauge Field Theory* goes into details about how to quantize gauge theories like QED using via Path Integral quantization, the addition of Faddeev-Popov ghosts and BRST formalism etc.

10.2.4 Feynman Gauge

We can somewhat fix this problem with $\Pi_0 = 0$ by introducing a new gauge fixing condition called the Feynman gauge, which stems from the Lorenz gauge condition. More specifically it is the R_ξ gauge fixing condition with $\xi = 1$. This means that we add the following gauge fixing term to our Lagrangian:

$$\mathcal{L}_{\text{gf}} = \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \overset{\text{Feynman}}{=} \frac{1}{2}(\partial_\mu A^\mu)^2 \, . \tag{10.2.16}$$

With this term added to the Lagrangian, our definition of the conjugate momenta becomes consistent:

$$\Pi_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\nu)} \neq 0 \quad \forall \nu \in \{0, 1, 2, 3\} \, , \tag{10.2.17}$$

and our equations of motions and solutions remain as before in the case of the Lorenz gauge condition. So:

$$\partial_\mu \partial^\mu A^\nu = \Box A^\nu = 0 \, , \tag{10.2.18}$$

$$A^\mu(x) = \varepsilon^\mu(k) e^{-ik\cdot x} \, . \tag{10.2.19}$$

This method is not without its own problems however. Most are going to be fixed down the line but I thought it was worth mentioning that while the Feynman gauge condition allows us to canonically quantize the theory, is still allows issues to lurk in the theory. For example, we still have time-like and longitudinal polarizations which are unphysical degrees of freedom and more scarily we have negative norm states if we were to try and naively construct a Hilbert space. We can still interpret the theory physically by imposing constraints on physical states (for example, Gupta-Bleuler formalism), but we will use a different trick in upcoming sections. Still in general it is better to quantize QED using path integrals.

10.3 The Ward Identity

One method of fixing the issues with unphysical states in QED when using the Feynman gauge and canonical quantization is to use the Ward identity. It arises from the gauge invariance of the theory and more widely from the Ward-Takahashi identity. It could be described as a specialization of the Ward-Takahashi identity to S-matrix elements and of course it deserves its own box.

Theorem: The Ward Identity

Let's define the invariant matrix element $\mathcal{M}$ as:

$$\mathcal{M}(k) = \varepsilon_\mu(k)\mathcal{M}^\mu(k) \, , \tag{10.3.1}$$

where $\varepsilon_\mu(k)$ is the polarization vector of the photon and $\mathcal{M}^\mu(k)$ is the invariant amplitude without dependence on the polarization vector. Then the Ward identity states that for on shell/real photons:

$$k_\mu \mathcal{M}^\mu(k) = 0 \, . \tag{10.3.2}$$

This means that the invariant amplitude $\mathcal{M}^\mu(k)$ is orthogonal to the wave vector k^μ of the photon. Practically this means that longitudinal polarizations, along with time-like polarizations (because $\varepsilon^0 = \varepsilon^3$) that arise from the Feynman gauge are unphysical and do not contribute to the invariant amplitude $\mathcal{M}^\mu(k)$, only transverse polarizations do.

10.4 EM Field Operator and Commutation Relations

Now that we've somewhat successfully quantized the theory, we can define the EM field operator which is important enough to warrant its own box.

Remember: EM Field Operator and Commutation Relations

The EM field operator $A_\nu(x)$ is defined as:

$$A_\nu(x) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3\sqrt{2E_k}} \sum_{r=0}^{3} \left[\varepsilon_\nu^r(k)e^{-ik\cdot x}a_{\boldsymbol{k}}^r + \varepsilon_\nu^{r\,*}(k)e^{ik\cdot x}a_{\boldsymbol{k}}^{r\,\dagger}\right] = A_\nu^\dagger(x)\,, \tag{10.4.1}$$

where we notice a very important property of the EM field operator - it is Hermitian, ie. self-adjoint. We've summed over all polarizations (one time-like, two transverse and one longitudinal) as it is necessary to do so in intermediate steps. The Ward identity guarantees that **only the transverse polarizations contribute to physical amplitudes**, so we can ignore the time-like and longitudinal modes in the final result. The EM field operator satisfies the free wave equation (its equation of motion):

$$\partial_\mu\partial^\mu A_\nu(x) = 0\,, \tag{10.4.2}$$

which is the same as the equation of motion for the EM field that we derived earlier, due to our choice of gauge fixing condition. This gives us Maxwell's equations in the absence of sources. We can postulate the following canonical commutation relations:

$$[a_k^r,\, a_{k'}^{r'\,\dagger}] = (2\pi)^3\delta^{(3)}(k-k')\delta_{rr'}\,, \tag{10.4.3}$$

$$[a_k^r,\, a_{k'}^{r'}] = [a_{k'}^{r'\,\dagger},\, a_k^{r\,\dagger}] = 0\,, \tag{10.4.4}$$

where a_k^r and $a_k^{r\,\dagger}$ are the annihilation and creation operators for the EM field, respectively. In the case of the above commutation relations, $r = 0$ would yield to big problems, ie. negative norm states which break unitarity and the probabilistic interpretation of the theory, but we can again ignore it due to the Ward identity.

10.5 EM Field Propagator

Like other fields, the EM field has a propagator which is defined as the vacuum expectation value of the time-ordered product of the field operators:

$$D_{\mu\nu}(x-y) = \overbrace{A_\mu(x)A_\nu}(y) = \langle 0|\mathcal{T}\big\{A_\mu(x)A_\nu(y)\big\}|0\rangle\,. \tag{10.5.1}$$

Using the definition of the EM field operator (10.4) we can compute the propagator in momentum space as:

$$\begin{aligned} D_{\mu\nu}(k) &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i\sum_{r=0}^3 \varepsilon_\mu^r(k)\varepsilon_\nu^r(k)}{k^2+i\varepsilon}e^{-ik\cdot(x-y)} \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}e^{-ik\cdot(x-y)}\,, \end{aligned} \tag{10.5.2}$$

where we used the identity/completeness relation:

$$\sum_{r=0}^{3}\varepsilon_\mu^r(k)\varepsilon_\nu^r(k) = -g_{\mu\nu}\,, \tag{10.5.3}$$

This is analogous to how for 3D space and an orthonormal basis $\big\{e_i\big\}$ we have the completeness relation:

$$\sum_{i=1}^{3} e_i^j e_i^k = \delta^{jk}\,, \tag{10.5.4}$$

where e_i^j is the j-th component of the i-th basis vector. We can also quickly sketch out the proof that such a propagator is really the Green's function of Maxwell's equations:

$$\partial_\sigma\partial^\sigma D_{\mu\nu}(x-y) \propto \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}(-ik)^2}{k^2+i\varepsilon}e^{-ik\cdot(x-y)} \propto \delta^{(4)}(x-y)\,. \tag{10.5.5}$$

For posterity, the propagator deserves its own box as well.

Remember: EM Field Propagator

The EM field propagator is defined as:

$$D_{\mu\nu}(x-y) = \contraction{}{A}{_\mu(x)}{A}A_\mu(x)A_\nu(y) = \langle 0|\mathcal{T}\{A_\mu(x)A_\nu(y)\}|0\rangle \,, \tag{10.5.6}$$

In momentum space it is given by:

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2+i\varepsilon} \,, \tag{10.5.7}$$

while in coordinate space it is given by the Fourier transform of the above expression:

$$D_{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2+i\varepsilon}e^{-ik\cdot(x-y)} \,, \tag{10.5.8}$$

where $g_{\mu\nu}$ is the metric tensor of Minkowski spacetime.

10.6 Feynman Rule for External Photon Lines

Now that we've implemented all this formalism, we can also easily compute the contraction of the EM field operator with external photons. Lets say we have an external photon with momentum k and polarization r. Just like we calculated the contraction of the scalar field operator with external scalar lines in Yukawa theory, we can do the same for the EM field operator:

$$\begin{aligned}
\contraction{}{A}{^\mu(x)|}{\gamma}A^\mu(x)|\gamma(k,r)\rangle \equiv A^{\mu -}(x)|\gamma(k,r)\rangle &= \int \frac{\mathrm{d}^3k'}{(2\pi)^3\sqrt{2E_{\boldsymbol{k'}}}}\sum_{r'=0}^{3}\varepsilon_\mu^{r'}(\boldsymbol{k'})a_{\boldsymbol{k'}}^{r'}e^{-i\boldsymbol{k'}\cdot\boldsymbol{x}}|\gamma(k,r)\rangle \\
&= \int \frac{\mathrm{d}^3k'}{(2\pi)^3\sqrt{2E_{\boldsymbol{k'}}}}\sum_{r'=0}^{3}\varepsilon_\mu^{r'}(\boldsymbol{k'})a_{\boldsymbol{k'}}^{r'}e^{-i\boldsymbol{k'}\cdot\boldsymbol{x}}\sqrt{2E_{\boldsymbol{k}}}a_{\boldsymbol{k}}^{r\,\dagger}|0\rangle \\
&= \int \frac{\mathrm{d}^3k'}{(2\pi)^3\sqrt{2E_{\boldsymbol{k'}}}}\sum_{r'=0}^{3}\varepsilon_\mu^{r'}(\boldsymbol{k'})(2\pi)^3\delta^{(3)}(\boldsymbol{k'}-\boldsymbol{k})\delta_{r,r'}e^{-i\boldsymbol{k'}\cdot\boldsymbol{x}}\sqrt{2E_{\boldsymbol{k}}}|0\rangle \\
&= \varepsilon_\mu^r(k)e^{-ik\cdot x}|0\rangle \,,
\end{aligned} \tag{10.6.1}$$

where we used the fact that the single photon state $|\gamma(k,r)\rangle = \sqrt{2E_{\boldsymbol{k}}}a_{\boldsymbol{k}}^{r\,\dagger}|0\rangle$ and the **positive** frequency part of the EM field operator as we've defined in (10.4). The same process can also be done to compute the outgoing photon state contraction:

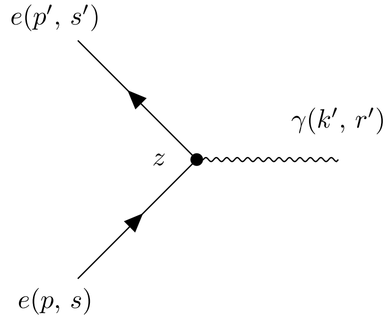
$$\begin{aligned}
\langle\contraction{}{\gamma}{(k,r)|}{A}\gamma(k,r)|A^\mu(x) \equiv \langle\gamma(k,r)|A^{\mu +}(x) &= \langle\gamma(k,r)|\int \frac{\mathrm{d}^3k'}{(2\pi)^3\sqrt{2E_{\boldsymbol{k'}}}}\sum_{r'=0}^{3}\varepsilon_\mu^{r'\,*}(\boldsymbol{k'})a_{\boldsymbol{k'}}^{r'\,\dagger}e^{i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&= \langle 0|\sqrt{2E_{\boldsymbol{k}}}a_{\boldsymbol{k}}^{r}\int \frac{\mathrm{d}^3k'}{(2\pi)^3\sqrt{2E_{\boldsymbol{k'}}}}\sum_{r'=0}^{3}\varepsilon_\mu^{r'\,*}(\boldsymbol{k'})a_{\boldsymbol{k'}}^{r'\,\dagger}e^{i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&= \langle 0|\sqrt{2E_{\boldsymbol{k}}}\int \frac{\mathrm{d}^3k'}{(2\pi)^3\sqrt{2E_{\boldsymbol{k'}}}}\sum_{r'=0}^{3}\varepsilon_\mu^{r'\,*}(\boldsymbol{k'})(2\pi)^3\delta^{(3)}(\boldsymbol{k'}-\boldsymbol{k})\delta_{r,r'}e^{i\boldsymbol{k'}\cdot\boldsymbol{x}} \\
&= \langle 0|\varepsilon_\mu^r(k)^*e^{ik\cdot x} \,.
\end{aligned} \tag{10.6.2}$$

Just as an additional note for the reader. We used the EM field commutation relations the get the delta functions in the above contractions. Like such:

$$a_{\boldsymbol{k}}^{r}a_{\boldsymbol{k'}}^{r'\,\dagger}|0\rangle = \left[(2\pi)^3\delta^{(3)}(\boldsymbol{k'}-\boldsymbol{k})\delta_{r,r'} + a_{\boldsymbol{k'}}^{r'\,\dagger}a_{\boldsymbol{k}}^{r}\right]|0\rangle \overset{a_{\boldsymbol{k}}^r|0\rangle=0}{=} (2\pi)^3\delta^{(3)}(\boldsymbol{k'}-\boldsymbol{k})\delta_{r,r'}|0\rangle \,. \tag{10.6.3}$$

10.7 Feynman Rule for QED Vertex

The tree-order Feynman diagram for the QED vertex is given by the 1st order diagram in the interaction Hamiltonian, which means we have to expand the time-ordered exponential to the first order. As with Yukawa theory, we do not need to worry about the identity term (ie. no interaction) since it does not contribute to the S-matrix element, which describes only interactions. The vertex is given diagrammatically as:



$$= {}_0\langle e(p',\, s')\gamma(k',\, r')| - i\mathcal{H}_{\text{int}}|e(p,\, s)\rangle_0 \;, \tag{10.7.1}$$

thus we need to evaluate the following:

$$\begin{aligned}
&{}_0\langle e(p',\, s')\gamma(k',\, r')| - i\mathcal{H}_{\text{int}}|e(p,\, s)\rangle_0 = \\
&= \frac{fp_W}{n!}(-ie)\int \mathrm{d}^4 z \;\; {}_0\langle e(p',\, s')\gamma(k',\, r')|\mathcal{N}\{\bar{\psi}(z)\gamma^\mu\psi(z)A_\mu(z)\}|e(p,\, s)\rangle_0 = \\
&= (-ie\gamma^\mu)\overline{u}^{s'}(p')u^s p\varepsilon_\mu^{r'\,*}(k') \;,
\end{aligned} \tag{10.7.2}$$

where we took into account that $\frac{fp_W}{n!} = 1$. If we remove the external contractions, we get the contribution of the QED vertex to the S-matrix element:

$$\mathcal{M}^\mu_{\text{Vertex}} = -ie\gamma^\mu \;, \tag{10.7.3}$$

which is the Feynman rule for the QED vertex.

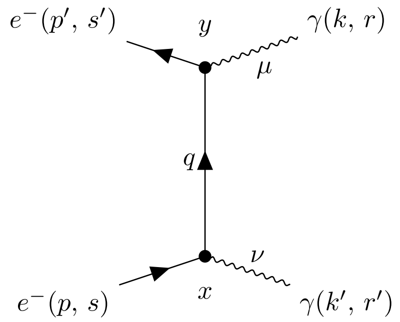
10.8 Assorted Scattering Examples

10.8.1 Compton Scattering

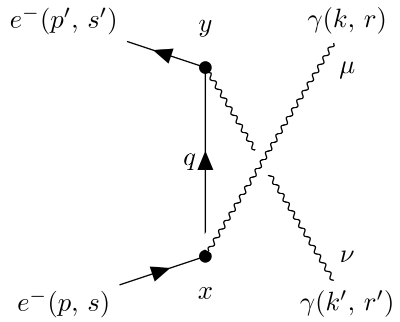
Say we want to compute the scattering amplitude for non-polarized Compton scattering, where a photon scatters of an electron. The process is given by:

$$e^-(p,\, s) + \gamma(k,\, r) \to e^-(p',\, s') + \gamma(k',\, r') \;, \tag{10.8.1}$$

which we can represent diagrammatically as:



or if $x \leftrightarrow y$



. (10.8.2)

Using Feynman rules we can determine that the scattering amplitude for this process is given by:

$$\begin{aligned}
i\mathcal{M} &= (-ie)^2\overline{u}^{s'}(p')\gamma^\mu\varepsilon_\mu^{r'\,*}(k')\frac{i(\not{q}_1 + m\mathbb{1})}{q_1^2 - m^2 + i\varepsilon}\varepsilon_\nu^r(k)\gamma^\nu u^s(p) \\
&+ (-ie)^2\overline{u}^{s'}(p')\gamma^\nu\varepsilon_\nu^r(k)\frac{i(\not{q}_2 + m\mathbb{1})}{q_2^2 - m^2 + i\varepsilon}\varepsilon_\mu^{r'\,*}(k')\gamma^\mu u^s(p) \;,
\end{aligned} \tag{10.8.3}$$

where α, β are spinor indices, $s = 1$ and $p_W = 1$. To make averaging over polarizations easier, lets define a tensor-valued **vertex kernel** $\Gamma^{\mu\nu}_{\alpha\beta}$ which will hold the full spinor and Lorentz structure in our scattering amplitude.

$$\Gamma^{\mu\nu}_{\alpha\beta} = \left[\left(\gamma^\mu \frac{\not{q}_1 + m}{q_1^2 - m^2 + i\varepsilon} \gamma^\nu \right)_{\alpha\beta} + \left(\gamma^\nu \frac{\not{q}_2 + m}{q_2^2 - m^2 + i\varepsilon} \gamma^\mu \right)_{\alpha\beta} \right] , \tag{10.8.4}$$

where $q_1 = p + k$ is the s-channel internal momentum and $q_2 = p - k'$ is the u-channel internal momentum. More about the so-called **Mandelstam variables** can be found in the Appendix (**??**). Using this vertex kernel, we can rewrite the scattering amplitude as:

$$i\mathcal{M} = -ie^2 \varepsilon^r_\nu(k) \varepsilon^{r'*}_\mu(k') \bar{u}^{s'}_\alpha(p') \Gamma^{\mu\nu}_{\alpha\beta} u^s_\beta(p) \tag{10.8.5}$$

To properly average over the polarizations of the external photons we need to calculate $|\mathcal{M}|^2$, which means we need to find the complex conjuugate of the scattering amplitude:

$$-i\mathcal{M}^* = ie^2 \varepsilon^{r\,*}_{\nu'}(k) \varepsilon^{r'}_{\mu'}(k') \bar{u}^s_{\alpha'}(p) \tilde{\Gamma}^{\mu'\nu'}_{\alpha'\beta'} u^{s'}_{\beta'}(p') , \tag{10.8.6}$$

where we defined the **dual vertex kernel** as:

$$\tilde{\Gamma}^{\mu\nu}_{\alpha\beta} = \gamma^0 \Gamma^\dagger \gamma^0 . \tag{10.8.7}$$

