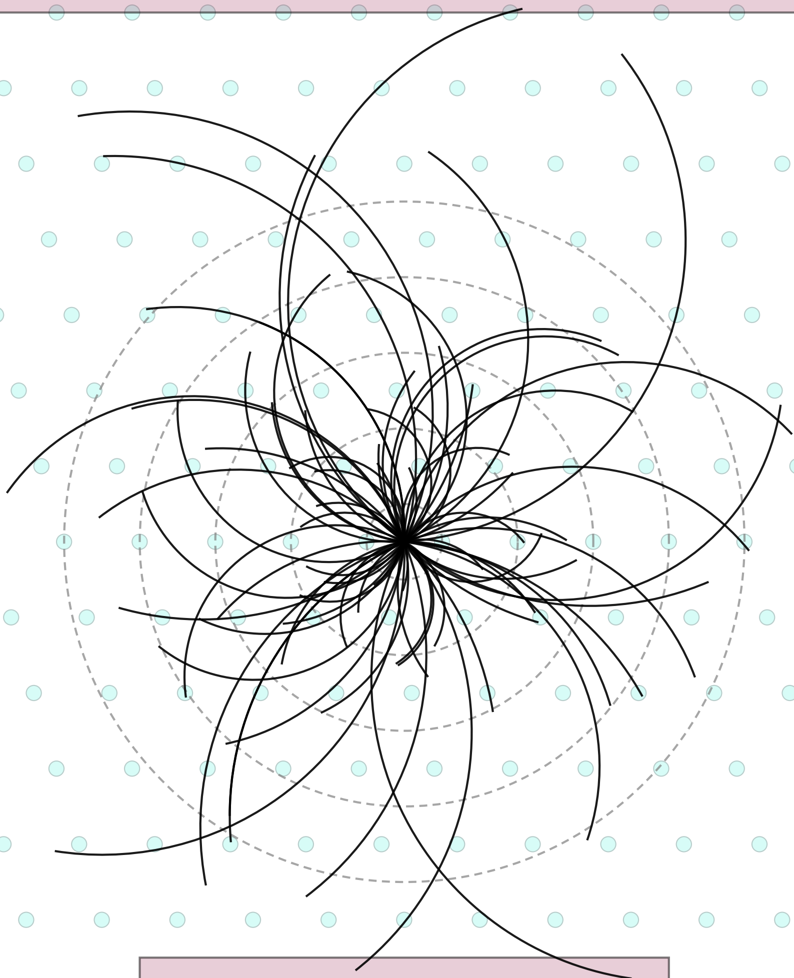
University of Ljubljana
Faculty of Mathematics and Physics

Department of Physics

Advanced Particle Detectors and Data Analysis

Notes for Exercises



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1 Interactions of Particles with Photons

1.1 Bethe-Bloch Equation

The Bethe-Bloch equation describes the mean energy loss per distance traveled while traversing through matter. We generally use the Bethe-Bloch equation when we are dealing with **thick absorbers**, such as the ones in calorimeters. Do note that the Bethe-Bloch equation does not accurately describe the energy loss of **electrons** and **positrons** due to their small mass and the fact that they suffer from much larger energy losses due to bremsstrahlung and pair production. For a particle with charge z and velocity $\beta = v/c$, the Bethe-Bloch equation is given as:

$$-\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle = 2\pi N_a r_e^2 m_e c^2 \rho \frac{Z}{A}\frac{z^2}{\beta^2}\left[\ln\left(\frac{2m_e c^2\beta^2\gamma^2 W_{\max}}{I^2}\right) - 2\beta^2 - \delta - 2\frac{C}{Z}\right] , \tag{1.1}$$

where δ is the **density effect correction** and C is the **shell correction**. The rest is as follows:

$$N_a = 6.022\times 10^{23}\,\mathrm{mol}^{-1} , \quad r_e = 2.818\times 10^{-15}\,\mathrm{m} , \quad m_e = 9.11\times 10^{-31}\,\mathrm{kg} , \quad c = 3\times 10^8\,\mathrm{m/s} ,$$
$$\rho = \text{density of the material} , \quad A = \text{atomic mass of the material} , \quad Z = \text{atomic number of the material} ,$$
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} , \quad W_{\max} = \text{maximum energy transfer in a single collision} , \quad I = \text{mean excitation energy} .$$