With all this we can now compute the squared scattering amplitude:

$$\begin{aligned} |\mathcal{M}|^2_{\text{non-pol}} = e^4 & \left[\frac{1}{2} \sum_{r=1,2} \varepsilon^r_\nu(k) \varepsilon^{r\,*}_{\nu'}(k) \right] \left[\sum_{r'=1,2} \varepsilon^{r'}_{\mu'}(k') \varepsilon^{r'*}_\mu(k') \right] \\ & \times \left[\frac{1}{2} \sum_{s=1,2} u^s_\beta(p) \bar{u}^s_{\alpha'}(p) \right] \tilde{\Gamma}^{\mu'\nu'}_{\alpha'\beta'} \left[\sum_{s'=1,2} u^{s'}_{\beta'}(p') \bar{u}^{s'}_\alpha(p') \right] \Gamma^{\mu\nu}_{\alpha\beta} , \end{aligned} \tag{10.8.8}$$

where the first two sums are over the polarizations of the external photons and the last two sums are over the spins of the external electrons. It holds that:

$$\left[\sum_{r'=1,2} \varepsilon^{r'}_{\mu'}(k') \varepsilon^{r'*}_\mu(k') \right] = -g_{\mu\mu'} , \tag{10.8.9}$$

$$\left[\sum_{s'=1,2} u^{s'}_{\beta'}(p') \bar{u}^{s'}_\alpha(p') \right] = (\not{p}' + m\mathbb{1})_{\beta'\alpha} . \tag{10.8.10}$$

Which means that we can rewrite the squared scattering amplitude as:

$$|\mathcal{M}|^2_{\text{non-pol}} = e^4 \frac{1}{4} g_{\nu\nu'} g_{\mu\mu'} (\not{p} + m)_{\beta\alpha'} \tilde{\Gamma}^{\mu'\nu'}_{\alpha'\beta'} (\not{p}' + m)_{\beta'\alpha} \Gamma^{\mu\nu}_{\alpha\beta} . \tag{10.8.11}$$

The contraction over spinor indices gives us the trace ie. $A_{\beta\alpha'} B_{\alpha'\beta} = \text{Tr}(AB)$. Making use of the metric tensors we can then simplify the expression to:

$$|\mathcal{M}|^2_{\text{non-pol}} = e^4 \frac{1}{4} \text{Tr} \left[(\not{p} + m) \tilde{\Gamma}^{\mu\nu} (\not{p}' + m) \Gamma^{\mu\nu} \right] . \tag{10.8.12}$$

From here we can use the following identify from the so-called **Trace Technology** (see Appendix A.1):

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu} , \tag{10.8.13}$$

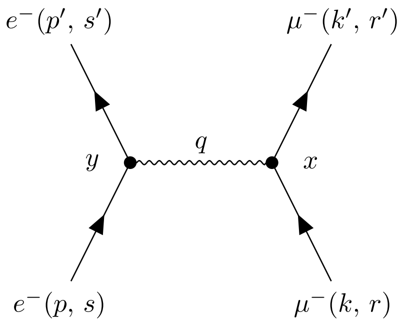
to evaluate the trace further and express it in terms of momenta and gamma matrices.

10.8.2 Coulomb Scattering

We want to compute the scattering amplitude for a process of distinguishable particles, such as an electron and a muon, which interact via the exchange of a virtual photon. The process is given by:

$$e^-(p,\, s) + \mu^-(k,\, r) \to e^-(p',\, s') + \mu^-(k',\, r')\;, \tag{10.8.14}$$

which is diagrammatically represented as:



$$= {}_0\langle f|\mathcal{T}\{-i\exp\int \mathrm{d}^4z\; \mathcal{H}_{\text{int}}(z)\}|i\rangle_0\;. \tag{10.8.15}$$

As we have two different fermionic fields that interact with the photonic field, our interaction Hamiltonian is given by:

$$\mathcal{H}_{\text{int}} = \mathcal{H}^e_{\text{int}} + \mathcal{H}^\mu_{\text{int}} = -e_\mu \bar{\psi}_\mu \gamma^\eta \psi_\mu A_\eta - e_e \bar{\psi}_e \gamma^\eta \psi_e A_\eta\;, \tag{10.8.16}$$

where we should stress that μ denotes the muon field and not a Lorentz index. As we saw, QED vertices are 3-point which means we need two vertices to connect 4 total external fermionic lines. This means we need to expand the time-ordered exponential to the second order:

$$\mathcal{T}\{-i\exp\int \mathrm{d}^4z\; \mathcal{H}_{\text{int}}(\boldsymbol{z})\} \approx \mathbb{1} + \frac{1}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y\; \mathcal{T}\{[\mathcal{H}^e_{\text{int}} + \mathcal{H}^\mu_{\text{int}}]^2\}\;. \tag{10.8.17}$$

In the second term we need to take the **mixed term** since the squares of either part of the interaction Hamiltonian will not correctly describe our process but would only describe the scattering of one fermion with another of its own kind. Thus we need to compute the following:

$$\begin{aligned}
&2\frac{p_W(-ie)^2}{s}\int \mathrm{d}^4x \int \mathrm{d}^4y\; {}_0\langle \mu^-(k',\, r')|\langle e^-(p',\, s')|\mathcal{N}\{\bar{\psi}^\mu_x \gamma^\eta \psi^\mu_x A^x_\eta \bar{\psi}^e_y \gamma^\nu \psi^e_y A^y_\nu\}|e^-(p,\, s)\rangle|\mu^-(k,\, r)\rangle_0 \\
&= \bar{u}^{r'}_\mu(k')(-ie\gamma^\eta)u^r_\mu(k)\bar{u}^{s'}_e(p')(-ie\gamma^\nu)u^s_e(p)\frac{-ig_{\eta\nu}}{q^2 + i\varepsilon}\;,
\end{aligned} \tag{10.8.18}$$

where we took into account that $e_e = e_\mu = e$ and that $\frac{p_W}{s} = 1/2$ as the operators are already in normal order and we have distinguishable particles, meaning **we cannot do $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$**. The additional factor of 2 comes from the fact that we used the mixed term of the 2nd order expansion of the time-ordered exponential.

10.8.3 Coulomb Scattering in the Non-Relativistic Limit

As an academic exercise, we can attempt to extract the form of the Coulomb potential from the scattering amplitude we computed above. First lets remember that in regular Quantum Mechanics we'd evaluate the scattering amplitude for such a process in the non-relativistic limit as:

$$iT_{fi} = -i\int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\; \psi^*_f(\boldsymbol{x}, t)V(\boldsymbol{x})\psi_i(\boldsymbol{x}, t)\;, \tag{10.8.19}$$

where $V(\boldsymbol{x})$ is the Coulomb potential. Using the QED Lagrangian (10.1.11):

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi\;, \tag{10.8.20}$$

we can derive the potential by writing the contraction explicitly and multiplying by γ^0:

$$\begin{aligned}
(i\gamma^\mu\partial_\mu - m)\psi &= \gamma^\mu A_\mu\psi \\
i\gamma^0\partial_0\psi + i\gamma^i\partial_i\psi - m\psi &= \gamma^\mu A_\mu\psi \\
i\partial_t\psi &= \left[-i\gamma^0\gamma^i\partial_i + m\gamma^0 + e\gamma^0\gamma^\mu A_\mu\right]\psi \\
i\partial_t\psi &= \left(-i\gamma^0\gamma^i\partial_i + m\gamma^0\right)\psi + V(\boldsymbol{x},t)\psi\,,
\end{aligned} \tag{10.8.21}$$

where we used $\gamma^0\gamma^0 = \mathbb{1}$ and $\gamma^0\gamma^i = -\gamma^i\gamma^0$. The expression we got is basically the non-relativistic Schrödinger's equation $i\partial_t\psi = [H_0 + V(\boldsymbol{x},t)]\,\psi$. Thus we can read the effective potential that is felt by one charged particle as:

$$V(\boldsymbol{x},t) = e\gamma^0\gamma^\mu A_\mu(\boldsymbol{x},t)\,. \tag{10.8.22}$$

In the non-relativistic limit, we can use the fact that $\gamma^0\gamma^i \approx 0$ and $\gamma^0\gamma^0 = \mathbb{1}$ to simplify the expression to:

$$V = eA^0 \tag{10.8.23}$$

In clasical electrodynamics, we can get the value of A^0 as the scalar potential generated by a static point charge by solving the Poisson equation. In QED this is analagous to using the propagator of the EM field:

$$A^0(\boldsymbol{x}) = \int \mathrm{d}^3\boldsymbol{y}\; D_{\mu\nu}(\boldsymbol{x}-\boldsymbol{y})\rho(\boldsymbol{y})\,, \tag{10.8.24}$$

where $J^0 = \rho(\boldsymbol{y})$. We also already know that both these problems equate to searching for the Green's function of the Poisson equation, the solution of which is well known in 3D space:

$$G = \frac{1}{4\pi|\boldsymbol{x}|}\,, \tag{10.8.25}$$

from which it follows that:

$$A^0 = \frac{e}{4\pi r}\,, \tag{10.8.26}$$

if we call $r = |\boldsymbol{x}|$ the distance from the charge. Inserting that into (10.8.23) we get the final form of the effective potential:

$$V(\boldsymbol{x}) = \frac{e^2}{4\pi r}\,. \tag{10.8.27}$$

This is the Coulomb potential that we were looking for and as we know it from classical electrodynamics. We can also derive this same result by using scattering amplitudes. For a non-relativistic case where:

$$\begin{aligned}
&p,\,k,\,p',\,k' \to 0\,,\\
&u^s(0) \approx \sqrt{2m}\begin{pmatrix}\xi_s\\0\end{pmatrix}\,,\\
&\bar{u}^{s'}(0)\gamma^0 u^s(0) \approx 2m\xi^\dagger_{s'}\xi_s = 2m\delta_{ss'}\,,\\
&\bar{u}^{s'}(0)\gamma^i u^s(0) \approx 0\,,
\end{aligned}$$

the scattering amplitude becomes:

$$\begin{aligned}
i\mathcal{M} &= (-ie)^2\bar{u}^{r'}_\mu(0)\gamma^\eta u^r_\mu(0)\frac{-ig_{\eta\nu}}{q^2}\bar{u}^{s'}_e(0)\gamma^\nu u^s_e(0)\\
&= (-ie)^2(2\pi m_\mu)\delta^{\eta 0}\delta rr'\frac{-ig^{00}}{0-|q|^2}(2\pi m_e)\delta^{\nu 0}\delta^{ss'}\,.
\end{aligned}$$

Using the **Born approximation** given as:

$$i\mathcal{M} = -\tilde{V}(\boldsymbol{q})\cdot\langle f|i\rangle = -\tilde{V}(\boldsymbol{q})(2m_e)(2m_\mu)\,, \tag{10.8.28}$$

we can identify the Fourier transform of the potential as:

$$\tilde{V}(\boldsymbol{q}) = -i\frac{e^2}{\boldsymbol{q}^2}\,. \tag{10.8.29}$$

Here we need to drop the imaginary prefactor of $-i$ which is a complex phase factor that we inherited from using QED's Minkowski-space propagators. Our potential needs to be real. To get the coordinate space potential we can use the Fourier transform:

$$V(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{e^2}{\boldsymbol{q}^2} = \frac{e^2}{4\pi|\boldsymbol{x}|} , \tag{10.8.30}$$

which is the Coulomb potential we were looking for. We can also very easily see how the sign would change if we were to consider scattering distinct particles of opposite charge. For example:

$$e^-(p,\, s) + \mu^+(k,\, r) \to e^-(p',\, s') + \mu^+(k',\, r') , \tag{10.8.31}$$

for which the scattering amplitude would be:

$$i\mathcal{M} \propto {}_0\langle \mu^+(k',\, r')|\langle e^-(p',\, s')|\mathcal{N}\{\bar{\psi}^\mu_x \gamma^\eta \psi^\mu_x A^x_\eta \bar{\psi}^e_y \gamma^\nu \psi^e_y A^y_\nu\}|e^-(p,\, s)\rangle|\mu^+(k,\, r)\rangle_0 . \tag{10.8.32}$$

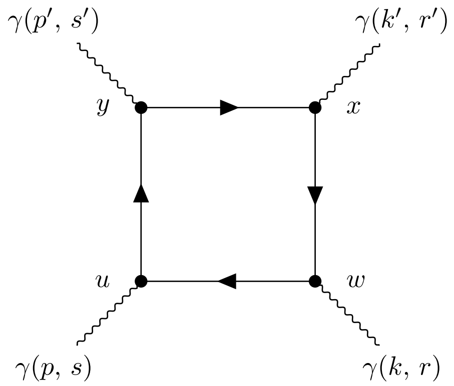
Performing such contractions would yield a $p_W = -1$ and thus the sign of the potential would be flipped as we'd expect for the interaction of opposite charges.

10.8.4 4-Photon Scattering

Continuing our set of examples, lets compute the scattering amplitude for the process:

$$\gamma(p,\, s) + \gamma(k,\, r) \to \gamma(p',\, s') + \gamma(k',\, r') , \tag{10.8.33}$$

which is diagramatically represented as:



(10.8.34)

It's evident that if we want to connect 4 external photon lines, we need to use 4 vertices, which means we need to expand the time-ordered exponential to the 4th order:

$$\begin{aligned}
\mathcal{M} &\propto \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4u\, \mathrm{d}^4w \;\; {}_0\langle\gamma\gamma|\mathcal{N}\{\bar{\psi}^\alpha_x \gamma^\mu_{\alpha\beta} \psi^\beta_x A^x_\mu \bar{\psi}^{\alpha'}_y \gamma^\nu_{\alpha'\beta'} \psi^{\beta'}_y A^y_\nu \bar{\psi}^{\gamma'}_u \gamma^\rho_{\gamma'\delta'} \psi^{\delta'}_u A^u_\rho \bar{\psi}^\gamma_w \gamma^\sigma_{\gamma\delta} \psi^\delta_w A^w_\sigma\}|\gamma\gamma\rangle_0 \\
&\propto \gamma_{\alpha\beta} \Delta_F^{\beta\alpha'}(x-y) \gamma_{\alpha'\beta'} \Delta_F^{\beta'\gamma'}(y-u) \gamma_{\gamma'\delta'} \Delta_F^{\delta'\gamma}(u-w) \gamma_{\gamma\delta} \Delta_F^{\delta\sigma}(w-x) \\
&\propto \mathrm{Tr}\big[\gamma\Delta_F(x-y)\gamma\Delta_F(y-u)\gamma\Delta_F(u-w)\gamma\Delta_F(w-x)\big] .
\end{aligned} \tag{10.8.35}$$

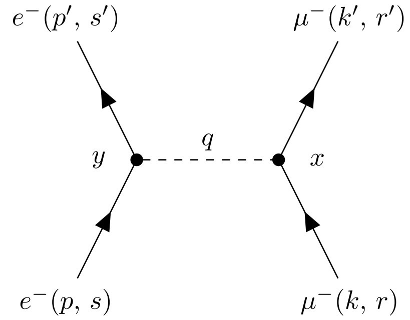
We can see from looking at the spinor indices or simply looking directly at the contractions that $p_W = -1$ for **any closed fermionic loop** due to the anti-commuting nature of fermionic fields. Additionally the corresponding amplitude for such a fermionic loop always involves **a trace over spinor indices** of the product of gamma matrices and the fermionic propagators that appear at the vertices and along the loop.

10.8.5 Yukawa Scattering

We've already solved some example cases for Yukawa scattering in the previously (Chapter 9.5), however we did not try to extract the effective potential from the scattering amplitude. Let's consider the same process of scattering distinguishable particles as for our Coulomb scattering example:

$$e^-(p,\, s) + \mu^-(k,\, r) \to e^-(p',\, s') + \mu^-(k',\, r') , \tag{10.8.36}$$

which is diagrammatically represented as:



(10.8.37)

The scattering amplitude for this process in the non-relativistic limit is proportional to:

$$i\mathcal{M} \propto g_e g_\mu \bar{u}_\mu^{r'}(0)\mathbb{1}u_\mu^r(0)\frac{i}{q^2 - m_\phi^2}\bar{u}_e^{s'}(0)\mathbb{1}u_e^s(0)\ . \tag{10.8.38}$$

Again using the Born approximation (10.8.28) we can identify the Fourier transform of the potential as:

$$\tilde{V}(\boldsymbol{q}) = g_e g_\mu \frac{-i}{|\boldsymbol{q}|^2 + m_\phi^2}\ , \tag{10.8.39}$$

which can be Fourier transformed to get the coordinate space potential:

$$V(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} \frac{g_e g_\mu}{|\boldsymbol{q}|^2 + m_\phi^2} = -g_e g_\mu \frac{e^{-m_\phi r}}{4\pi r}\ . \tag{10.8.40}$$

We see that the potential becomes much more short-ranged the heavier the scalar particle is. Alternatively if we wanted to consider the scattering of opposite charges we would get the same effective potential without any sign change. This means that Yukawa scattering is always attractive, regardless of the charges of the fermions involved.

	ff or $\bar{f}\bar{f}$ Scattering	$f\bar{f}$ Scattering
Coulomb	Repulsive	Attractive
Yukawa	Attractive	Attractive

Table 10.1: Sign of the effective potential for Coulomb and Yukawa scattering.

11 Radiative Corrections and Renormalization

11.1 Renormalization Crash Course

Renormalization is a process in quantum field theory that deals with infinities arising in calculations of physical quantities, such as scattering amplitudes and correlation functions. The main idea is to absorb these infinities into redefined parameters of the theory, such as masses and coupling constants, thus allowing for finite predictions of observables.

11.1.1 Lagrangian with Bare Parameters

For a given problem, we start with a Lagrangian that contains **bare parameters**, which are the original parameters of the theory before renormalization. Say for example $\mathcal{L}(e, m)$ is the Lagrangian of a theory with a coupling constant e and a mass m, sometimes denoted e_0 and m_0. We attempt to calculate some physical quantity, such as a scattering amplitude, using perturbation theory, however we encounter a divergence.

11.1.2 Regularization

To handle the divergence, we introduce a **regularization scheme**, which modifies the theory such that we extract the divergent part. Common regularization methods include:

- **Cutoff Regularization**: Introduces a cutoff scale Λ that limits the energy or momentum in the theory.
- **Dimensional Regularization**: Extends the theory to d dimensions, where d is not necessarily an integer, allowing for the separation of divergent and finite parts.
- **Pauli-Villars Regularization**: Introduces additional fields with large masses to cancel divergences.

In our case we will mostly make use of **dimensional regularization**, where we will imagine that we live in $d = 4 - \varepsilon_d$ dimensions where $\varepsilon_d \to 0$ instead of $d = 4$. This will allow us to calculate observables as a series in ε_d ie.:

$$\mathcal{M} = \frac{1}{\varepsilon_d} A + \text{finite terms} \ldots \,. \tag{11.1.1}$$

11.1.3 Renormalization Conditions

After regularization, we have a theory with a divergent part. We need to impose **renormalization conditions** to define how to absorb these divergences into the parameters of the theory. We will find that the renormalized parameters are commonly infinite. Popular renormalization conditions include:

- **On-shell Renormalization**: Fixes the parameters such that physical quantities match experimental values at specific kinematic points (eg. $p^2 = m^2_{\text{rest}}$).
- **Minimal Subtraction Scheme (MS)**: Removes the divergent part of the parameters, leaving only finite contributions.

From this we can then calculate the **renormalized parameters** and use them to evaluate corrected diagrams, which will now be finite.

11.2 Feynman Parameters

Feynman parameters are a technique used to simplify the evaluation of loop integrals in quantum field theory. We will encounter such integrals further in this chapter so lets tackle them here. For a case of two factors A, B in the denominator we can perform:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \, \frac{1}{[xA + (1-x)B]^2} = \int_0^1 \mathrm{d}x\mathrm{d}y \, \delta(x + y - 1) \frac{1}{[xA + yB]^2} \,, \tag{11.2.1}$$

where we've introduced the Feynman parameters x and y. By induction one could derive the following formula for n factors

$$\frac{1}{A_1 A_2 \ldots A_n} = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \, \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots + x_n A_n]^n} \,, \tag{11.2.2}$$

and for n factors raised to the powers m_i:

$$\frac{1}{A_1^{m_1} A_2^{m_2} \ldots A_n^{m_n}} = \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n \, \delta\left(\sum x_i - 1\right) \frac{\prod x_i^{m_i - 1}}{[\sum x_i A_i]^{\sum m_i}} \frac{\Gamma(m_1 + \cdots + m_n)}{\Gamma(m_1) \ldots \Gamma(m_n)} \,, \tag{11.2.3}$$

where Γ is the well-known gamma function, defined as:

$$\Gamma(z) = \int_0^\infty \mathrm{d}t \, t^{z-1} e^{-t} \,, \quad z > 0 \,. \tag{11.2.4}$$

11.3 Wick Rotation

Using a **Wick rotation** we can transform integrals over Minkowski space to integrals over Euclidean space. Wick rotations are important since we can use them to avoid divergences in loop integrals (as seen in Figure (11.1)). Note however that the Wick rotation cannot be viewed as a simple rotation in a complex vector space if that space is equipped with a conventional (Euclidean or Hilbert space) norm and a metric induced by the inner product. A Wick rotation is **not** a unitary transformation in an Euclidean space but rather a complex analytic continuation (a technique used to extend the domain) that changes the metric signature of the space from a Minkowski signature to a Euclidean signature.

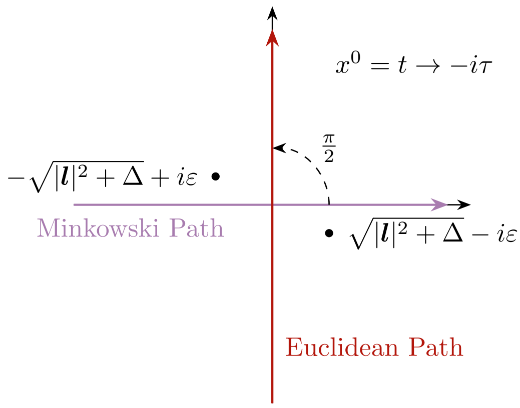


Figure 11.1: Applying a Wick rotation to the time component of the four-position. The Minkowski path is transformed into a Euclidean path, allowing us to avoid divergences in loop integrals.

The location of the poles here is not apparent but will become clear when used on an example like in Equation (11.4.11). To perform a Wick rotation, we replace a component of the four-position with an imaginary component, most commonly the time component:

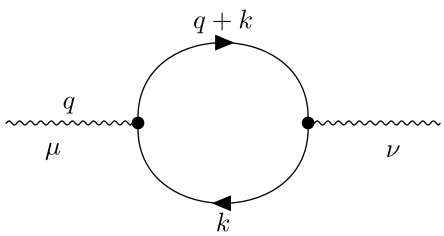
$$t \to -i\tau \, . \tag{11.3.1}$$

Another use of Wick rotations is to link Quantum Field Theory to Statistical Mechanics.

11.4 One-Loop Radiative Correction to the Photon Propagator

11.4.1 Regularization

It is possible for a photon to emit and reabsorb a virtual electron-positron pair as it travels through spacetime. This leads to a loop correction to the photon propagator. The corresponding Feynman diagram for such a process is given as:



$$= i\Pi_2^{\mu\nu}(q) \, , \tag{11.4.1}$$

where we've used the notation $\Pi_2^{\mu\nu}(q)$ to denote the **vacuum polarization tensor** (photon self-energy tensor) such that the corrected photon propagator can then be written as:

$$D_{\mu\nu}^{\text{1-loop}}(q) = D_{\mu\nu}(q) + D_{\mu\rho}(q) i\Pi_2^{\rho\sigma}(q) D_{\sigma\nu}(q) + \dots \, . \tag{11.4.2}$$

As we've seen in Equation (10.8.35), the amplitude for a process that involves a fermionic loop contains a trace over spinor indices of the gamma matrices and the fermionic propagators. Performing the necessary

Wick contractions to form the loop introduces a factor of $p_W = -1$. In our case, the amplitude for the photon self-energy correction is given by:

$$\begin{aligned} i\Pi_2^{\mu\nu} &= (-1)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left[(-ie\gamma^\mu)\frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}(-ie\gamma^\nu)\frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\varepsilon}\right] \\ &= (-1)(-1)(-ie)^2\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{m^2g^{\mu\nu}+k^\mu(k+q)^\nu+k^\nu(k+q)^\mu-g^{\mu\nu}k\cdot(k+q)}{[k^2-m^2+i\varepsilon]\,[(k+q)^2-m^2+i\varepsilon]}\,, \end{aligned} \tag{11.4.3}$$

where we used various Dirac trace identities to systematically evaluate the trace (see Appendix A.1). From here we will denote our numerator and denominators as:

$$N^{\mu\nu} = m^2g^{\mu\nu}+k^\mu(k+q)^\nu+k^\nu(k+q)^\mu-g^{\mu\nu}k\cdot(k+q)\,, \tag{11.4.4}$$

$$A = (k+q)^2-m^2+i\varepsilon\,, \tag{11.4.5}$$

$$B = k^2-m^2+i\varepsilon\,. \tag{11.4.6}$$

This situation is precisely one in which we can use Feynman parameters to simplify our integrals as we've seen in Equation (11.2.1). Using a Feynman parameter x we can see that the denominator can be rewritten as:

$$\begin{aligned} D = x(A-B)+B &= x\left(k^2+2kq+q^2-m^2+i\varepsilon-(k^2-m^2+i\varepsilon)\right)+k^2-m^2+i\varepsilon \\ &= k^2+2xkq+q^2x-m^2+i\varepsilon \\ &= [k+xq]^2-q^2x^2+q^2x-m^2+i\varepsilon \\ &= [k+xq]^2-\left[m^2-x(1-x)q^2\right]+i\varepsilon\,, \end{aligned} \tag{11.4.7}$$

From here we will call:

$$l^2 = [k+xq]^2\,, \tag{11.4.8}$$

$$\Delta(q,x) = m^2-x(1-x)q^2\,, \tag{11.4.9}$$

with which we can rewrite (11.4.3) as:

$$i\Pi_2^{\mu\nu} = -4e^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{N^{\mu\nu}}{[l^2-\Delta+i\varepsilon]^2}\,. \tag{11.4.10}$$

We've now encountered an issue with the integral, since it is divergent when integrating over l^0. To resolve this, we will perform a Wick rotation (as seen in Figure 11.1) to transform the integral into Euclidean space. If we separate the temporal and spatial components we can see where the poles are:

$$\begin{aligned} l^2-\Delta+i\varepsilon &= l_0^2+\boldsymbol{l}^2-\Delta+i\varepsilon \\ &= \left[l^0-(E_{\boldsymbol{l}}-i\varepsilon)\right]\left[l^0+(E_{\boldsymbol{l}}-i\varepsilon)\right]\,, \end{aligned} \tag{11.4.11}$$

where $E_{\boldsymbol{l}} = \sqrt{|\boldsymbol{l}|^2+\Delta}$. This places the poles as demonstrated above in Figure 11.1. To perform the Wick rotation we will replace the temporal component:

$$l^0 \equiv il_E^0\,, \tag{11.4.12}$$

where l_E^0 is the Euclidean time component. With this we've shifted $k = l - qx$ with Δ that is not dependent on l. This is important since it centers our integrand and makes the use of symmetry possible. To be extremely explicit, the denominator changes as follows:

$$\frac{1}{[l^2-\Delta+i\varepsilon]^2} \to \frac{1}{(-1)(-1)\,[l_E^2+\Delta]^2}\,, \tag{11.4.13}$$

where one minus comes from the Wick rotation and the other from the Minkowski metric signature converting $l^2 = (l^0)^2-|\boldsymbol{l}|^2 = (il_E^0)^2-|\boldsymbol{l}|^2 = -l_E^2$. Our integral transforms to:

$$i\Pi_2^{\mu\nu} = -4e^2\int_0^1\mathrm{d}x\;i\int_{-\infty}^{\infty}\mathrm{d}l_E^0\,\frac{\int\mathrm{d}\boldsymbol{l}}{(2\pi)^4}\frac{N^{\mu\nu}}{[l_E^2+\Delta]^2}\,. \tag{11.4.14}$$

We need to use a couple of tricks to simplify this integral in Euclidean 4-space, even though we will only illustrate its result and not fully solve it initially. First of all, make note that any **odd integral** over the Euclidean space will vanish, ie.:

$$\int \mathrm{d}^4 l_E \; l_E^\mu F(l_E^2) = 0 \,, \tag{11.4.15}$$

where $F(l_E^2)$ is any function that is even in l_E. This means that any linear term in l_E integrates to zero after performing the shift and Wick rotation. Quadratic terms in l_E will not vanish, however, and we can use the fact that:

$$\int \mathrm{d}^4 l_E \; l_E^\mu l_E^\nu F(l_E^2) = \frac{1}{4}\delta^{\mu\nu} \int \mathrm{d}^4 l_E \; l_E^2 F(l_E^2) \,. \tag{11.4.16}$$

Our previous numerator can be rewritten as:

$$N^{\mu\nu} = 2l^\mu l^\nu - g^{\mu\nu} l^2 + \left\{-2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)\right\} + (\text{terms linear in } l) \,. \tag{11.4.17}$$

Written this way we can apply the previously mentioned tricks to the numerator. The whole integration process is a tad bit involved so we will skip the integration of the numerator and focus on the final result. We can substitute the spatial integral in Euclidean space with a spherical integral:

$$\int \mathrm{d}\boldsymbol{l} = \int \mathrm{d}\Omega_3 \int \mathrm{d}l \; l^2 \,, \tag{11.4.18}$$

where $\mathrm{d}\Omega_3$ is the solid angle in three dimensions, to obtain:

$$i\Pi_2^{\mu\nu} \propto \cdots - 4e^2 \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \int_\Omega \frac{\mathrm{d}\Omega_4 \mathrm{d}l_E \; l_E^3}{(2\pi)^4 (l_E^2 + \Delta)^m} \,, \tag{11.4.19}$$

If we were to integrate this straight away we'd encounter the very well known **UV divergence**:

$$\cdots = \cdots \int_{-\infty}^{\infty} \frac{\mathrm{d}l_E}{l_E} = \ldots \ln l_E \Big|_{-\infty}^{\infty} = \infty \,. \tag{11.4.20}$$

To avoid this divergence we will make use of **dimensional regularization**. We'll imagine that we live in $d = 4 - \varepsilon_d$ dimensions, where $\varepsilon_d \to 0$, which will allow us to calculate the integral as a series in ε_d. The problem we need to solve is:

$$i\Pi_2^{\mu\nu} = -4ie^2 \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 l_E}{(2\pi)^4} \frac{N^{\mu\nu}}{\left[l_E^2 - \Delta + i\varepsilon\right]^2} \,, \tag{11.4.21}$$

which we can rewrite as an integral over $4 - \varepsilon_d$ dimensions as:

$$i\Pi_2^{\mu\nu} = -4ie^2 \int_0^1 \mathrm{d}x \int_{-\infty}^{\infty} \frac{\mathrm{d}^d l_E \; M^{d-4}}{(2\pi)^d} \frac{N^{\mu\nu}}{\left[l_E^2 - \Delta + i\varepsilon\right]^2} \,, \tag{11.4.22}$$

where M is a mass scale that we will use to keep the dimensions of the integral correct. If we also want to modify the numerator accordingly we need to modify Equation (11.4.16) to:

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d} l^\mu l^\nu F(l^2) = \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{1}{d} g^{\mu\nu} l^2 F(l^2) \,, \tag{11.4.23}$$

with which we can now rewrite our numerator as:

$$N^{\mu\nu} = \frac{2}{d} g^{\mu\nu}(-l_E^2) + g^{\mu\nu} l_E^2 + \{\ldots\} + \ldots \,, \tag{11.4.24}$$

where the $(-l_E^2)$ comes from the metric signature conversion. Now we can use the previously solved integral from Equation (11.2.3) to rewrite our integral part as:

$$\begin{aligned}
\int \frac{\mathrm{d}^d l_E}{(2\pi)^d} \frac{\left(1 - \frac{2}{d}\right) g^{\mu\nu} l_E^2}{(l_E^2 + \Delta)^2} &= \frac{-1}{(4\pi)^{d/2}} \left(1 - \frac{2}{d}\right) \Gamma\left(1 - \frac{d}{2}\right) \left[\frac{1}{\Delta}\right]^{1-\frac{d}{2}} g^{\mu\nu} \\
&= \frac{1}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \left[\frac{1}{\Delta}\right]^{2-\frac{d}{2}} (-\Delta g^{\mu\nu}) \,,
\end{aligned} \tag{11.4.25}$$

where we've made use of the Gamma function property that $n\Gamma(n) = \Gamma(n+1)$. Finally, we get the result:

$$i\Pi_2^{\mu\nu} = -4ie^2 \int_0^1 \mathrm{d}x \, \frac{M^{4-d}}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} \left(-\Delta g^{\mu\nu} + \left\{-2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)\right\}\right), \tag{11.4.26}$$

which after some sweat and tears can be simplified to:

$$i\Pi_2^{\mu\nu} = -4ie^2 \int_0^1 \mathrm{d}x \, \frac{M^{4-d}}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} \, 2x(1-x)(q^2 g^{\mu\nu} - q^\mu q^\nu) \, . \tag{11.4.27}$$

We can check that the **Generalized Ward Identity** holds for this result:

$$q_\mu \Pi_2^{\mu\nu} \propto q_\mu (q^2 g^{\mu\nu} - q^\mu q^\nu) = q^2 q^\nu - q^2 q^\nu = 0 \, , \tag{11.4.28}$$

where I remind the reader that $g^{\mu\nu}$ works sort of like a Kronecker delta but also raises the covariant index to a contravariant one. If we focus again on the integral part of our result we can finally do $d \to 4$. If we take a look at the mass scale M we can now apply the limit:

$$M^{4-d} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} \to \frac{\Gamma\left(\frac{\varepsilon_d}{2}\right)}{(4\pi)^2} \left(\frac{\Delta}{4\pi M^2}\right)^{-\varepsilon_d/2} \, . \tag{11.4.29}$$

So the important part is the following limit:

$$\lim_{\varepsilon_d \to 0} a^{-\frac{\varepsilon_d}{2}} = 1 - \frac{\varepsilon_d}{2} \ln a \, . \tag{11.4.30}$$

The value of the Gamma function in the limit is:

$$\Gamma\left(\frac{\varepsilon_d}{2}\right) = \frac{2}{\varepsilon_d} - \gamma_{EM} + \mathcal{O}(\varepsilon_d) \, , \tag{11.4.31}$$

where $\gamma_{EM} \approx 0.577$ is the **Euler-Mascheroni constant**. Applying these limits to Equation (11.4.29) we get:

$$\begin{aligned} &\lim_{\varepsilon_d \to 0} \frac{1}{(4\pi)^2} \left(1 - \frac{\varepsilon_d}{2} \ln\left[\frac{\Delta}{4\pi M^2}\right] + \dots\right) \left(\frac{2}{\varepsilon_d} - \gamma_{EM} + \dots\right) \\ &= \frac{1}{(4\pi)^2} \left(\frac{2}{\varepsilon_d} - \ln\frac{\Delta}{M^2} - \gamma_{EM} + \ln(4\pi)\right) \, . \end{aligned} \tag{11.4.32}$$

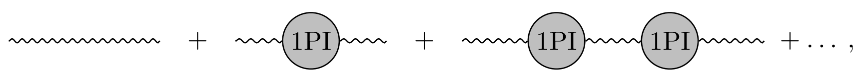
This is the result of the regularization procedure. We can pack it into a final expression as:

$$\begin{aligned} i\Pi_2^{\mu\nu} &= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi_2(q^2) \\ &= -\frac{4ie^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \; 2x(1-x) \left(\frac{2}{\varepsilon_d} - \ln\frac{\Delta}{M^2} - \gamma_{EM} + \ln(4\pi)\right) (q^2 g^{\mu\nu} - q^\mu q^\nu) \, , \end{aligned} \tag{11.4.33}$$

where $\Pi_2(q^2)$ is the scalar part of the vacuum polarization tensor.