The constant factor in the equation can be written as:

$$\Xi = \boldsymbol{2\pi N_a r_e^2 m_e c^2 = 0.1535\,\mathrm{MeV\ cm^2 mol^{-1}}} , \tag{1.2}$$

where I've chosen to mark this constant factor as Ξ for easier reference in further calculations. We can find the mean excitation energy I from the following experimentally determined formula:

$$I = \begin{cases} Z(12 + \frac{7}{Z})\,\mathrm{eV} & \text{for } Z < 13 , \\ Z(9.76 + 58.8Z^{-1.19})\,\mathrm{eV} & \text{for } Z \geq 13 . \end{cases} \tag{1.3}$$

The maximum energy transfer in a single collision $W_{\max}$ can be calculated as:

$$W_{\max} = \frac{2m_e c^2\beta^2\gamma^2}{1 + 2\gamma\left({}^{m_e}/_{M}\right) + \left({}^{m_e}/_{M}\right)^2} \approx 2m_e c^2\beta^2\gamma^2 . \tag{1.4}$$

For our purposes we will ignore the density effect correction δ and the shell correction C.

1.1.1 Energy Loss of Charged Kaons

Let us calculate the energy losses for charged kaons K^+ and K^- with a rest mass of $0.493\,\mathrm{GeV}$ and momentum of $2.5\,\mathrm{GeV}$ in copper which has the following properties:

$$\rho = 8.92\,\mathrm{g/cm^3} ,$$
$$Z = 29 ,$$
$$A = 63.5\,\mathrm{g/mol} .$$

First let us calculate the velocity β and the Lorentz factor γ. We know that

$$\beta = \frac{pc}{E} = \frac{pc}{\sqrt{(pc)^2 + (Mc^2)^2}} , \tag{1.5}$$

where M is the mass of the particle. Thus:

$$\beta = \frac{2.5\,\frac{\mathrm{GeV}}{c}\,c}{\sqrt{\left(2.5\,\frac{\mathrm{GeV}}{c}\,c\right)^2 + \left(0.493\,\frac{\mathrm{GeV}}{c^2}\,c^2\right)^2}} \approx 0.981031 . \tag{1.6}$$

Remember to take at least 4 significant digits for the velocity β! This is due to the logarithm in the Bethe-Bloch equation. The Lorentz factor γ is then:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 5.159 . \tag{1.7}$$

Next let us calculate the maximum energy transfer in a single collision $W_{\max}$:

$$W_{\max} = 2m_e c^2 \beta^2 \gamma^2 = 2 \cdot 0.511 \,\frac{\text{MeV}}{c^2}\, c^2 \cdot (0.981031)^2 (5.159)^2 = 26.7\,\text{MeV}\,. \tag{1.8}$$

Last prerequisite is the mean excitation energy I which we can calculate using the formula (1.3) for $Z \geq 13$:

$$I = 29(9.76 + 58.8 \cdot 29^{-1.19})\,\text{eV} = 313.9\,\text{eV}\,. \tag{1.9}$$

Now all that is left is to plug in the values into the Bethe-Bloch equation (1.1):

$$\begin{aligned}
-\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle &= \Xi\,\rho \frac{Z}{A}\frac{z^2}{\beta^2}\left[\ln\left(\frac{W_{\max}^2}{I^2}\right) - 2\beta^2\right] \\
&= 0.1535\,\frac{\text{MeV}\,\text{cm}^2}{\text{mol}} \cdot 8.92\,\frac{\text{g}}{\text{cm}^3} \cdot \frac{29}{63.5}\,\frac{\text{mol}}{\text{g}} \cdot \frac{1}{0.981031^2} \cdot \left[\ln\left(\frac{(26.7 \cdot 10^6\,\text{eV})^2}{(313.9\,\text{eV})^2}\right) - 2 \cdot (0.981031)^2\right] \\
&= 13.47\,\frac{\text{MeV}}{\text{cm}}\,.
\end{aligned} \tag{1.10}$$

Thus the energy loss of charged kaons K^+ and K^- with a momentum of 2.5 GeV in copper is 13.47 MeV/cm.