11.4.2 Renormalization

The next step is to **renormalize** the photon propagator. We will do this by defining renormalized parameters e_R and M_R and relating them to the bare parameters e_0 and M_0. The corrected photon propagator can then be written as:

$D_{\mu\nu}^{\text{corrected}}(q) =$  (11.4.34)

or explicitly as in Equation (11.4.2) for the one-loop correction:

$$\begin{aligned} D_{\mu\nu}^{\text{1-loop}}(q) &= D_{\mu\nu}(q) + D_{\mu\rho}(q) i\Pi_2^{\rho\sigma}(q) D_{\sigma\nu}(q) + \dots \\ &= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2} \left[i(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi_2(q^2)\right] \frac{-ig_{\sigma\nu}}{q^2} + \dots \, , \end{aligned} \tag{11.4.35}$$

where $\Pi_2(q^2)$ is the vacuum polarization tensor from Equation (11.4.33) without the tensor part $(q^2 g^{\mu\nu} - q^\mu q^\nu)$. Since the Ward identity imposes that terms proportional to q^μ vanish we can rewrite the photon propagator as:

$$\begin{aligned} D_{\mu\nu}^{\text{1-loop}}(q) &= \frac{-ig_{\mu\nu}}{q^2}\left[1 + \Pi_2(q^2) + \Pi_2^2(q^2) + \dots\right] \\ &= \frac{-ig_{\mu\nu}}{q^2}\sum_{n=0}^{\infty}\Pi_2^n(q^2) \\ &= \frac{-ig_{\mu\nu}}{q^2}\frac{1}{1-\Pi_2(q^2)}\,, \end{aligned} \tag{11.4.36}$$

where we've used the geometric series sum:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,, \tag{11.4.37}$$

to rewrite the propagator in a more compact form. Photon mass is related to the position of the pole of the propagator, thus we want $m_\gamma = 0$ thus a pole at $q^2 = 0$ for a real photon (this way the propagator is the Green's function of Maxwell's equations). Fortunately this condition is automatically satisfied due to the Ward identity:

$$q^2(1 - \Pi_2(q^2))\Big|_{q^2=0} = 0\,. \tag{11.4.38}$$

Let's expand the propagator as a series in $\Pi_2(q^2)$ when $q \simeq 0$:

$$\begin{aligned} D_{\mu\nu}^{\text{1-loop}}(q) &= \frac{-ig_{\mu\nu}}{q^2(1-\Pi_2(q^2))} = \frac{-ig_{\mu\nu}}{q^2(1-[\Pi_2(q^2)-\Pi_2(0)]-\Pi_2(0))} \\ &\simeq \frac{-ig_{\mu\nu}}{q^2(1-[\Pi_2(q^2)-\Pi_2(0)])}\frac{1}{1-\Pi_2(0)} \\ &= \frac{-ig_{\mu\nu}}{q^2(1-[\Pi_2(q^2)-\Pi_2(0)])}\,Z_3\,, \end{aligned} \tag{11.4.39}$$

where we have defined the **EM field-strength renormalization** Z_3 as:

$$Z_3 = \frac{1}{1-\Pi_2(0)}\,. \tag{11.4.40}$$

This factor Z_3 includes both the tree order and the one-loop correction to the photon propagator. If we want to explicitly see just the one-loop correction we can write:

$$\delta Z_3 \equiv Z_3 - 1 = -\Pi_2(0)\,. \tag{11.4.41}$$

We will get the evaluated value of $\Pi_2(0)$ after we perform the renormalization procedure for the vertex correction (see Equation (11.6.49)), from which we will derive the relation between the bare and renormalized charge. Using that we will also be able to write the finite renormalized version as:

$$\Pi_{2,R}(q^2) \equiv \Pi_2(q^2) - \Pi_2(0)\,, \tag{11.4.42}$$

which is why it will be used in the denominator of the propagator. With that the renormalized photon propagator can be written as:

$$\int \mathrm{d}^4x\,\langle\Omega|\mathcal{T}\{A^\mu(x)A^\nu(0)\}|\Omega\rangle = \frac{-ig^{\mu\nu}Z_3}{q^2(1-[\Pi_2(q^2)-\Pi_2(0)])} + \mathcal{O}(e^4) \equiv \frac{-ig^{\mu\nu}}{\Gamma_\gamma^{(2)}(q)}\,, \tag{11.4.43}$$

or if we prefer to use the renormalized photon field $A_R^\mu = Z_3^{-1/2}A^\mu$:

$$\int \mathrm{d}^4x\,\langle\Omega|\mathcal{T}\{A_R^\mu(x)A_R^\nu(0)\}|\Omega\rangle = \frac{-ig^{\mu\nu}}{q^2(1-[\Pi_2(q^2)-\Pi_2(0)])} + \mathcal{O}(e^4) \equiv \frac{-ig^{\mu\nu}}{\Gamma_{\gamma,R}^{(2)}(q)}\,. \tag{11.4.44}$$

To clear up confusion, these two are identical. It's only a matter of where we put the Z_3 factor is placed and bookkeeping:

$$\Gamma_\gamma^{(2)}(q) = \frac{1}{Z_3}(q^2(1 - [\Pi_2(q^2) - \Pi_2(0)])) + \mathcal{O}(e^4)\,, \tag{11.4.45}$$

$$\Gamma_{\gamma,R}^{(2)}(q) = q^2(1 - [\Pi_2(q^2) - \Pi_2(0)]) + \mathcal{O}(e^4) \quad \text{but} \quad A_R^\mu = Z_3^{-1/2} A^\mu\,, \tag{11.4.46}$$

$$\Rightarrow \Gamma_{\gamma,R}^{(2)}(q) = \sqrt{Z_3}\sqrt{Z_3}\Gamma_\gamma^{(2)}(p) \tag{11.4.47}$$

$\Gamma_\gamma^{(2)}(p)$ is only called *bare* since it contains the bare mass m_0 and charge e_0 even though it is fully dressed with the one-loop correction. For posterity lets repeat the renormalization conditions in terms of $\Gamma^{(2)}$:

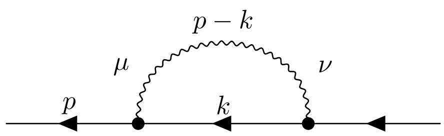
$$\Gamma_{\gamma,R}^{(2)}(q^2 = 0) = 0\,, \tag{11.4.48}$$

$$\frac{\partial \Gamma_{\gamma,R}^{(2)}}{\partial q^2}(q^2 = 0) = 1\,. \tag{11.4.49}$$

11.5 One-Loop Radiative Correction to the Electron Propagator

11.5.1 Regularization

During travel an electron can emit and reabsorb a virtual photon, leading to a loop correction to the electron propagator. The corresponding Feynman diagram for such a process is given as:



$$= i\Sigma_2(p)\,, \tag{11.5.1}$$

where we've used the notation $\Sigma_2(p)$ to denote the **electron self-energy** such that the corrected momentum space electron propagator can then be written as:

$$\begin{aligned} \Delta_F^{\text{1-loop}}(p) &= \Delta_F(p) + \Delta_F(p)(-i\Sigma_2(p))\Delta_F(p) + \Delta_F(p)(-i\Sigma_2(p))\Delta_F(p)(-i\Sigma_2(p))\Delta_F(p) + \ldots \\ &= \Delta_F(p)\sum_{n=0}^{\infty}\left[(-i\Sigma_2(p))\Delta_F(p)\right]^n\,, \end{aligned} \tag{11.5.2}$$

Using our knowledge of Feynman rules we can write the electron self-energy as:

$$i\Sigma_2(p) = (-ie)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon}\gamma^\nu \frac{-ig_{\mu\nu}}{(p-k)^2 + i\varepsilon}\,, \tag{11.5.3}$$

where we note that we've evaluated the amputated Feynman diagram (ie. without the external legs). This can be rewritten in d dimensions as:

$$i\Sigma_2(p) = (-ie)^2 M^{4-d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon}\gamma^\mu \frac{-ig_{\mu\nu}}{(p-k)^2 + i\varepsilon}\gamma^\nu\,. \tag{11.5.4}$$

Contracting the photon propagator yields $\gamma^\mu g_{\mu\nu}\gamma^\nu = \gamma^\mu\gamma_\mu$ with which we can then act on $\not{k}$ as described in the Gamma matrix identity from Equation (A.1.7) to obtain $-(d-2)\not{k}$. With this the self-energy can be rewritten as:

$$i\Sigma_2(p) = (-ie)^2 M^{4-d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i\left[-(d-2)\not{k} + dm\right]}{k^2 - m^2 + i\varepsilon}\frac{-i}{(p-k)^2 + i\varepsilon}\,. \tag{11.5.5}$$

Now we can use a Feynman parameter x (as in Equation (11.2.1)) to rewrite the denominator:

$$\begin{aligned} \frac{1}{k^2 - m^2 + i\varepsilon}\frac{1}{(p-k)^2 + i\varepsilon} &= \int_0^1 \mathrm{d}x\, \frac{1}{\left[(k^2 - m^2 + i\varepsilon)(1-x) + (p^2 - 2pk + k^2 + i\varepsilon)x\right]^2} \\ &= \int_0^1 \mathrm{d}x\, \frac{1}{\left[(k - px)^2 - \Delta + i\varepsilon\right]^2} \\ &= \int_0^1 \mathrm{d}x\, \frac{1}{\left[l^2 - \Delta + i\varepsilon\right]^2}\,, \end{aligned} \tag{11.5.6}$$

where we have defined $l = k - px$ and $\Delta = p^2(x^2 - x) + m^2(1-x)$. Now in a similar manner to the photon self-energy we can rewrite the self-energy as:

$$i\Sigma_2(p) = (-ie)^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l}{(2\pi)^d} M^{4-d} \frac{-(d-2)(\not{l} + x\not{p}) + dm}{\left[l^2 - \Delta + i\varepsilon\right]^2} \,. \tag{11.5.7}$$

Next we perform a Wick rotation to transform the integral into Euclidean space, just as we did for the photon self-energy.

$$i\Sigma_2(p) = i(-ie)^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d l_E}{(2\pi)^d} M^{4-d} \frac{-(d-2)x\not{p} + md}{(-1)(-1)\left(l_E^2 + \Delta\right)^2} \,, \tag{11.5.8}$$

where we have used the fact that $l^2 = -l_E^2$ in Euclidean space and that the denominator changes as in Equation (11.4.13). The $\not{l}$ term vanishes after the Wick rotation since the integral over an odd function in Euclidean space vanishes, like so:

$$\int \mathrm{d}^d l_E \, \not{l} F(l_E^2) = \int \mathrm{d}^d l_E \, \gamma^\mu l_{E\mu} \frac{1}{\left(l_E^2 + \Delta\right)^2} = 0 \,. \tag{11.5.9}$$

Now using the result from Equation (11.2.3) we can rewrite the integral as:

$$i\Sigma_2(p) = -ie^2 \int_0^1 \mathrm{d}x \left[M^{4-d} \frac{1}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \right] \left(-(d-2)x\not{p} + md\right) . \tag{11.5.10}$$

Finally, we can take the limit $d \to 4$ and apply the same tricks as in the photon self-energy case seen in Equation (11.4.32) to obtain:

$$i\Sigma_2(p) = -\frac{4ie^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \left(\frac{2}{\varepsilon_d} - \ln \frac{\Delta}{M^2} - \gamma_{EM} + \ln(4\pi) \right) (4m - 2x\not{p}) \,. \tag{11.5.11}$$

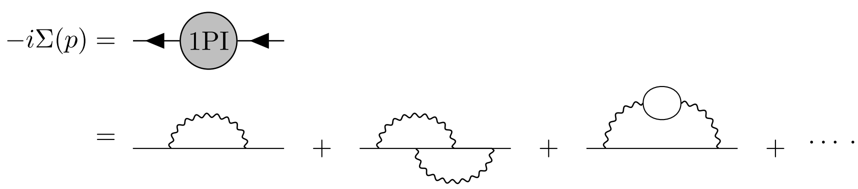
11.5.2 Renormalization

After regularization we can proceed to renormalization. Our experiments always measure the fully dressed/corrected vertex or propagators. We need to relate the bare parameters of the theory m_0, e_0 to the physical renormalized parameters m_R, e_R that we measure in experiments. The bare parameters are just parameters of the Lagrangian that we use to describe the theory and do not need to correspond to the measured mass or charge. Moreover, if we keep such parameters finite, we will run into divergences when calculating observables. We'd like to have:

$$\frac{e_R^2}{4\pi} = \frac{1}{137} \,, \tag{11.5.12}$$

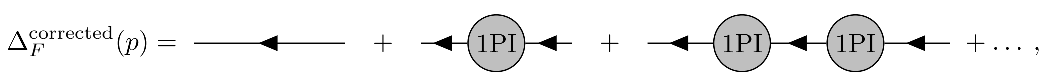
$$m_R c^2 = 0.511 \text{ MeV} \,. \tag{11.5.13}$$

Using an **on-shell renormalization scheme** the renormalization conditions require that the corrected electron propagator has a pole at $p^2 = m_R^2$ (same as $\not{p} = m_R$) and that the corrected vertex is related to the measured charge $(-ie_R\gamma^\mu)$. In general Σ is given by the sum of all **one particle-irreducible** (1PI) diagrams, which are diagrams that do not divide into two separate parts if one line is cut:



(11.5.14)

We've calculated the one-loop correction to the electron propagator in Equation (11.5.11) for which $\Sigma = \Sigma_2$. The corrected electron propagator can then be written as:



(11.5.15)

or explicitly for the one-loop correction as stated in Equation (11.5.2):

$$\begin{aligned}
\Delta_F^{\text{1-loop}}(p) &= \Delta_F(p) + \Delta_F(p) i\Sigma_2(p)\Delta_F(p) + \Delta_F(p)(-i\Sigma_2(p))\Delta_F(p)(-i\Sigma_2(p))\Delta_F(p) + \dots \\
&= \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}(-i\Sigma_2)\frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}(-i\Sigma_2)\frac{i}{\not{p}-m}(-i\Sigma_2)\frac{i}{\not{p}-m} + \dots \\
&= \frac{i}{\not{p}-m}\sum_{n=0}^{\infty}\left[(-i\Sigma_2)\frac{i}{\not{p}-m}\right]^n \\
&= \frac{i}{\not{p}-m}\frac{1}{1+\frac{\Sigma_2(p)}{\not{p}-m}} \\
&= \frac{i}{\not{p}-m-\Sigma_2(p,m)}\,,
\end{aligned} \tag{11.5.16}$$

where we've used the sum of a geometric series from Equation (11.4.37) to rewrite the propagator. Since Σ_2 is infinite, m is obviously not the physical mass of the electron. As stated in Equation (11.5.13) we want the corrected propagator to have a pole at $p^2 = m_R^2$ (which is the same as $\not{p} = m_R$). Thus:

$$\not{p} - m - \Sigma_2(\not{p}, m)\Big|_{\not{p}=m_R} = 0\,, \tag{11.5.17}$$

from which we can derive the relation between the bare and renormalized mass:

$$m_R - m_0 - \Sigma_2(m_R, m) = 0 \quad \Rightarrow \quad m_R = m_0 + \Sigma_2(m_R, m)\,, \tag{11.5.18}$$

Now we can expand the propagator for when $\not{p} \simeq m_R$ in terms of $(\not{p} - m_R)$:

$$\begin{aligned}
\Delta_F^{\text{1-loop}}(p) &= \frac{i}{\not{p} - m_0 - \Sigma_2(m_R, m) - \frac{\partial \Sigma}{\partial \not{p}}\Big|_{\not{p}=m_R}(\not{p} - m_R)} \\
&= \frac{i}{\not{p} - m_R}\frac{1}{1 - \frac{\partial \Sigma}{\partial \not{p}}\Big|_{\not{p}=m_R}} \\
&= \frac{i}{\not{p} - m_R} Z_2\,,
\end{aligned} \tag{11.5.19}$$

where we have defined the **electron field-strength renormalization** Z_2 as:

$$Z_2 = \frac{1}{1 - \frac{\partial \Sigma}{\partial \not{p}}\Big|_{\not{p}=m_R}}\,. \tag{11.5.20}$$

This means that the renormalized electron propagator is given by:

$$\int \mathrm{d}^4x \,\langle\Omega|\mathcal{T}\left\{\psi(x)\bar{\psi}(0)\right\}|\Omega\rangle \simeq \frac{iZ_2}{\not{p} - m_R} + \mathcal{O}((\not{p} - m_R)^2) \equiv \frac{i}{\Gamma_e^{(2)}(p)}\,, \tag{11.5.21}$$

or if we prefer to use the renormalized electron field $\psi_R = Z_2^{-1/2}\psi$:

$$\int \mathrm{d}^4x \,\langle\Omega|\mathcal{T}\left\{\psi_R(x)\bar{\psi}_R(0)\right\}|\Omega\rangle \simeq \frac{i}{\not{p} - m_R} + \mathcal{O}((\not{p} - m_R)^2) \equiv \frac{i}{\Gamma_{e,R}^{(2)}(p)}\,. \tag{11.5.22}$$

Just as with the photon propagator, these two are identical. It's only a matter of where we put the Z_2 factor is placed and bookkeeping:

$$\Gamma_e^{(2)}(p) = \frac{1}{Z_2}(\not{p} - m_R) + \mathcal{O}((\not{p} - m_R)^2)\,, \tag{11.5.23}$$

$$\Gamma_{e,R}^{(2)}(p) = \not{p} - m_R + \mathcal{O}((\not{p} - m_R)^2) \quad \text{but} \quad \psi_R = Z_2^{-1/2}\psi\,, \tag{11.5.24}$$

$$\Rightarrow \Gamma_{e,R}^{(2)}(p) = \sqrt{Z_2}\sqrt{Z_2}\Gamma_e^{(2)}(p) \tag{11.5.25}$$

$\Gamma_e^{(2)}(p)$ is only called *bare* since it contains the bare mass m_0 and charge e_0 even though it is fully dressed with the one-loop correction. Its name is purely a bookkeeping choice. For posterity lets repeat the renormalization conditions in terms of $\Gamma^{(2)}$:

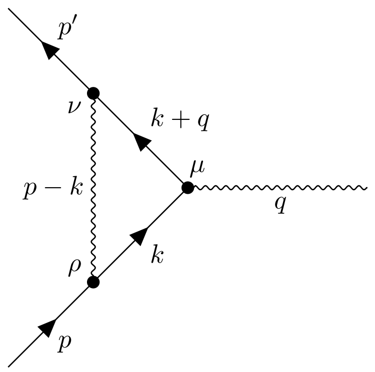
$$\Gamma_{e,R}^{(2)}(\not{p} = m_R) = 0 \,, \tag{11.5.26}$$

$$\frac{\partial \Gamma_{e,R}^{(2)}}{\partial \not{p}}(\not{p} = m_R) = 1 \,. \tag{11.5.27}$$

11.6 One-Loop Radiative Correction to the QED Vertex

11.6.1 Regularization

The one-loop radiative correction to the QED vertex is a bit more involved than the previous two examples. The corresponding Feynman diagram for such a process is given as:



$$= (-ie)\delta\Gamma^\mu(p,p') \,, \tag{11.6.1}$$

where we've used the notation $\delta\Gamma^\mu$ to denote the **one-loop vertex correction** such that the corrected QED vertex can then be written as:

$$\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu \,. \tag{11.6.2}$$

Since Γ^μ must be a Lorentz 4-vector which is sandwiched between two Dirac spinors ($\overline{u}(p')$ and $u(p)$) the most general form of the vertex, that is still consistent with Lorentz invariance, is:

$$\Gamma^\mu(p,p') = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m} + F_3(q^2)q^\mu + F_4(q^2)(\not{p}' + m)\gamma^\mu + \dots \,. \tag{11.6.3}$$

TODO: Paragraph Needs FIX In principle this vertex decomposition could contain any term that is a Lorentz 4-vector, however lots of these terms can be eliminated by the fact that the QED Lagrangian is invariant under **CP transformations** (Charge conjugation and Parity transformations):

$$\psi(t,\boldsymbol{x}) \xrightarrow{\text{CP}} i\gamma^0 C \bar{\psi}^\intercal(t,-\boldsymbol{x}) \,, \tag{11.6.4}$$

where C is the charge conjugation matrix and $P = \gamma^0$ is the parity operator. This means that the vertex must also transform such that electromagnetic current J^μ is invariant under such transformations:

$$J^\mu = \bar{\psi}\Gamma^\mu\psi \,, \tag{11.6.5}$$

which adds additional constraints on the form of the vertex. For example it is not possible to have a term of the form $\gamma_5\gamma_\mu$ since γ_5 changes parity behavior meaning that $\bar{\psi}\gamma_5\gamma^\mu\psi$ transforms as an axial vector which does not change sign under parity transformations. Such a term would then violate CP-symmetry of the electromagnetic current J^μ. Terms proportional to p^μ or p'^μ without gamma matrices are not allowed since while they are Lorentz 4-vectors, they cannot form proper vertex operators on Dirac spinors.

The above general form can be further simplified by the fact that for *on-shell* external fermions the free Dirac equation holds true, thus:

$$(\not{p} - m)u(p) = 0 \,, \tag{11.6.6}$$

$$\overline{u}(p')(\not{p}' - m) = 0 \,. \tag{11.6.7}$$

Applying the **Ward-Takahashi identity** (see (10.3)) we get:

$$q^\mu \Gamma^\mu(p,p') = \Delta_F^{-1}(p') - \Delta_F^{-1}(p) = \left[\not{p}' - m - \Sigma(p')\right] - \left[\not{p} - m - \Sigma(p)\right] \,, \tag{11.6.8}$$

where $\Delta_F^{-1}(p) = \left[\not{p} - m - \Sigma(p)\right]$ is the inverse of the full/dressed fermion propagator and $\Sigma(p)$ is the fermionic self-energy. This means that for a properly renormalized vertex it must hold that:

$$\bar{u}(p')\Delta_F^{-1}(p') = 0 \,, \tag{11.6.9}$$

$$\Delta_F^{-1}(p)u(p) = 0 \,. \tag{11.6.10}$$

In turn that ensures that the following holds:

$$\bar{u}(p')q_\mu \Gamma^\mu(p,p')u(p) = 0 \,, \tag{11.6.11}$$

which means that any terms that are linear in q^μ will vanish (ie. there are no longitudinal terms). Terms containing q^2 will not vanish, however, for in scattering processes we usually consider an *on-shell* incoming photon with $q^2 = 0$. This does not exclude functions of q^2 like the form factors $F(q^2)$. With all this in mind the general form of the vertex is simplified to:

$$\Gamma^\mu(p,p') = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m} \,, \tag{11.6.12}$$

where $F_1(q^2)$ and $F_2(q^2)$ are the **Dirac** and **Pauli form factors** respectively (for more details see [2]) and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ is **the commutator of gamma matrices**. The Dirac form factor $F_1(q^2)$ is related to the charge of the fermion and is usually $F_1(0) = 1$. The Pauli form factor $F_2(q^2)$ is related to the anomalous magnetic moment of the fermion, which is usually $F_2(0) = a = (g-2)/2$, where g is the gyromagnetic ratio of the fermion.

We can write the diagram using the Feynman rules as:

$$\bar{u}(p')\delta\Gamma^\mu u(p) = (-ie)^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\gamma^\nu \frac{i\bar{u}(p')\left[(\not{k}+\not{q}) + m\right]u(p)}{(k+q)^2 - m^2 + i\varepsilon}\gamma^\mu \frac{i(\not{k}+m)}{k^2 - m^2 + i\varepsilon}\gamma^\rho \frac{-ig_{\nu\rho}}{(p-k)^2 + i\varepsilon} \,, \tag{11.6.13}$$

where this expression corresponds to the amputated diagram (ie. without the external legs) but we've manually added the external contractions for fermions, which will become relevant later. We can expand this to:

$$\bar{u}(p')\delta\Gamma^\mu u(p) = 2ie^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p)}{((k-p)^2 + i\varepsilon)(k'^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} \,, \tag{11.6.14}$$

where we have defined $k' = k + q$. We can now use 3 Feynman parameters x, y, z as in Equation (11.2.2) to get to:

$$\frac{1}{((k-p)^2 + i\varepsilon)(k'^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)} = \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\delta(x+y+z-1)\,\frac{2}{D^3} \,, \tag{11.6.15}$$

where the denominator D is defined as:

$$\begin{aligned} D &= x(k^2 - m^2) + y(k'^2 - m^2) + z(k-p)^2 + (x+y+z)i\varepsilon \\ &= k^2 + 2k\cdot(yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\varepsilon \,, \end{aligned} \tag{11.6.16}$$

and where we've used the fact that $x + y + z = 1$ to simplify the denominator. To find the complete square we need to shift k to:

$$l = k + yq - zp \,. \tag{11.6.17}$$

We can write the denominator in the standard form for Feynman parameters $D = l^2 - \Delta + i\varepsilon$ if we define:

$$\Delta = -xyq^2 + (1-z)^2m^2 \,. \tag{11.6.18}$$

Using symmetry identities for integrals in Euclidean space (Equations (11.4.15) and (11.4.16)) we can rewrite the numerator as:

$$\begin{aligned}N^\mu &= \bar{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p) \\ &= \bar{u}(p')\left[-\frac{1}{2}\gamma^\mu l^2 + (-y\not{q} + z\not{p})\gamma^\mu\left((1-y)\not{q} - z\not{p}\right) + m^2\gamma^\mu - 2m\left((1-2y)q^\mu + 2zp^\mu\right)\right]u(p)\,.\end{aligned} \tag{11.6.19}$$

With quite a bit of algebra and manipulation we can rewrite the numerator as:

$$\begin{aligned}N^\mu = \bar{u}(p')\Big[\gamma^\mu\cdot\left(-\frac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\right) \\ + (p'^\mu + p^\mu)\cdot mz(z-1) + q^\mu\cdot m(z-2)(x-y)\Big]u(p)\,.\end{aligned} \tag{11.6.20}$$

Next we can use the **Gordon identity**:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)\,, \tag{11.6.21}$$

which stems from the fact that the Dirac equation holds for both $u(p)$ and $\bar{u}(p')$, to replace the $(p'+p)$ term with one containing $\sigma^{\mu\nu}q_\nu$. Thus our entire expression becomes:

$$\bar{u}(p')\delta\Gamma^\mu u(p) = 2ie^2\int\frac{\mathrm{d}^4l}{(2\pi)^4}\int_0^1\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\delta(x+y+z-1)\frac{2N^\mu}{D^3}\,, \tag{11.6.22}$$

where N^μ is given by Equation (11.6.20) and the denominator D is defined as above:

$$\begin{aligned}D &= l^2 - \Delta + i\varepsilon \\ &= (k+yq-zp)^2 - \left[-xyq^2 + (1-z)^2m^2\right] + i\varepsilon\,.\end{aligned} \tag{11.6.23}$$

Instead of using dimensional regularization like we did for the previous two examples, we will use **Pauli-Villars regularization** to regularize the integral. Before we can do that, however, we need to perform a Wick rotation to transform the integral into Euclidean space over 4-dimensional spherical coordinates:

$$x = (r\sin\omega\sin\theta\cos\phi, r\sin\omega\sin\theta\sin\phi, r\sin\omega\cos\theta, r\cos\omega)\,, \tag{11.6.24}$$

$$\mathrm{d}^4x = r^3\sin^2\omega\sin\theta\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \tag{11.6.25}$$

$$\Omega_4 = \sin^2\omega\sin\theta\mathrm{d}r\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}\phi\,, \tag{11.6.26}$$

$$\mathrm{d}^4l_E = l_E^3\mathrm{d}l_E\,\mathrm{d}\Omega_4\,, \tag{11.6.27}$$

where $\mathrm{d}\Omega_4$ is the solid angle in four dimensions, whose integral is $2\pi^2$. Now after performing the Wick rotation $l^0 \equiv il_E^0$ we need to evaluate an integral of the form:

$$\begin{aligned}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{\left[l^2-\Delta\right]^3} &= \frac{i}{(-1)^3(2\pi)^4}\int\mathrm{d}^4l_E\,\frac{1}{\left[l_E^2+\Delta\right]^3} \\ &= \frac{i}{(-1)^3(2\pi)^4}\int\mathrm{d}\Omega_4\int_0^\infty\mathrm{d}l_E\,\frac{l_E^3}{\left[l_E^2+\Delta\right]^3} \\ &= \frac{i}{(-1)^3(2\pi)^4}\cdot 2\pi^2\cdot\frac{1}{4\Delta} = -\frac{i}{32\pi^2\Delta}\,,\end{aligned} \tag{11.6.28}$$

where we've rewritten the integral in spherical coordinates in the second line. For a general power m of the denominator, this integral can be evaluated as:

$$\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{1}{\left[l^2-\Delta\right]^m} = \frac{i}{(-1)^m(2\pi)^4}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}}\,. \tag{11.6.29}$$

We also will need to evaluate a similar integral of the form:

$$\int\frac{\mathrm{d}^4l}{(2\pi)^4}\frac{l^2}{\left[l^2-\Delta\right]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}}\,, \tag{11.6.30}$$

where we've again used m to mark the power of the denominator. As we can see, this integral diverges for $m = 3$ and thus we will need to regularize it. To do so using **Pauli-Villars regularization** we will return to the original expression for the vertex correction and add a massive term Λ to the photon propagator:

$$\frac{1}{(k-p)^2 + i\varepsilon} \to \frac{1}{(k-p)^2 + i\varepsilon} - \frac{1}{(k-p)^2 - \Lambda^2 + i\varepsilon} \,. \tag{11.6.31}$$

Here Λ acts as a very large mass which will not affect the integrand for small k, but will smoothly cutoff when k approaches Λ. We can effectively imagine the second term as the propagator of a fictitious heavy photon. Taking this change into account our numerator stays the same, however Δ in the denominator of the second term changes to:

$$\Delta \to \Delta_\Lambda = -xyq^2 + (1-z)^2 m^2 + z\Lambda^2 \,. \tag{11.6.32}$$

This way our divergent integral from Equation (11.6.30) becomes:

$$\begin{aligned}
\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left(\frac{l^2}{[l^2-\Delta]^3} - \frac{l^2}{[l^2-\Delta_\Lambda]^3}\right) &= \frac{i}{(2\pi)^4}\int_0^\infty \mathrm{d}^4 l_E \left(\frac{l_E^2}{[l_E^2+\Delta]^3} - \frac{l_E^2}{[l_E^2+\Delta_\Lambda]^3}\right) \\
&= \frac{2i\pi^2}{(2\pi)^4}\int_0^\infty \mathrm{d}l_E\, l_E^3 \left(\frac{l_E^2}{[l_E^2+\Delta]^3} - \frac{l_E^2}{[l_E^2+\Delta_\Lambda]^3}\right) \\
&= \frac{i}{(4\pi)^2}\int_0^\infty \mathrm{d}l_E^2 \left(\frac{l_E^4}{[l_E^2+\Delta]^3} - \frac{l_E^4}{[l_E^2+\Delta_\Lambda]^3}\right) \\
&= \frac{i}{(4\pi)^2}\ln\left(\frac{\Delta_\Lambda}{\Delta}\right) ,
\end{aligned} \tag{11.6.33}$$

where we used a integration parameter swap $t = l_E^2$, where $l_E^3 \mathrm{d}l_E = (t\mathrm{d}t)/2$ in the third line. With the addition of Λ the integral that was already convergent is now modified with terms of the order Λ^{-2}, which we can ignore since we're interested in the limit where Λ is large. Now we can finally write the explicit expression for the one-loop vertex correction using the evaluated integrals from Equations (11.6.28) and (11.6.33):

$$\begin{aligned}
\overline{u}(p')\delta\Gamma^\mu(p,p')u(p) = \frac{\alpha}{2\pi}\int_0^1 &\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\delta(x+y+z-1) \\
\times\, \overline{u}(p')\Bigg(&\gamma^\mu\left[\ln\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}\left((1-x)(1-y)q^2 + (1-4z+z^2)m^2\right)\right] \\
&+ \frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta}2m^2 z(1-z)\right]\Bigg)u(p) \,,
\end{aligned} \tag{11.6.34}$$

where we have used the fact that $\alpha = e^2/(4\pi)$ is the fine-structure constant. The form factors $F_1(q^2)$ and $F_2(q^2)$ can then be read off from the above expression as:

$$F_1(q^2) = \int \ldots \left[\ln\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}\left((1-x)(1-y)q^2 + (1-4z+z^2)m^2\right)\right] , \tag{11.6.35}$$

$$F_2(q^2) = \int \ldots \left[\frac{1}{\Delta}2m^2 z(1-z)\right] . \tag{11.6.36}$$

Unfortunately we run into trouble if we want to evaluate $F_1(q^2)$ since we have an **infrared divergence** when $q^2 = 0$. There are ways to deal with this. Peskin introduces a small non-zero mass μ to the denominator of the photon propagator, however I don't think going through that is necessary here. Instead we can evaluate $F_2(q^2)$ at $q^2 = 0$ to obtain the anomalous magnetic moment of the electron, which is unaffected by both the infrared and ultraviolet divergences:

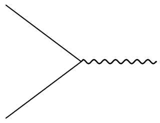
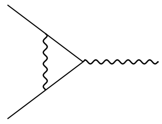
$$\begin{aligned}
F_2(q^2 = 0) &= \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\delta(x+y+z-1)\frac{2m^2 z(1-z)}{(1-z)^2 m^2} \\
&= \frac{\alpha}{\pi}\int_0^1 \mathrm{d}z\int_0^{1-z}\mathrm{d}y\;\frac{z}{1-z} = \frac{\alpha}{\pi}\frac{1}{2} \\
&= \frac{\alpha}{2\pi} \approx 0.00116 \,.
\end{aligned} \tag{11.6.37}$$

This is the famous **anomalous magnetic moment** of the electron, for which Julian Schwinger received the Nobel Prize in Physics in 1965. Schwinger passed away in 1994, however this discovery was so important that it remains engraved on his gravestone.

11.6.2 Renormalization

Again, now that we've regularized the vertex correction we can proceed to renormalize it. The one-loop corrected vertex can be written as:

$$\bar{u}(p')\Gamma^{(3),\mu}(q)u(p) = \bar{u}(p')\left[(-ie)\left\{\gamma^\mu + \delta\Gamma^\mu(q)\right\}\right]u(p) =$$

 +  . (11.6.38)

Using what we've learned from the previous two examples we can write the renormalized vertex by guessing the way we renormalize n-point gamma functions (we will do this explicitly later):

$$\begin{aligned}\Gamma_R^{(3),\mu}(q) &= \sqrt{Z_2}\sqrt{Z_2}\sqrt{Z_3}\Gamma^{(3),\mu}(q) \\ &= \sqrt{Z_2}\sqrt{Z_2}\sqrt{Z_3}\left[(-ie)\left\{\gamma^\mu(1+\delta F_1(q^2)) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right\}\right] ,\end{aligned} \tag{11.6.39}$$

where we used Equation (11.6.12) to rewrite the vertex in terms of the form factors $\delta F_1(q^2)$ and $F_2(q^2)$. Since $\delta F_1(q^2)$ is infinite the bare three-point function $\Gamma^{(3),\mu}(q)$ is also infinite. The renormalized version has to be finite and related to the measurable charge e_R. Therefore we impose a renormalization condition on the vertex which states that the charge at low energies (ie. $q = 0$) has to be the measured charge e_R:

$$\Gamma_R^{(3),\mu}(q=0) = (-ie_R)\gamma^\mu \qquad \frac{e_R^2}{4\pi} = \frac{1}{137} . \tag{11.6.40}$$

This renormalization condition will give us the relation between the bare and renormalized charge which has been missing up until now. From the condition we can derive:

$$\Gamma_R^{(3),\mu}(q=0) = \sqrt{Z_2}\sqrt{Z_2}\sqrt{Z_3}(-ie)\gamma^\mu(1+\delta F_1(0)) , \tag{11.6.41}$$

where $F_2(0) = 0$ as it is not divergent. Lets take a look at the perturbative correction to Z_2. The α-order correction is $\delta Z_2 = (Z_2 - 1)$. Having calculated this before we can explicitly try to evaluate this correction:

$$\begin{aligned}\delta Z_2 &= \left.\frac{\mathrm{d}\Sigma_2}{\mathrm{d}\not{p}}\right|_{\not{p}=m_R} \\ &= \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\left[-x\ln\frac{x\Lambda^2}{(1-x)^2m^2 + x\mu^2} + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2m^2 + x\mu^2}\right] ,\end{aligned} \tag{11.6.42}$$

if we would have gone through the regularization procedure with both μ and Λ. If we were to do this also for $\delta F_1(0)$ we'd arrive at the expression:

$$\delta F_1(0) = \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}z\,(1-z)\left[\ln\frac{z\Lambda^2}{(1-z)^2m^2 + z\mu^2} + \frac{(1-4z+z^2)m^2}{(1-z)^2m^2 + z\mu^2}\right] , \tag{11.6.43}$$

from which it can be shown via integration by parts that:

$$\delta F_1(0) = -\delta Z_2 . \tag{11.6.44}$$

I apologize for the hand-wavy derivation, but fully explaining this would take too long and is not even really a part of this course directly. Simply put, this is another consequence of the Ward-Takahashi identity. Taking this into account we can calculate the following:

$$Z_2(1+\delta F_1(0)) = (1+\delta Z_2)(1+\delta F_1(0)) = 1 + \delta Z_2 + \delta F_1(0) + \mathcal{O}(e^4) = 1 , \tag{11.6.45}$$

with which we can simplify the renormalized vertex to:

$$\Gamma_R^{(3),\mu}(q) = \sqrt{Z_3}(-ie)\gamma^\mu \,. \tag{11.6.46}$$

In literature it is common to denote the vertex correction by Z_1, defined as:

$$Z_1 \equiv \frac{1}{1+\delta F_1(0)} \,, \tag{11.6.47}$$

which is in our case equal to Z_2. This means that the correction to electron legs cancels out the vertex correction and that the remaining renormalization condition for the charge is simply:

$$e_R = \sqrt{Z_3}e \,. \tag{11.6.48}$$

This means that the bare charge e has to be infinite in order to ensure a finite renormalized charge e_R. Now lets call back to the photon self-energy $\Pi_2(q^2)$. With the renormalized charge we can write $\Pi_2(q^2=0)$ as:

$$\Pi_2(q^2=0) = -\frac{2\alpha_R}{6\pi}\left[\frac{2}{\varepsilon_d} - \ln\frac{m_R^2}{M^2} - \gamma_{EM} + \ln(4\pi)\right] \,, \tag{11.6.49}$$

where $\alpha_R = e_R^2/(4\pi)$ is the renormalized fine-structure constant. Likewise, as we've said before, the renormalized photon propagator is then given by Equation (11.6.50), which can be explicitly written as:

$$\Pi_{2,R}(q^2) = \Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha_R}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\ln\frac{m_R^2}{m_R^2 - x(1-x)q^2} \,, \tag{11.6.50}$$

which is now independent of ε_d in the limit $\varepsilon_d \to 0$. This means that the renormalized photon self-energy is finite as we use $\Pi_2(q^2=0)$ to subtract the divergent part.