1.1.2 What is the Energy Resolution of the Detector from the Previous Example?

Let's calculate the energy resolution of the detector from the previous example, assuming that the length of the particle track through the detector is $d = 5\,\text{cm}$ and that energy is measured based on all deposited energy without any additional losses. Using the result from the previous example (1.10), we can calculate the average energy deposited in the detector as:

$$\Delta E = \bar{\Delta} = -\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle \cdot d = 13.47\,\frac{\text{MeV}}{\text{cm}} \cdot 5\,\text{cm} = 67.35\,\text{MeV}\,. \tag{1.11}$$

This is an approximation since we are assuming that β is constant throughout the detector, which is not true. In reality we'd have to integrate the energy loss over the path of the particle, however at $p \sim$ GeV additional losses of $\sim$ MeV are negligible. Measurements of energy are dependant on the energy resolution R which is defined as:

$$R = \frac{\sigma_E}{\bar{\Delta}}\,, \tag{1.12}$$

where σ_E is the standard deviation of the energy measurement which we assume to have a Gaussian distribution like such:

$$p(\Delta) = \frac{1}{\sqrt{2\pi}\sigma_E} \exp\left(-\frac{\Delta - \bar{\Delta}^2}{2\sigma_E^2}\right)\,. \tag{1.13}$$

σ_E is determined empirically. For **non-relativistic** particles it can be calculated as the variance of the Bethe-Bloch equation as:

$$\sigma_0^2 = 4\pi N_a r_e^2 (m_e c^2)^2 \rho \frac{Z}{A}\,\Delta x\,. \tag{1.14}$$

For **relativistic** particles we can correct the variance from (1.14) as such:

$$\sigma_E^2 = \sigma_0^2\,\frac{1 - \frac{1}{2}\beta^2}{1 - \beta^2}\,. \tag{1.15}$$

In our case this gives us:

$$\begin{aligned}
\sigma_E^2 &= 2 \cdot 0.511\,\frac{\text{MeV}}{c^2}\,c^2 \cdot 0.1535\,\frac{\text{MeVcm}^2}{\text{mol}} \cdot 8.92\,\frac{\text{g}}{\text{cm}^3} \cdot \frac{29}{63.5}\,\frac{\text{mol}}{\text{g}} \cdot 5\,\text{cm} \cdot \frac{1 - \frac{1}{2}(0.981031)^2}{1 - (0.981031)^2} \\
&= 44.12\,\text{MeV}^2\,.
\end{aligned} \tag{1.16}$$

Thus the energy resolution of the detector is:

$$R = \frac{\sqrt{44.12\,\text{MeV}^2}}{67.35\,\text{MeV}} = 9.9\%\,. \tag{1.17}$$

1.1.3 What if the Detector is Made of a Molecule?

Let's assume now that our detector is made of lead(II) fluoride PbF_2 in a cubic crystal form which has the following properties:

$$\rho = 7.77\,\mathrm{g/cm^3}\,, \quad Z = 100\,, \quad A = 245.2\,\mathrm{g/mol}\,. \quad A_{\mathrm{Pb}} = 207.2\,\mathrm{u}\,, \quad A_{\mathrm{F}} = 19\,\mathrm{u}\,,$$

$$Z_{\mathrm{Pb}} = 82\,, \quad Z_{\mathrm{F}} = 9\,, \quad \rho_{\mathrm{Pb}} = 11.34\,\mathrm{g/cm^3}\,, \quad \rho_{\mathrm{F}} = 0.001696\,\mathrm{g/cm^3}\,.$$

We are interested in the energy loss of protons with a momentum of 3 GeV in such a detector. The difference between calculating the energy loss in a compound material is that we have to calculate the energy loss for each element in the compound. This sum is weighted by the fraction of the element in the compound. As such:

$$\frac{1}{\rho}\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_{\mathrm{compound}} = \frac{w_1}{\rho_1}\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_1 + \frac{w_2}{\rho_2}\left\langle\frac{\mathrm{d}E}{\mathrm{d}x}\right\rangle_2 + \dots\,, \tag{1.18}$$

where we calculate w_i as:

$$w_i = \frac{a_i \cdot A_i}{\sum a_i \cdot A_i}\,, \tag{1.19}$$

here a_i is the number of atoms of the element in the compound and A_i is the atomic mass of the element. Our professor stated that such problems will not be present on the exam and that we should not worry about them. However it is still good to know how to calculate the energy loss in a compound. In our case we can expect to get effective values if the detector is made of a compound. If we calculate the weights for lead and fluorine in lead(II) fluoride we get:

$$w_{\mathrm{Pb}} = \frac{1\cdot 207.2\,\mathrm{u}}{1\cdot 207.2\,\mathrm{u} + 2\cdot 19\,\mathrm{u}} = 0.845\,,$$
$$w_{\mathrm{F}} = \frac{2\cdot 19\,\mathrm{u}}{1\cdot 207.2\,\mathrm{u} + 2\cdot 19\,\mathrm{u}} = 0.154\,,$$

where we used the atomic masses of lead and fluoride in atomic mass units. Next we need to calculate the velocity β and the Lorentz factor γ for protons. So using (1.5) we get:

$$\beta = \frac{3\,\frac{\mathrm{GeV}}{c}\,c}{\sqrt{\left(3\,\frac{\mathrm{GeV}}{c}\,c\right)^2 + \left(0.938\,\frac{\mathrm{GeV}}{c^2}\,c^2\right)^2}} \approx 0.95443\,, \tag{1.20}$$

which gives us a Lorentz factor of:

$$\gamma = \frac{1}{\sqrt{1-(0.95443)^2}} \approx 3.351\,. \tag{1.21}$$

Next we need to calculate the maximum energy transfer in a single collision $W_{\max}$ using (1.4) as:

$$W_{\max} = 2\cdot 0.511\,\frac{\mathrm{MeV}}{c^2}\,c^2\cdot(0.95443)^2(3.351)^2 = 10.5\,\mathrm{MeV}\,, \tag{1.22}$$

and the mean excitation energy I using (1.3) for each component:

$$I_{\mathrm{Pb}} = 82\left(9.76 + 58.8\cdot 82^{-1.19}\right)\mathrm{eV} = 825.8\,\mathrm{eV}\,,$$
$$I_{\mathrm{Cu}} = 9\left(12 + \frac{7}{9}\right)\mathrm{eV} = 115\,\mathrm{eV}\,.$$

Now we can calculate the energy loss for each component using the Bethe-Bloch equation (1.1) and sum them up:

$$\begin{aligned}
-\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_{\mathrm{Pb}} &= \Xi\, \rho_{\mathrm{Pb}} \frac{Z_{\mathrm{Pb}}}{A_{\mathrm{Pb}}} \frac{1}{0.95443^2} \left[\ln\left(\frac{(10.5 \cdot 10^6\,\mathrm{eV})^2}{(825.8\,\mathrm{eV})^2} \right) - 2 \cdot (0.95443)^2 \right] \\
&= \Xi\, \rho_{\mathrm{Pb}} \frac{Z_{\mathrm{Pb}}}{A_{\mathrm{Pb}}} \cdot 18.7488 \\
&= 0.1535\, \frac{\mathrm{MeV\,cm^2}}{\mathrm{mol}} \cdot 11.34\, \frac{\mathrm{g}}{\mathrm{cm^3}} \cdot \frac{82}{207.2} \frac{\mathrm{mol}}{\mathrm{g}} \cdot 18.7488 \\
&= 12.9\, \frac{\mathrm{MeV}}{\mathrm{cm}} \,, \qquad (1.23) \\
-\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_{\mathrm{F}} &= \Xi\, \rho_{\mathrm{F}} \frac{Z_{\mathrm{F}}}{A_{\mathrm{F}}} \frac{1}{0.95443^2} \left[\ln\left(\frac{(10.5 \cdot 10^6\,\mathrm{eV})^2}{(115\,\mathrm{eV})^2} \right) - 2 \cdot (0.95443)^2 \right] \\
&= \Xi\, \rho_{\mathrm{F}} \frac{Z_{\mathrm{F}}}{A_{\mathrm{F}}} \cdot 23.0773 \\
&= 0.1535\, \frac{\mathrm{MeV\,cm^2}}{\mathrm{mol}} \cdot 0.001696\, \frac{\mathrm{g}}{\mathrm{cm^3}} \cdot \frac{9}{19} \frac{\mathrm{mol}}{\mathrm{g}} \cdot 23.0773 \\
&= 0.002846\, \frac{\mathrm{MeV}}{\mathrm{cm}} \,. \qquad (1.24)
\end{aligned}$$