11.7 Observable Consequences

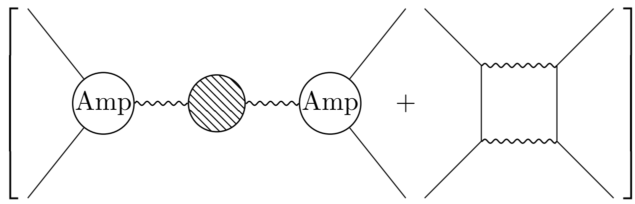
11.7.1 Example: Electron-Muon Coulomb Scattering

Lets take another look at the electron-muon Coulomb scattering process, which we've already discussed earlier (see Section 10.8.2). The process is given by:

$$e^-(p,\, s) + \mu^-(k,\, r) \to e^-(p',\, s') + \mu^-(k',\, r') \,. \tag{11.7.1}$$

There we've already calculated the scattering amplitude in the tree order. Now using our one-loop corrected vertex let's evaluate the scattering amplitude to the order of $\mathcal{O}(e^4)$. The scattering amplitude we want to evaluate is:

$$i\mathcal{M} = \left[\sqrt{Z_2^e Z_2^\mu}\right]^2$$



. (11.7.2)

We'll ignore the second term for now, despite it being of the same order, since it is not relevant for finding the change of the central Coulomb potential static term (which gives a $1/r$ dependence). The first term with one-loop corrections is:

$$\begin{aligned} i\mathcal{M}_1 = &\left[\sqrt{Z_2^e Z_2^\mu}\right]^2 \bar{u}(p')(-ie)\left[\gamma^\mu(1+\delta F_1^e(q^2) - \delta F_1^e(0)) + \delta F_1^e(0) + i\frac{\sigma^{\mu\alpha}q_\alpha}{2m_e}F_2^e(q^2)\right]u(p) \\ &\times \bar{u}(k')(-ie)\left[\gamma^\nu(1+\delta F_1^\mu(q^2) - \delta F_1^\mu(0)) + \delta F_1^\mu(0) + i\frac{\sigma^{\nu\beta}q_\beta}{2m_\mu}F_2^\mu(q^2)\right]u(k) \\ &\times \frac{-ig_{\mu\nu}Z_3}{q^2(1-[\Pi_2(q^2)-\Pi_2(0)])} \,, \end{aligned} \tag{11.7.3}$$

where we've used the corrected vertex and corrected photon propagator. Using, the relations we've derived earlier:

$$Z_2(1+\delta F_1(0)) = 1\,, \tag{11.7.4}$$
$$Z_3 e^2 = e_R^2\,, \tag{11.7.5}$$

we can simplify the amplitude to:

$$\begin{aligned} i\mathcal{M}_1 &= \bar{u}(p')(-ie_R)\left[\gamma^\mu(1+\delta F_1^e(q^2)-\delta F_1^e(0)) + i\frac{\sigma^{\mu\alpha}q_\alpha}{2m_{R_e}}F_2^e(q^2)\right]u(p) \\ &\times \bar{u}(k')(-ie_R)\left[\gamma^\nu(1+\delta F_1^\mu(q^2)-\delta F_1^\mu(0)) + i\frac{\sigma^{\nu\beta}q_\beta}{2m_{R_\mu}}F_2^\mu(q^2)\right]u(k) \\ &\times \frac{-ig_{\mu\nu}}{q^2(1-[\Pi_2(q^2)-\Pi_2(0)])}\,. \end{aligned} \tag{11.7.6}$$

Using the same trick in the non-relativistic static limit where $q^0 \approx 0$, we can use the **Born approximation**, as we did in Section 10.8.3, to extract the Fourier transform of the new effective interaction potential:

$$\tilde{V}(\boldsymbol{q}) = \frac{-ie_R^2}{q^2\left[1-\Pi_{2,R}(q^2)\right]}\,, \tag{11.7.7}$$

which we see has gained a correction term in the denominator compared to our tree order effective potential from Equation (10.8.29) which is called the **Uehling correction**. Remember here that the $-i$ factor is manually removed before the inverse Fourier transform as it is an artifact of bookkeeping. Our potential needs to be real. Earlier we skipped the second box diagram, which if we'd evaluate, we'd come to find the full **Breit potential** [3] which includes spin-orbit and spin-spin interactions alongside the Uehling correction.

11.7.2 Example: Lamb Shift

The **Lamb shift** is a small anomalous energy shift in the energy levels between the $2s_{1/2}$ and $2p_{1/2}$ electron orbitals in the hydrogen atom. We can calculate the expected value of the Lamb shift using the Uehling correction to the Coulomb potential from Equation (11.7.7). It will be practical to write the potential as a series expansion in α_R:

$$\tilde{V}(\boldsymbol{q}) = -\frac{e_R^2}{q^2\left[1-\Pi_{2,R}(q^2)\right]} \simeq -\frac{e_R^2}{q^2}\left[1+\Pi_{2,R}(q^2)+\mathcal{O}(\alpha_R^2)\right]\,. \tag{11.7.8}$$

Note: We've manually removed the i factor from the denominator, as it is an artifact of bookkeeping with Feynman diagrams and changed the sign to get the potential between opposite charges as we expect in the hydrogen atom. Our previous expression in Equation (11.7.7) was derived for two like-charged particles.

Performing an inverse Fourier transform on the Uehling correction term:

$$\delta\tilde{V}(\boldsymbol{q}) = -\frac{e_R^2}{q^2}\Pi_{2,R}(q^2)\,, \tag{11.7.9}$$

yields the familiar Uehling potential [4]:

$$\delta V(\boldsymbol{r}) = -\frac{2\alpha_R^2}{3\pi}\frac{1}{\boldsymbol{r}}\int_1^\infty \mathrm{d}t\, e^{-2m_R(\boldsymbol{r}\cdot t)}\left(1+\frac{1}{2t^2}\right)\frac{\sqrt{t^2-1}}{t^2}\,. \tag{11.7.10}$$

For our purposes $\delta\tilde{V}(\boldsymbol{q})$ is sufficient though. Let us expand the photon self-energy $\Pi_{2,R}(q^2)$ in the non-relativistic static limit $|\boldsymbol{q}|^2 \ll m_R^2$, $q^0 = 0$, as a series in $|\boldsymbol{q}|^2/m_R^2$:

$$\Pi_{2,R}(q^2=-|\boldsymbol{q}|^2) \simeq -\frac{2\alpha_R}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)x(1-x)\frac{|\boldsymbol{q}|^2}{m_R^2} = -A\,\alpha_R\frac{|\boldsymbol{q}|^2}{m_R^2} + \mathcal{O}\left(\frac{|\boldsymbol{q}|^4}{m_R^4}\right)\,, \tag{11.7.11}$$

where A is some proportionality constant and we've made use of the series expansion of the logarithm:

$$\ln\left(1+\frac{|\boldsymbol{q}|^2}{m_R^2}\right) \simeq \frac{|\boldsymbol{q}|^2}{m_R^2} + \ldots \,. \tag{11.7.12}$$

Plugging in this approximation into the momentum-space Uehling potential we get:

$$\delta\tilde{V}(\boldsymbol{q}) \simeq -\frac{e_R^2}{|\boldsymbol{q}|^2} A\,\alpha_R \frac{|\boldsymbol{q}|^2}{m_R^2} = -e_R^2 A \frac{\alpha_R}{m_R^2}\,, \tag{11.7.13}$$

where we see that what we get is a **momentum-independent constant** in momentum space. The inverse Fourier transform of a constant is a delta function. Were we to also properly perform the integral in Equation (11.7.11) such that we explicitly calculate $A = 1/15$, we would find that this coordinate-space potential is:

$$\delta V(\boldsymbol{r}) = -\frac{4}{15}\frac{\alpha_R^2}{m_R^2}\delta^{(3)}(\boldsymbol{r})\,. \tag{11.7.14}$$

Now remember that hydrogen s states (when $l = 0$) have a non-zero probability density at the origin:

$$\psi_{n,0,0}(\boldsymbol{r}) = \frac{1}{\sqrt{\pi a_0^3}}\left(\frac{1}{n}\right)^{3/2} e^{-r/a_0}\,, \tag{11.7.15}$$

where $a_0 = (\alpha_R m_R)^{-1}$ is the Bohr radius. While it's p states (when $l \neq 0$) are suppressed as $r \to 0$. This means that p are not affected by the aforementioned delta function, while s states are. The Lamb shift is the difference in eigenenergies between the $2s_{1/2}$ and $2p_{1/2}$ states, which can be calculated as:

$$\begin{aligned}\Delta E^{\mathrm{U}}_{\mathrm{Lamb}} &= \int \mathrm{d}^3 r\, |\psi_{2,0,0}(\boldsymbol{r})|^2 \delta V(\boldsymbol{r}) = \delta V(0)|\psi(0)|^2 = -\frac{4}{15}\frac{\alpha_R^2}{m_R^2}\frac{\alpha_R^3 m_R^3}{8\pi} \\ &= -1.123\cdot 10^{-7}\ \mathrm{eV}\,,\end{aligned} \tag{11.7.16}$$

where $|\psi_{2,0,0}(0)|^2$:

$$|\psi_{2,0,0}(0)|^2 = \frac{1}{8\pi a_0^3} = \frac{(\alpha_R m_R)^3}{8\pi}\,. \tag{11.7.17}$$

This result only captures the vacuum polarization contribution (Uehling correction) to the Lamb shift which is in general much larger at $\sim 4.4\cdot 10^{-6}$ eV due to the electron self-energy which dominates.

11.7.3 Example: Running Coupling Constant

Another interesting consequence of the one-loop vertex correction is the **running coupling constant** which is effectively the same story as with the Lamb shift except where we reinterpret the Uehling correction as a renormalization of the charge itself. We can expand the logarithm term in the photon self-energy (vacuum polarization) from Equation (11.6.50), this time in the limit of highly virtual photons, ie. when $|\boldsymbol{q}|^2 \gg m_R^2$ as a series in $|\boldsymbol{q}|^2/m_R^2$:

$$\ln\frac{m_R - x(1-x)q^2}{m_R} \simeq \ln\left(x(1-x)\right) + \ln\left(-\frac{q^2}{m_R^2}\right) + \mathcal{O}\left(\frac{q^4}{m_R^4}\right)\,. \tag{11.7.18}$$

Thus if we evaluate $\Pi_{2,R}(q^2)$ in this limit we get:

$$\begin{aligned}\Pi_{2,R}(q^2) &= \frac{2\alpha_R}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\left[\ln\left(x(1-x)\right) + \ln\left(-\frac{q^2}{m_R^2}\right)\right] \\ &\approx \frac{\alpha_R}{3\pi}\left[\ln\left(-\frac{q^2}{m_R^2}\right) - \frac{5}{3}\right]\,.\end{aligned} \tag{11.7.19}$$

Plugging this approximation into the corrected effective potential from Equation (11.7.7) we get:

$$\tilde{V}(\boldsymbol{q}) = \frac{|e_R|^2}{q^2\left[1 - \frac{2\alpha_R}{6\pi}\ln\left(-\frac{q^2}{m_R^2}\right)\right]}\,, \tag{11.7.20}$$

where we can interpret this as the renormalized charge e_R being dependent on the virtuality of the photon q^2. Thus the effective coupling constant is q^2-dependent:

$$e_R^{\mathrm{eff}\,2}(q^2) = \frac{|e_R|^2}{1 - \frac{2\alpha_R}{6\pi}\ln\left(-\frac{q^2}{m_R^2}\right)} \,. \tag{11.7.21}$$

We can easily see that for an entirely real photon, ie. $q^2 = 0$, the effective coupling constant is exactly the renormalized charge $e_R^{\mathrm{eff}}(0) = e_R$. Physically the running coupling constant reflects vacuum polarization effects, where virtual electron-positron pairs screen the bare charge of the electron. At low photon virtualities ($|\boldsymbol{q}|^2 \to 0$, ie. long wavelengths) the screening is maximal and the effective coupling constant is at its lowest value, which is the renormalized charge e_R. As the virtuality of the photon increases (ie. shorter wavelengths, higher energies) , the photon probes deeper into the vacuum polarization cloud, which reduces the screening effect and causes the effective coupling constant to increase.

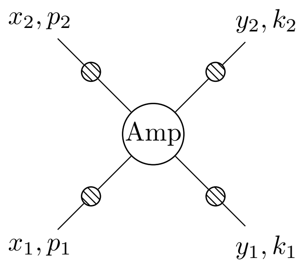
11.8 Renormalization of n-point Gamma Function

11.8.1 For Scalar Fields

Using what we've learned from the previous examples we can now generalize the renormalization procedure to any n-point gamma function $\Gamma^{(n)}$ with the help of the LSZ reduction formula which we discussed earlier (see Section 9.3.2). We want to prove that a n-point gamma function for scalar fields is renormalized as:

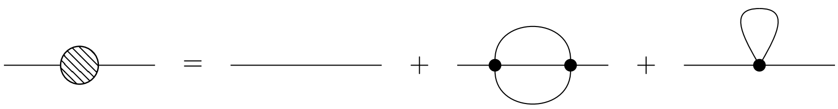
$$\Gamma^{(n)}_{\phi,R}(p_1,\ldots,p_n) = Z^{n/2}\Gamma^{(n)}_{\phi}(p_1,\ldots,p_n) \,, \tag{11.8.1}$$

where Z is the scalar field renormalization constant. This follows because each external scalar leg in the LSZ reduction formula contributes a factor of $\sqrt{Z}$ to the renormalized correlation function. Equivalently this can be seen by defining the renormalized field as $\phi_R = Z^{-1/2}\phi$ and then having $\Gamma^{(n)}_{\phi,R}$ be the n-point function of the renormalized field ϕ_R. In fact, we've practically already proven how scalar fields are renormalized in the example after our derivation of the LSZ procedure (see Section 9.3.3). Let us diagramatically represent the left-hand side of the LSZ reduction formula for a 4-point correlation function:



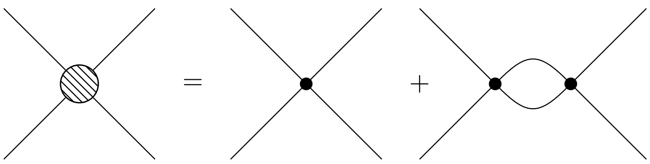
$$= C^{(4)}(x_1,x_2,y_1,y_2) \,. \tag{11.8.2}$$

Here the blobs on the external legs represent all possible self-energy corrections to the external legs:



$$+ \cdots = \frac{i\sqrt{Z}}{p^2 - m_R^2 + i\varepsilon} \,, \tag{11.8.3}$$

and the center blob represents the 4-point function $\Gamma^{(4)}_{\phi}$:



$$+ \cdots = \Gamma^{(4)}_{\phi} \,. \tag{11.8.4}$$

Now lets recall the result from the LSZ reduction formula for the 4-point correlation function (see Equation (9.3.28)):

$$\tilde{C}^{(4)}(p_1,p_2;k_1,k_2) = \frac{i\sqrt{Z}}{p_1^2 - m_R^2 + i\varepsilon}\frac{i\sqrt{Z}}{p_2^2 - m_R^2 + i\varepsilon}\langle \boldsymbol{p}_1,\boldsymbol{p}_2|\hat{S}|\boldsymbol{k}_1,\boldsymbol{k}_2\rangle\frac{i\sqrt{Z}}{k_1^2 - m_R^2 + i\varepsilon}\frac{i\sqrt{Z}}{k_2^2 - m_R^2 + i\varepsilon} \,, \tag{11.8.5}$$

where the S-matrix element is given by:

$$\begin{aligned}\langle \boldsymbol{p}_1, \boldsymbol{p}_2|\hat{S}|\boldsymbol{k}_1, \boldsymbol{k}_2\rangle &= (2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)i\mathcal{M}(p_1,p_2\to k_1,k_2)\\ &= (2\pi)^4\delta^{(4)}(p_1+p_2-k_1-k_2)i\Gamma^{(4)}_\phi(p_1,p_2;k_1,k_2)\,.\end{aligned} \tag{11.8.6}$$

Here $\mathcal{M}$ is the invariant matrix element for the $2\to 2$ scattering process. At tree level $\mathcal{M}=\Gamma^{(4)}_\phi$ but beyond that, they are not the same. The invariant matrix element $\mathcal{M}$ is actually the **sum of all amputated connected** diagrams, which includes chains of 1PI vertices connected by internal propagators:

$$\mathcal{M}=\Gamma^{(4)}_{\text{1PI}}+(\text{diagrams built from 1PI vertices connected by propagators})\,. \tag{11.8.7}$$

On the other hand, $\Gamma^{(4)}_\phi$ contains only the **1PI diagrams**, without any external propagators attached:

$$\Gamma^{(4)}_\phi=\Gamma^{(4)}_{\text{1PI}}=(\text{1PI diagrams only, no external propagators attached})\,. \tag{11.8.8}$$

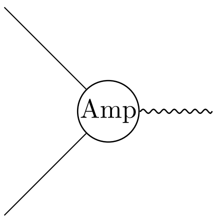
LSZ amputation ensures that the external propagators are removed and that the poles are put on-shell. Consequently the renormalized 1PI vertex function is related to the bare 1PI vertex function by:

$$\Gamma^{(4)}_{\phi,R}(p_1,p_2;k_1,k_2)=Z^2\Gamma^{(4)}_\phi(p_1,p_2;k_1,k_2)\,. \tag{11.8.9}$$

To summarize: $\mathcal{M}\neq\Gamma^{(4)}_\phi$ beyond tree level but LSZ amputation ensures that the two relate correctly.

11.8.2 In General

I just wanted to quickly note, without proof, that renormalization of n-point gamma functions for other fields follow the same logic. Each external leg contributes a factor of $\sqrt{Z}$, where Z is the renormalization constant for that particular field. So as we saw earlier for QED in Equation (11.6.39) the renormalization of the QED vertex function up to one-loop is:



$$=\Gamma^{(3)}_R(p,p';q)=\sqrt{Z_2}\sqrt{Z_2}\sqrt{Z_3}\Gamma^{(3)}(p,p';q)\,, \tag{11.8.10}$$

where each $\sqrt{Z_2}$ comes from the electron legs, and hence contains Z_2 the electron field-strength renormalization constant, and $\sqrt{Z_3}$ comes from the photon leg, which contains Z_3 the electromagnetic field-strength renormalization constant. This pattern holds at higher loop orders as well, so that the renormalized vertex function is always related to the bare one by the product of such factors.

11.9 QED Renormalization via Counterterms

11.9.1 Introduction of Counterterms

Earlier we discussed how to renormalize QED at the one-loop level directly by absorbing the divergences into redefined parameters and fields. However a more systematic and formal approach to renormalization is to introduce **counterterms** into the Lagrangian. These counterterms are designed to cancel the divergences that arise in loop calculations, ensuring that physical observables remain finite. Let's illustrate this procedure using QED as an example. The bare QED Lagrangian is given by:

$$\begin{aligned}\mathcal{L}_{\text{QED}}&=\bar\psi(i\gamma^\mu D_\mu-m)\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ &=\bar\psi\left(i\gamma^\mu\partial_\mu-m\right)\psi-e\bar\psi\gamma^\mu A_\mu\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,,\end{aligned} \tag{11.9.1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. To renormalize this Lagrangian we introduce renormalization constants for the fields which contain δ-terms that will act as counterterms, they are chosen to exactly cancel the divergences that arise in loop diagrams:

$$Z_1 = 1 + \delta_1 \,, \tag{11.9.2}$$
$$Z_2 = 1 + \delta_2 \,, \tag{11.9.3}$$
$$Z_3 = 1 + \delta_3 \,. \tag{11.9.4}$$

We also introduce a mass counterterm δm to renormalize the electron mass:

$$Z_2 m = m_R + \delta m \,. \tag{11.9.5}$$

Now we can rewrite the QED Lagrangian in terms of renormalized fields and parameters:

$$\begin{aligned}\mathcal{L}^{\text{ren}}_{\text{QED}} &= Z_2\bar{\psi}_R(i\gamma^\mu\partial_\mu - m)\psi_R - Z_2\sqrt{Z_3}\bar{\psi}_R\gamma^\mu A^R_\mu\psi_R - \frac{Z_3}{4}F_{R,\mu\nu}F^{\mu\nu}_R \\ &= \bar{\psi}_R(i\gamma^\mu\partial_\mu - m_R)\psi_R - e_R\bar{\psi}_R\gamma^\mu A^R_\mu\psi_R - \frac{1}{4}F^R_{\mu\nu}F^{\mu\nu}_R \\ &\quad + \bar{\psi}_R(i\delta_2\gamma^\mu\partial_\mu - \delta_m)\psi_R - \delta_1 e_R\bar{\psi}_R\gamma^\mu A^R_\mu\psi_R - \frac{\delta_3}{4}F_{R,\mu\nu}F^{\mu\nu}_R \,,\end{aligned} \tag{11.9.6}$$

where the renormalized fields and parameters are defined as:

$$\psi_R = Z_2^{-1/2}\psi \,, \tag{11.9.7}$$
$$A^R_\mu = Z_3^{-1/2}A_\mu \,, \tag{11.9.8}$$
$$e_R = Z_1^{-1}Z_2 Z_3^{1/2}e \,, \tag{11.9.9}$$
$$m_R = Z_2 m - \delta m \,. \tag{11.9.10}$$

This ensures that m_R is finite and corresponds to the physical electron mass and that e_R is the finite physical charge.

11.9.2 Electron Self-Energy Counterterms (δ_2, δ_m)

For the electron, the relevant Lagrangian part is:

$$\mathcal{L}_e = \bar{\psi}_R(i\gamma^\mu\partial_\mu - m_R)\psi_R + \bar{\psi}_R(i\delta_2\gamma^\mu\partial_\mu - \delta_m)\psi_R \,. \tag{11.9.11}$$

The first term gives us the renormalized free electron propagator, while the second term acts as a vertex insertion into the Feynman diagrams which is automatically 1PI since it cannot be disconnected by cutting a single internal line. 1PI electron self-energy $\Sigma(p)$ at one-loop order is defined as the sum of all 1PI diagrams with two external fermion legs, including the counterterm insertion:

$$-i\Sigma(p) = -i\Sigma_2(p) + i\delta_2(i\delta_2\gamma^\mu\partial_\mu - \delta_m) \,, \tag{11.9.12}$$

From this we can derive the renormalized electron propagator by resuming the geometric series of self-energy insertions:

$$\begin{aligned}\Delta^{\text{1-loop}}_F(p) &= \Delta_F(p) + \Delta_F(p)i\Sigma(p)\Delta_F(p) + \Delta_F(p)(-i\Sigma(p))\Delta_F(p)(-i\Sigma(p))\Delta_F(p) + \dots \\ &= \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}(-i\Sigma)\frac{i}{\not{p} - m} + \frac{i}{\not{p} - m}(-i\Sigma)\frac{i}{\not{p} - m}(-i\Sigma)\frac{i}{\not{p} - m} + \dots \\ &= \frac{i}{\not{p} - m}\sum_{n=0}^{\infty}\left[(-i\Sigma)\frac{i}{\not{p} - m}\right]^n \\ &= \frac{i}{\not{p} - m}\frac{1}{1 + \frac{\Sigma(p)}{\not{p} - m}} \\ &= \frac{i}{\not{p} - m - \Sigma(p, m)} \,,\end{aligned} \tag{11.9.13}$$

where each denominator technically also contains an $i\varepsilon$ term, which we've omitted for clarity. This is identical to the expression we had earlier in Equation (11.5.16) except now $\Sigma(p)$ includes the counterterms. Now we can impose the renormalization conditions for on-shell renormalization. We want a pole at the physical mass $\not{p} = m_R$ which has a residue of 1:

$$\not{p} = m_R : \quad \delta_m = \Sigma_2(\not{p} = m_R) \,, \tag{11.9.14}$$

$$\text{Res} = 1 : \quad \delta_2 = \left. \frac{\partial \Sigma_2(\not{p})}{\partial \not{p}} \right|_{\not{p}=m_R} . \tag{11.9.15}$$

11.9.3 Photon Self-Energy Counterterm (δ_3)

The relevant Lagrangian part for the photon is:

$$\mathcal{L}_\gamma = -\frac{1}{4} F^R_{\mu\nu} F^{\mu\nu}_R - \frac{\delta_3}{4} F^R_{\mu\nu} F^{\mu\nu}_R \,. \tag{11.9.16}$$

The first term gives us the renormalized free photon propagator. The second term acts as a photon two-point vertex insertion, which is again automatically 1PI. The 1PI photon self-energy $\Pi_{\mu\nu}(q^2)$ is then:

$$i\Pi_{\mu\nu}(q^2) = i\Pi_{2,\mu\nu}(q^2) + i\delta_3(q^2 g_{\mu\nu} - q_\mu q_\nu) \,, \tag{11.9.17}$$

where $\Pi_{2,\mu\nu}(q^2)$ is the one-loop photon self-energy. In the Feynman gauge the self-energy is:

$$\Pi_{\mu\nu,R}(q^2) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_{2,R}(q^2) = (q^2 g_{\mu\nu} - q_\mu q_\nu) \left[\Pi_2(q^2) - \delta_3\right] . \tag{11.9.18}$$

where $\Pi_2(q^2)$ is the scalar part of the unrenormalized one-loop photon self-energy. It follows then that:

$$\Pi_{2,R}(q^2) = \Pi_2(q^2) - \delta_3 \,. \tag{11.9.19}$$

We will borrow the result for the corrected photon propagator from Equation (11.4.36) with which the full renormalized photon propagator at one-loop order is:

$$D^{\text{1-loop}}_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2\left[1 - \Pi_{2,R}(q^2)\right] + i\varepsilon} = \frac{-ig_{\mu\nu}}{q^2\left[1 - \Pi_2(q^2) + \delta_3\right] + i\varepsilon} \,. \tag{11.9.20}$$

We fix the value of the counterterm δ_3 by imposing the renormalization condition that the photon remains massless, ie. that there is a pole at $q^2 = 0$ and that the residue of this pole is 1. Near the pole we can expand $\Pi_2(q^2)$ as a Taylor series:

$$\left. \Pi_2(q^2) \right|_{q^2 \to 0} = \Pi_2(0) + q^2 \Pi_2'(0) + \mathcal{O}(q^4) \,, \tag{11.9.21}$$

where $\Pi_2'(0)$ denotes the derivative of the scalar part of the photon self-energy w.r.t. q^2 as:

$$\Pi_2'(0) \equiv \left. \frac{\mathrm{d}\Pi_2(q^2)}{\mathrm{d}q^2} \right|_{q^2=0} . \tag{11.9.22}$$

The two renormalization conditions are then:

$$q^2 = 0 : \quad \Pi_2(0) - \delta_3 = 0 \quad \Rightarrow \quad \delta_3 = \Pi_2(0) \,, \tag{11.9.23}$$

$$\text{Res} = 1 : \quad \left. \frac{\partial}{\partial q^2} D^{\text{1-loop}\,-1}_{\mu\nu}(q) \right|_{q^2=0} = -ig_{\mu\nu} \quad \Rightarrow \quad \text{Satisfied Automatically} \,. \tag{11.9.24}$$

The second condition is very non-trivial and stems from the residue requirement which we can explicitly evaluate as:

$$\text{Res}[D^{\text{1-loop}}_{\mu\nu}(q)] = \lim_{q^2 \to 0} q^2 D^{\text{1-loop}}_{\mu\nu}(q) = \lim_{q^2 \to 0} \frac{-ig_{\mu\nu} q^2}{q^2\left[1 - \Pi_2(q^2) + \delta_3\right] + i\varepsilon} = -ig_{\mu\nu} \,, \tag{11.9.25}$$

where the square bracket in the denominator can be expanded using Equations (11.9.21) and (11.9.23) as:

$$\lim_{q^2 \to 0} \left[1 - \Pi_2(q^2) + \delta_3\right] = \left[1 - \Pi_2(0) + \delta_3 + \mathcal{O}(q^4)\right] = 1 + \mathcal{O}(q^4) \,, \tag{11.9.26}$$

and we've dropped $q^2 \Pi_2'(0)$ since it is $\mathcal{O}(q^4)$.

11.9.4 Vertex Counterterm (δ_1)

The renormalized QED vertex function is given by:

$$\Gamma^\mu_R(p,p') = \Gamma^\mu(p,p') + \delta_1\gamma^\mu \,, \tag{11.9.27}$$

where $\Gamma^\mu(p,p')$ is the one-loop vertex function without the counterterm. The renormalization condition we want to impose is that at zero momentum transfer ($q^2 = 0$) the vertex function reduces to the tree-level vertex:

$$\Gamma^\mu_R(p,p')\Big|_{q^2=0} = (-ie_R)\gamma^\mu \,. \tag{11.9.28}$$

We will borrow the result of our QED vertex decomposition from Equation (11.6.12). We see that

$$\Gamma^\mu(q^2=0) = -ie_R\gamma^\mu F_1(0) \,. \tag{11.9.29}$$

We can use the condition above to solve for the counterterm δ_1:

$$-ie_R\gamma^\mu = \Gamma^\mu(q^2=0) + \delta_1\gamma^\mu = -ie_R\gamma^\mu F_1(0) + \delta_1\gamma^\mu \,, \tag{11.9.30}$$

which yields:

$$\delta_1 = -\left[ie_R - ieF_1(0)\right] = ie_R\delta F_1(0) \,, \tag{11.9.31}$$

where $F_1(0) = 1 + \delta F_1(0)$ is the form factor at zero momentum transfer from Equation (11.6.43). At the one-loop level Γ^μ matches $\partial\Sigma_2/\partial\not{p}$ at $\not{p} = m_R$ due to the Ward identity (see Section 10.3), which means that $\delta_1 = \delta_2$.

12 Quantization with Feynman Path Integrals

12.1 Short Introduction

So far our entire approach to quantum field theory has been based on the canonical quantization of fields. This approach is very powerful, but it has its limitations. In this section we will introduce an alternative approach to quantization, which is based on the Feynman path integral formulation of quantum mechanics. The key idea in quantization with path integrals is that instead of promoting fields to operators as in canonical quantization, we encode quantum dynamics directly through a sum of all possible configurations weighted by the exponential of the action by means of a functional integral. In this formulation, time-ordered operator products are represented as insertions of classical field functions inside the functional integral. This approach is particularly useful for theories which cannot be solved perturbatively, such as Quantum Chromodynamics (QCD). However this does not mean that we can calculate such theories by hand, but rather that we can use numerical methods to approximate the path integral.

Since the path integral can be seen as a sum over all possible field configurations, there exists a natural connection to Statistical Physics. This analogy follows strictly after Wick rotation where transform from Minkowski to Euclidean spacetime. In fact, the path integral can be seen as a partition function of a statistical system after a Wick rotation, where the fields are the degrees of freedom of the system. This connection allows us to use powerful numerical methods from Statistical Physics, such as Monte Carlo methods, to approximate the path integral, which is excellent in the case of non-perturbative theories.

12.2 Feynman Path Integral for Scalar Fields

The first step is to work in the eigenbasis of the field operators. In this basis, the field operator $\hat{\phi}(\boldsymbol{x},t)$ acts by multiplication with its eigenvalue:

$$\hat{\phi}(\boldsymbol{x},t)|\phi,t\rangle = \phi(\boldsymbol{x},t)|\phi,t\rangle \,, \tag{12.2.1}$$

where $\phi(\boldsymbol{x},t)$ is the eigenvalue. These eigenstates need to satisfy orthogonality via a functional version of the usual orthogonality relation:

$$\langle\phi',t|\phi,t\rangle = \prod_{\boldsymbol{x}} \delta\left(\phi'(\boldsymbol{x},t) - \phi(\boldsymbol{x},t)\right) \,. \tag{12.2.2}$$

Now if we imagine that we discretize spacetime into a lattice of hypercubes. We want to find the field theory analogue of the well-known completeness relation from quantum mechanics:

$$\int \mathrm{d}x\, |x\rangle\langle x| = 1\,. \tag{12.2.3}$$

In the case of fields, we have to integrate over all possible field configurations $\phi(\boldsymbol{x},t)$ instead of just over a single coordinate x. On a spatial lattice with N sites, a field configuration is specified by N values of the field $\{\phi(\boldsymbol{x}_1,t),\phi(\boldsymbol{x}_2,t),\ldots,\phi(\boldsymbol{x}_N,t)\}$, where $\boldsymbol{x}_i$ are the lattice points. The field theory analogue of the completeness relation then takes the form of a product of integrals over each of these field values:

$$\mathbb{1}(t) = \prod_{\boldsymbol{x}} \int_{-\infty}^{\infty} \mathrm{d}\phi(\boldsymbol{x},t)|\phi,t\rangle\langle\phi,t|\,. \tag{12.2.4}$$

In the continuum limit, where $N\to\infty$ and the lattice spacing goes to zero, this product of ordinary integrals becomes a **functional integral**, which we denote as:

$$\mathbb{1}(t) = \int \mathcal{D}\phi(\boldsymbol{x},t)|\phi,t\rangle\langle\phi,t|\,, \tag{12.2.5}$$

where $\mathcal{D}\phi(\boldsymbol{x},t)$ denotes the functional measure ie. integration over all possible field configurations $\phi(\boldsymbol{x},t)$ at a fixed time t. The full path integral is defined as the **transition amplitude** between the initial and final states:

$$\langle\phi_f,t_f|\phi_i,t_i\rangle = \langle\phi_f|U(t_f,t_i)|\phi_i\rangle\,, \tag{12.2.6}$$

where $U(t_f,t_i)=\exp(-iH(t_f-t_i))$ is the time evolution operator. Now since we've discretized spacetime we need to apply time-slicing to this evolution. Splitting the interval $[t_i,t_f]$ into N_t slices of equal length $\Delta t=(t_f-t_i)/N_t$, we can write the transition amplitude as:

$$\langle\phi_f|U(t_f,t_i)|\phi_i\rangle = \langle\phi_f|\left[e^{-iH\Delta t}\right]^N|\phi_i\rangle\,. \tag{12.2.7}$$

Next we insert the field completeness relation from Equation (12.2.5) at each time slice:

$$\langle\phi_f|U(t_f,t_i)|\phi_i\rangle = \int \prod_{k=1}^{N-1}\mathcal{D}\phi_k \prod_{j=0}^{N-1}\langle\phi_{j+1}|e^{-iH\Delta t}|\phi_j\rangle\,, \tag{12.2.8}$$

where $\phi_0=\phi_i$ and $\phi_N=\phi_f$. Next we need to split the Hamiltonian into two parts via a Trotter decomposition:

$$e^{-iH\Delta t} \approx e^{-iH_\pi\Delta t}e^{-iH_\phi\Delta t} + \mathcal{O}(\Delta t^2)\,, \tag{12.2.9}$$

where H_π is the part of the Hamiltonian that depends on the square of the conjugate momentum $\pi(\boldsymbol{x},t)$ and H_ϕ is the part that depends on the field $\phi(\boldsymbol{x},t)$. For a real scalar field, we have:

$$H_\pi = \frac{1}{2}\int \mathrm{d}^3x\,\hat{\pi}^2(\boldsymbol{x},t)\,, \tag{12.2.10}$$

$$H_\phi = \int \mathrm{d}^3x\,\mathcal{H}_\phi(\phi(\boldsymbol{x},t)) = \int \mathrm{d}^3x\left[\frac{1}{2}(\nabla\hat{\phi}(\boldsymbol{x},t))^2 + \frac{1}{2}m^2\hat{\phi}^2(\boldsymbol{x},t) + V(\hat{\phi}(\boldsymbol{x},t))\right]\,. \tag{12.2.11}$$

Next we insert the momentum completeness relation (analogue of (12.2.5)) for conjugate momenta eigenstates:

$$\mathbb{1} = \int \mathcal{D}\pi_k(\boldsymbol{x},t)|\pi_k,t\rangle\langle\pi_k,t|\,, \tag{12.2.12}$$

in between the two exponentials in the Trotter decomposition, such that each exponential acts in its own eigenbasis:

$$\langle\phi_{j+1}|e^{-iH\Delta t}|\phi_j\rangle \approx \int \mathcal{D}\pi_j\langle\phi_{j+1}|e^{-iH_\pi\Delta t}|\pi_j\rangle\langle\pi_j|e^{-iH_\phi\Delta t}|\phi_j\rangle\,. \tag{12.2.13}$$