Now all that is left is to compute the weighted sum as stated in (1.18):

$$\begin{aligned}
-\left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_{\mathrm{compound}} &= -\frac{\rho \cdot w_{\mathrm{Pb}}}{\rho_{\mathrm{Pb}}} \cdot \left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_{\mathrm{Pb}} - \frac{\rho \cdot w_{\mathrm{F}}}{\rho_{\mathrm{F}}} \cdot \left\langle \frac{\mathrm{d}E}{\mathrm{d}x} \right\rangle_{\mathrm{F}} \\
&= \frac{7.77\, \frac{\mathrm{g}}{\mathrm{cm^3}} \cdot 0.845}{11.34\, \frac{\mathrm{g}}{\mathrm{cm^3}}} \cdot 12.9\, \frac{\mathrm{MeV}}{\mathrm{cm}} + \frac{7.77\, \frac{\mathrm{g}}{\mathrm{cm^3}} \cdot 0.154}{0.001696\, \frac{\mathrm{g}}{\mathrm{cm^3}}} \cdot 0.002846\, \frac{\mathrm{MeV}}{\mathrm{cm}} \\
&= 9.47\, \frac{\mathrm{MeV}}{\mathrm{cm}} \,. \qquad (1.25)
\end{aligned}$$

1.2 Landau Distribution

For detectors of moderate thickness, which we can consider as **thin absorbers**, we can use a highly-skewed Landau-Vavilov distribution to describe the energy loss of particles. The most probable energy loss Δ_p is given as:

$$\bar{\Delta} = \Delta_p = \xi \left[\ln \frac{2 m_e c^2 \beta^2 \gamma^2}{I} + \ln \frac{\xi}{I} + j - \beta^2 - \delta(\beta\gamma) \right] \,, \qquad (1.26)$$

where $\delta(\beta\gamma)$ represents corrections due to the density effect, $j = 0.200$ and ξ is given as:

$$\xi = \frac{K}{2} \left\langle \frac{Z}{A} \right\rangle \frac{x}{\beta^2} \,, \qquad (1.27)$$

for x in $\mathrm{g/cm^2}$ and $K = 0.3\,\mathrm{MeV\,cm^2/g}$. **Warning:** x here is normalized with density $\mathrm{g/cm^2}$. This means that $x = \rho \cdot d$ where d is the thickness of the detector. To know which distribution to use, we can use the following rule of thumb:

$$\kappa = \frac{\bar{\Delta}}{W_{\max}} \begin{cases} > 10 & \text{use Bethe-Bloch} \,, \\ < 0.01 & \text{use Landau} \,. \end{cases} \qquad (1.28)$$

To determine the energy resolution of such a detector we can use the following formula:

$$R_{\mathrm{FWHM}} = \frac{4\xi}{\Delta_p} \,. \qquad (1.29)$$

1.2.1 What is the Most Probable Energy Loss of a Charged Pion in Silicon?

Let's calculate the most probable energy loss of a charged pion with a rest mass of 139.57 MeV and momentum of 0.5 GeV in a silicon based detector which has a 320 μm thick silicon layer. Silicon has the following properties:

$$\rho = 2.32\,\mathrm{g/cm^3}\,,$$
$$Z = 14\,,$$
$$A = 28\,\mathrm{g/mol}\,.$$

As before we first calculate the velocity β and the Lorentz factor γ for pions. Using (1.5) we get:

$$\beta = \frac{0.5\,\frac{\mathrm{GeV}}{c}\,c}{\sqrt{\left(0.5\,\frac{\mathrm{GeV}}{c}\,c\right)^2 + \left(0.13957\,\frac{\mathrm{GeV}}{c^2}\,c^2\right)^2}} \approx 0.96318\,, \tag{1.30}$$

which gives us a Lorentz factor of:

$$\gamma = \frac{1}{\sqrt{1-(0.96318)^2}} \approx 3.72\,. \tag{1.31}$$

Likewise as before we want to calculate the mean excitation energy I using (1.3) for $Z \geq 13$:

$$I = 14\left(9.76 + 58.8 \cdot 14^{-1.19}\right)\,\mathrm{eV} = 172.3\,\mathrm{eV}\,. \tag{1.32}$$