Because of diagonality in the eigenbasis, we can easily evaluate the two matrix elements:

$$\langle\phi_{j+1}|e^{-iH_\pi\Delta t}|\pi_j\rangle = \exp\left\{-\frac{i\Delta t}{2}\int \mathrm{d}^3x\,\pi_j^2\right\}\langle\phi_{j+1}|\pi_j\rangle\,, \tag{12.2.14}$$

$$\langle\pi_j|e^{-iH_\phi\Delta t}|\phi_j\rangle = \exp\left\{-i\Delta t\int \mathrm{d}^3x\,\mathcal{H}_\phi(\phi_j)\right\}\langle\pi_j|\phi_j\rangle\,. \tag{12.2.15}$$

What remains is to evaluate the overlaps $\langle\phi|\pi\rangle$ and $\langle\pi|\phi\rangle$. From canonical quantization we know that the field and its conjugate momentum satisfy the commutation relation:

$$[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}) \,, \tag{12.2.16}$$

and of course that operators work on their eigenstates as multiplication with the eigenvalue:

$$\hat{\phi}(\boldsymbol{x})|\phi\rangle = \phi(\boldsymbol{x})|\phi\rangle \,, \tag{12.2.17}$$
$$\hat{\pi}(\boldsymbol{x})|\pi\rangle = \pi(\boldsymbol{x})|\pi\rangle \,. \tag{12.2.18}$$

From this it holds that:

$$\langle\phi|\pi\rangle = A_{\text{norm}} \exp\left\{i\int \mathrm{d}^3x \; \pi(\boldsymbol{x})\phi(\boldsymbol{x})\right\} \,, \tag{12.2.19}$$

where A_{norm} is some field-independent normalization constant. Using this result we can combine the two overlaps into:

$$\langle\phi_{j+1}|\pi_j\rangle\langle\pi_j|\phi_j\rangle \propto \exp\left\{i\int \mathrm{d}^3x \; \pi_j(\boldsymbol{x})\left(\phi_{j+1}(\boldsymbol{x}) - \phi_j(\boldsymbol{x})\right)\right\} \,. \tag{12.2.20}$$

Then when we put everything together, we find that the matrix element can be written as:

$$\langle\phi_{j+1}|e^{-iH\Delta t}|\phi_j\rangle \approx \int \mathcal{D}\pi_j \exp\left\{i\int \mathrm{d}^3x \; \left[\pi_j\frac{\phi_{j+1} - \phi_j}{\Delta t} - \left(\frac{1}{2}\pi_j^2 + \mathcal{H}_\phi(\phi_j)\right)\Delta t\right]\right\} \,. \tag{12.2.21}$$

Now to evaluate the continuum limit when $\Delta \to 0$, we need to perform the functional integral over π_j. This is a Gaussian functional integral, which we can evaluate point-wise in coordinate space:

$$\frac{1}{2}\pi_j^2 - \pi_j\frac{\phi_{j+1} - \phi_j}{\Delta t} = \frac{1}{2}(\pi_j - v_j)^2 - \frac{1}{2}v_j^2 \,, \tag{12.2.22}$$

where $v_j = (\phi_{j+1} - \phi_j)/\Delta t$, such that we complete the square. From this we see that the functional integral over π_j reduces to:

$$\int \mathcal{D}\pi_j \exp\left\{i\int \mathrm{d}^3x \; \left[-\frac{1}{2}(\pi_j - v_j)^2\Delta t\right]\right\} \propto \exp\left\{i\int \mathrm{d}^3x \; \frac{1}{2}v_j^2\Delta t\right\} \,. \tag{12.2.23}$$

With this our matrix element from Equation (12.2.21) becomes:

$$\langle\phi_{j+1}|e^{-iH\Delta t}|\phi_j\rangle \propto \exp\left\{i\int \mathrm{d}^3x \; \Delta t\left[\frac{1}{2}\left(\frac{\phi_{j+1} - \phi_j}{\Delta t}\right)^2 - \mathcal{H}_\phi(\phi_j)\right]\right\} \,. \tag{12.2.24}$$

Now if we multiply over slices $j = 0, \ldots, N-1$ and recall that $\phi_0 = \phi_i$ and $\phi_N = \phi_f$, we find that the transition amplitude can be written as:

$$\langle\phi_f|U(t_f, t_i)|\phi_i\rangle = A'_{\text{norm}}\int\prod_{k=1}^{N-1}\mathcal{D}\phi_k \exp\left\{i\sum_{j=0}^{N-1}\Delta t\int \mathrm{d}^3x \; \left[\frac{1}{2}\left(\frac{\phi_{j+1} - \phi_j}{\Delta t}\right)^2 - \mathcal{H}_\phi(\phi_j)\right]\right\} \,. \tag{12.2.25}$$

Finally we can take the continuum limit $\Delta t \to 0$ and $N \to \infty$, with that:

$$\frac{\phi_{j+1} - \phi_j}{\Delta t} \quad \to \quad \partial_t\phi(\boldsymbol{x}, t) \,, \tag{12.2.26}$$
$$\sum_{j=0}^{N-1}\Delta t \quad \to \quad \int_{t_i}^{t_f}\mathrm{d}t \,, \tag{12.2.27}$$

giving us the following for the transition amplitude:

$$\langle\phi_f, t_f|\phi_i, t_i\rangle = A'_{\text{norm}}\int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f}\mathcal{D}\phi \; \exp\left\{i\int_{t_i}^{t_f}\mathrm{d}t\int \mathrm{d}^3x \; \left[\frac{1}{2}\left(\partial_t\phi(\boldsymbol{x}, t)\right)^2 - \mathcal{H}_\phi(\phi(\boldsymbol{x}, t))\right]\right\} \,. \tag{12.2.28}$$

We can identify the Lagrangian density in the square brackets:

$$\mathcal{L} = \frac{1}{2}\left(\partial_t \phi(\boldsymbol{x}, t)\right)^2 - \mathcal{H}_\phi(\phi(\boldsymbol{x}, t)) , \tag{12.2.29}$$

from which we can recover the **standard action**:

$$S[\phi] = \int \mathrm{d}^4x \; \mathcal{L}(\phi, \partial_t\phi) = \int \mathrm{d}^4x \; \left[\frac{1}{2}\left(\partial_t\phi\right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)\right] . \tag{12.2.30}$$

Hence we see that the transition amplitude is given by the **Feynman path integral**:

$$\langle \phi_f, t_f | \phi_i, t_i \rangle = A'_{\text{norm}} \int_{\phi(t_i)=\phi_i}^{\phi(t_f)=\phi_f} \mathcal{D}\phi \; \exp\left\{iS[\phi]\right\} , \tag{12.2.31}$$

where A'_{norm} is a normalization constant which comes from the product of Gaussian integrals. Generally it will cancel out in normalized correlators.

12.3 Correlation Functions

The two-point correlation function is defined as the transition amplitude between two field configurations at different times with a time-ordered product inserted in between:

$$\langle \phi_B, t_B | e^{-iHt_B} \mathcal{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\} e^{iHt_A} | \phi_A, t_A \rangle . \tag{12.3.1}$$

Assuming $t_A < t_1$ and $t_2 < t_B$, we can write this as:

$$\langle \phi_B, t_B | e^{-iHt_B} \hat{\phi}(x_1)\hat{\phi}(x_2) e^{iHt_A} | \phi_A, t_A \rangle = \int_{\phi(t_A)=\phi_A}^{\phi(t_B)=\phi_B} \mathcal{D}\phi \; \phi(x_1)\phi(x_2) e^{iS[\phi]} . \tag{12.3.2}$$

Time-ordering is automatic because the path integral builds the fields in time order. To obtain the vacuum two-point correlation function we project onto the vacuum by sending the boundary times to asymptotic past and future with the usual $i\varepsilon$ prescription to avoid contributions from excited states:

$$\langle \Omega | \mathcal{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\} | \Omega \rangle = \lim_{\substack{t_A \to -\infty(1-i\varepsilon) \\ t_B \to +\infty(1+i\varepsilon)}} \frac{\int_{\phi(t_A)=\phi_A}^{\phi(t_B)=\phi_B} \mathcal{D}\phi \; \phi(x_1)\phi(x_2) e^{iS[\phi]}}{\int_{\phi(t_A)=\phi_A}^{\phi(t_B)=\phi_B} \mathcal{D}\phi \; e^{iS[\phi]}} . \tag{12.3.3}$$

Since both the numerator and the denominator include the same asymptotic vacuum projection factors, the result is independent of the particular fixed boundary conditions ϕ_A and ϕ_B, so long as they are non-orthogonal to the vacuum and the $i\varepsilon$ prescription is used. In compact notation then we can write:

$$\langle \Omega | \mathcal{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\} | \Omega \rangle = \frac{\int \mathcal{D}\phi \; \phi(x_1)\phi(x_2) e^{iS[\phi]}}{\int \mathcal{D}\phi \; e^{iS[\phi]}} = \frac{1}{Z[0]} \int \mathcal{D}\phi \; \phi(x_1)\phi(x_2) e^{iS[\phi]} , \tag{12.3.4}$$

where $Z[0]$ is the partition function without sources, defined as:

$$Z[0] = \int \mathcal{D}\phi \; e^{iS[\phi]} . \tag{12.3.5}$$

We properly derive correlation functions and generating functionals in path integral formalism in the subject *Gauge Field Theory* with prof. Kamenik. Just as a quick illustration however, path integral formalism allows us to compute time-ordered correlation functions of fields in a very straightforward manner. To generate correlation functions, we introduce a classical source term $J(x)$ which couples linearly to the field $\phi(x)$ in the action. With this addition we define the **generating functional** as:

$$Z[J] = \int \mathcal{D}\phi \; \exp\left\{ i \int \mathrm{d}^4x \; \left[\mathcal{L}(\phi, \partial_t\phi) + J(x)\phi(x)\right] \right\} . \tag{12.3.6}$$

From this generating functional we can obtain time-ordered correlation functions by taking **functional derivatives** with respect to the source $J(x)$:

$$\langle \Omega | \mathcal{T}\left\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\right\} | \Omega \rangle = \frac{1}{Z[0]} \frac{\delta^n Z[J]}{i^n \delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\bigg|_{J=0} . \tag{12.3.7}$$

12.4 Wick Rotation to Euclidean Space

The path integral in Minkowski spacetime is highly oscillatory due to the e^{iS} factor, which makes numerical evaluation very difficult. To make the path integral more amenable to numerical methods, we perform a Wick rotation to Euclidean spacetime by transforming the time coordinate as $t \to -i\tau$. Under this transformation, the action transforms as:

$$\begin{aligned} S &= \int \mathrm{d}t \int \mathrm{d}^3x \, \mathcal{L} \\ &= \int (-i\mathrm{d}\tau) \int \mathrm{d}^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial(-i\tau)} \right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \right] \\ &= \int (-i\mathrm{d}\tau) \int \mathrm{d}^3x \left[-\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \right] \\ &= \int (-i\mathrm{d}\tau) \int \mathrm{d}^3x \, (-\mathcal{L}_E) \\ &\equiv iS_E \, , \end{aligned} \tag{12.4.1}$$

where $\mathcal{L}_E$ is the Euclidean Lagrangian density (signs all equal due to metric swap) and S_E is the Euclidean action. So the weight factor in the path integral transforms simply as:

$$e^{iS} \quad \to \quad e^{-S_E} \, . \tag{12.4.2}$$

12.5 QFT at Finite Temperature

Quantum field theory at finite temperatures can easily be formulated in the path integral formalism. The key idea is to use the analogy between the path integral and the partition function from Statistical Physics. We've already seen how we can rotate to Euclidean spacetime, which transforms the weight factor in the path integral to e^{-S_E}, which is reminiscent of the Boltzmann factor $e^{-\beta H}$ in Statistical Physics.

Mathematically we start from Equation (12.3.2) where instead of projecting onto the vacuum in the asymptotic past and future, we consider a thermal ensemble where all states contribute with Boltzmann weights:

$$e^{-\beta H} \quad \text{where} \quad \beta = \frac{1}{T} \, . \tag{12.5.1}$$

In the language of path integral formalism, this corresponds to compactifying Euclidean time on a circle of circumference β, with fields satisfying periodic boundary conditions in the Euclidean time:

$$\phi(\boldsymbol{x}, \tau) = \phi(\boldsymbol{x}, \tau + \beta) \, . \tag{12.5.2}$$

With such a compactification the partition function becomes:

$$Z_\beta \equiv \sum_n \langle n | e^{-\beta H} | n \rangle = \mathrm{Tr}(e^{-\beta H}) \, . \tag{12.5.3}$$

The Euclidean path integral then becomes:

$$Z(\beta) = \int_{\phi(\tau)}^{\phi(\tau+\beta)} \mathcal{D}\phi \, e^{-S_E} \, . \tag{12.5.4}$$

Observables at finite temperature can then be computed as thermal expectation values:

$$\langle \mathcal{O} \rangle_\beta = \frac{1}{Z(\beta)} \int_{\phi(\tau)}^{\phi(\tau+\beta)} \mathcal{D}\phi \, \mathcal{O}[\phi] e^{-S_E[\phi]} \, . \tag{12.5.5}$$

In fact we can calculate the trace of any operator $\hat{\mathcal{O}}$ as:

$$\mathrm{Tr}(\hat{\mathcal{O}}) \equiv \sum_m \langle m | \hat{\mathcal{O}} | m \rangle \, , \tag{12.5.6}$$

where $\{|m\rangle\}$ is any complete orthonormal basis. Basically, the key idea behind finite-temperature QFT is to evolve the system in imaginary time over a finite interval $\tau \in [0, \beta]$, where $\beta = 1/T$ is the inverse temperature.

12.6 Example: Positronium Mass

As an example of how to use the path integral formalism, let us try and calculate the mass of a positronium e^-e^+ bound state in the context of QED. We will see how correlation functions encode physical observables like the masses of bound states. Lets imagine we have a suitable operator $\hat{\mathcal{O}}$ with well-defined coordinates:

$$\hat{\mathcal{O}}(\boldsymbol{x}, t) = \hat{\bar{\psi}}(\boldsymbol{x}, t)\gamma_z\hat{\psi}(\boldsymbol{x}, t) \,. \tag{12.6.1}$$

We can make use of a Fourier transform to express the operator in momentum space:

$$\hat{\mathcal{O}}(\boldsymbol{p}, t) = \int \mathrm{d}^3x \; e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\mathcal{O}}(\boldsymbol{x}, t) \,. \tag{12.6.2}$$

With this we can then calculate the two-operator correlation function (essentially the propagator of the composite state):

$$\begin{aligned}
C(t) &= \langle\Omega|\hat{\mathcal{O}}(\boldsymbol{p}, t)\hat{\mathcal{O}}^\dagger(\boldsymbol{p}, 0)|\Omega\rangle = \qquad\qquad (12.6.3)\\
&= \int \mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}_2 \; e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\langle\Omega|\hat{\bar{\psi}}(\boldsymbol{x}_1, t)\gamma_z\hat{\psi}(\boldsymbol{x}_1, t)\left[\hat{\bar{\psi}}(\boldsymbol{x}_2, 0)\gamma_z\hat{\psi}(\boldsymbol{x}_2, 0)\right]^\dagger|\Omega\rangle \\
&= \int \mathrm{d}\boldsymbol{x}_1\mathrm{d}\boldsymbol{x}_2 \; \frac{e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}e^{iS[\psi,\bar{\psi},A^\mu]}}{Z[0]} \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A^\mu \; \hat{\bar{\psi}}(\boldsymbol{x}_1, t)\gamma_z\hat{\psi}(\boldsymbol{x}_1, t)\left[\hat{\bar{\psi}}(\boldsymbol{x}_2, 0)\gamma_z\hat{\psi}(\boldsymbol{x}_2, 0)\right]^\dagger \,.
\end{aligned}$$

Using a Wick rotation to Euclidean space, we can write this as:

$$\begin{aligned}
C(\tau) &= \langle\Omega|\hat{\mathcal{O}}(\boldsymbol{p}=0, \tau)\hat{\mathcal{O}}^\dagger(\boldsymbol{p}=0, 0)|\Omega\rangle = \\
&= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A^\mu \; \hat{\mathcal{O}}(\boldsymbol{p}=0, \tau)\hat{\mathcal{O}}^\dagger(\boldsymbol{p}=0, 0)e^{-S_E[\psi,\bar{\psi},A^\mu]} \\
&= \sum_n \langle\Omega|\hat{\mathcal{O}}(\boldsymbol{p}=0, \tau)|n\rangle\langle n|\hat{\mathcal{O}}^\dagger(\boldsymbol{p}=0, 0)|\Omega\rangle \exp\left[-E_n\tau\right] \,,
\end{aligned} \tag{12.6.4}$$

where in the last step we inserted a complete set of energy eigenstates $\sum_n |n\rangle\langle n|$. $C(\tau)$ can be interpreted as the Euclidean-time propagator of the positronium state from time 0 to time τ. As we limit $\tau \to \infty$ the lightest state dominates because all higher-energy states are exponentially suppressed, thus:

$$C(\tau) \xrightarrow{\tau\to\infty} A_0 e^{-m_{\mathrm{Ps}}\tau} \,, \tag{12.6.5}$$

where $m_{\mathrm{Ps}} = E_0$ is mass of the positronium in the ground state and $A_0 = |\langle\Omega|\hat{\mathcal{O}}|\mathrm{Ps}\rangle|^2$ is a normalization factor. Thus to extract the positronium mass we do:

$$m_{\mathrm{Ps}} = -\lim_{\tau\to\infty} \frac{1}{\tau}\ln C(\tau) \,. \tag{12.6.6}$$

This is the standard technique used in lattice QCD or Euclidean path integral calculations to extract masses from correlation functions.

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A Appendix

A.1 Dirac Trace Technology

Dirac trace technology is a powerful tool in quantum field theory, particularly in the context of QED. It allows us to compute traces of products of Dirac gamma matrices, which arise in calculations involving fermions.

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \,, \tag{A.1.1}$$

$$\mathrm{Tr}(\text{odd } \# \text{ of } \gamma) = 0 \,, \tag{A.1.2}$$

$$\mathrm{Tr}(\gamma^\mu k_\alpha \gamma^\alpha \gamma^\nu (k+q)_\beta \gamma^\beta) = 4k_\alpha (k+q)_\beta \left[g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta} \right] \,, \tag{A.1.3}$$

$$\tag{A.1.4}$$

And other useful gamma matrix identities:

$$\gamma_\mu \gamma^\mu = d \,, \tag{A.1.5}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \,, \tag{A.1.6}$$

$$\gamma^\mu k_\alpha \gamma^\alpha \gamma_\nu = -(d-2) k_\alpha \gamma^\alpha \,, \tag{A.1.7}$$