We can also calculate our approximation for the maximum energy transfer in a single collision W_{max} using (1.4), since we can spot it in the Landau distribution (1.26):

$$W_{\mathrm{max}} = 2 \cdot 0.511\,\frac{\mathrm{MeV}}{c^2}\,c^2 \cdot (0.96318)^2(3.72)^2 = 13.12\,\mathrm{MeV}\,. \tag{1.33}$$

Next we calculate ξ using (1.27):

$$\xi = \frac{0.3\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}}}{2}\,\frac{14}{28}\,\frac{320 \cdot 10^{-4}\,\mathrm{cm} \cdot 2.32\,\frac{\mathrm{g}}{\mathrm{cm^3}}}{(0.96318)^2} = 0.0060\,\mathrm{MeV}\,. \tag{1.34}$$

Now we can calculate the most probable energy loss using the Landau distribution (1.26):

$$\begin{aligned}\Delta_p &= 0.006\,\mathrm{MeV}\left[\ln\frac{13.12 \cdot 10^6\,\mathrm{eV}}{172.3\,\mathrm{eV}} + \ln\frac{0.006 \cdot 10^6\,\mathrm{eV}}{172.3\,\mathrm{eV}} + 0.2 - (0.96318)^2\right] \\ &= 0.0844\,\mathrm{MeV}\,.\end{aligned} \tag{1.35}$$

From this we can now also calculate the energy resolution of the detector using the formula (1.29):

$$R_{\mathrm{FWHM}} = \frac{4 \cdot 0.006\,\mathrm{MeV}}{0.0844\,\mathrm{MeV}} = 28.5\%\,. \tag{1.36}$$

If we're paranoid if we've used the right distribution, we can calculate κ as:

$$\kappa = \frac{0.0844\,\mathrm{MeV}}{13.12\,\mathrm{MeV}} = 0.0064\,, \tag{1.37}$$

which is less than 0.01 so we've used the right distribution. Alternatively if we magically procure the result from the Bethe-Bloch equation, we'd get $\bar{\Delta} = 126\,\mathrm{keV}$ and $\sigma = 402\,\mathrm{keV}$ which would give us $\kappa = 0.0105$ which still hints that we should use the Landau distribution.

1.3 Cherenkov Radiation

Charged particles moving through a medium with a velocity greater than the speed of light in that medium emit Cherenkov radiation. The angle of the emitted radiation is given by the Cherenkov angle which is defined as:

$$\cos\theta = \frac{1}{\beta n}\,, \tag{1.38}$$

where n is the refractive index of the medium. The threshold velocity for Cherenkov radiation is given as:

$$\beta_{\mathrm{thr}} = \frac{1}{n}\,. \tag{1.39}$$

We can calculate the number of produced Cherenkov photons per unit length x with the following formula:

$$\frac{\mathrm{d}^2 N}{\mathrm{d}E\,\mathrm{d}x} = \frac{\alpha z^2}{\hbar c} \sin^2\theta \,, \tag{1.40}$$

which we can approximate for $z = 1$ as:

$$\frac{\mathrm{d}^2 N}{\mathrm{d}E\,\mathrm{d}x} = \frac{370}{\text{eV cm}} \sin^2\theta \,. \tag{1.41}$$

if we assume that $\beta \approx$ const. and that the refractive index is not a function of the wavelength $n \neq n(\lambda)$. Thus if we'd like to calculate the total number of Cherenkov photons produced in our detector we can use the following formula:

$$N_{\text{Cherenkov}} = \frac{370}{\text{eV cm}} \Delta x \, \Delta E \left(1 - \frac{1}{\beta^2 n^2}\right) \,, \tag{1.42}$$

where Δx is the thickness of the detector and ΔE is the energy of an emitted Cherenkov photon which is simply calculated as:

$$\Delta E = h\nu = \frac{hc}{\lambda} = \frac{1240\,\text{eV nm}}{\lambda} \,, \tag{1.43}$$

where λ is the wavelength of the emitted Cherenkov photon.

1.3.1 How many Cherenkov Photons are Produced in Water by a Proton?

Let's calculate the number of Cherenkov photons produced in 1 cm of water by a proton with a rest mass of 0.938 GeV and a momentum of 2 GeV. The refractive index of water is $n = 1.33$. How many photons are detected with a photodetector which is sensitive to light between 250 nm and 800 nm with an average efficiency of 10%?

Using the formula (1.5) we can calculate the velocity β for the proton:

$$\beta = \frac{2\,\frac{\text{GeV}}{c}\,c}{\sqrt{\left(2\,\frac{\text{GeV}}{c}\,c\right)^2 + \left(0.938\,\frac{\text{GeV}}{c^2}\,c^2\right)^2}} \approx 0.90537 \,. \tag{1.44}$$

The Cherenkov angle θ can be calculated using the formula (1.38):

$$\theta = \arccos \frac{1}{0.90537 \cdot 1.33} = 33.85° \,. \tag{1.45}$$

From this we can calculate the number of Cherenkov photons produced per unit length using the formula (1.41):

$$\frac{\mathrm{d}^2 N}{\mathrm{d}E\,\mathrm{d}x} = \frac{370}{\text{eV cm}} \sin^2(33.85°) = 114.8\,\text{eV}^{-1}\text{cm}^{-1} \,. \tag{1.46}$$

Now we can calculate the total number of Cherenkov photons produced in 1 cm of water using the formula (1.42):

$$N_{\text{min}} = \frac{114.8}{\text{eV cm}} \cdot 1\,\text{cm} \cdot \frac{1240\,\text{eV nm}}{250\,\text{nm}} = 569.4 \,, \tag{1.47}$$

$$N_{\text{max}} = \frac{114.8}{\text{eV cm}} \cdot 1\,\text{cm} \cdot \frac{1240\,\text{eV nm}}{800\,\text{nm}} = 177.9 \,. \tag{1.48}$$

Using these two values we can very roughly estimate the number of detected Cherenkov photons in the range 250 nm to 800 nm. Essentially we use a linear approximation of the integral of the number of Cherenkov photons produced per unit length over the range of wavelengths and multiply it by the efficiency of the photodetector. Thus the number of detected Cherenkov photons is:

$$N_{\text{det}} = 0.1 \cdot (N_{\text{max}} - N_{\text{min}}) \approx 39 \,. \tag{1.49}$$

1.4 Neutrinos and Dark Matter

Neutrinos are elementary particles that interact only very weakly with matter. They are produced in nuclear reactions and in the decay of particles. Neutrinos are classified into three types: electron neutrinos ν_e, muon neutrinos ν_μ and tau neutrinos ν_τ. There are three possible interactions we can detect:

$$\begin{aligned} &\nu_x + e^- \to \nu_x + e^- && \text{elastic scattering} ,\\ &\nu_e + n \to p + e^- && \text{charged current interaction} ,\\ &\bar{\nu}_e + p \to n + e^+ && \text{inverse beta decay} . \end{aligned}$$

All of these interactions produce electrons/positrons which emit Cherenkov radiation. For an incoming flux of neutrinos F we can calculate the interaction cross-section as:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = F \, \frac{\mathrm{d}N}{\mathrm{d}\Omega} \, , \tag{1.50}$$

where $\frac{\mathrm{d}N}{\mathrm{d}\Omega}$ is the number of Cherenkov photons produced per unit angle. We often measure cross-sections in barns where $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^2$. For neutrinos that only interact via the weak force the interaction cross-section is about:

$$\boldsymbol{\sigma_\nu = 10^{-20}\,\mathrm{b} = 10^{-44}\,\mathrm{cm}^2 = 10^{-48}\,\mathrm{m}^2} \, . \tag{1.51}$$

The number of interactions can be calculated as:

$$N = t\phi\sigma_\nu N_{\mathrm{sc}} \, , \tag{1.52}$$

where t is the time of exposure, ϕ is the flux of neutrinos, σ_ν is the interaction cross-section and N_{sc} is the number of target particles (scattering centers). Both σ_ν and N_{sc} are in general dependant on the type of interaction. We can calculate the number of number of scattering centers as:

$$N_{\mathrm{sc}} = \frac{mN_a}{A} \, , \tag{1.53}$$

if we assume that the number of scattering centers is the same as the number of nucleons in the target material.

1.4.1 What mass of pure atomic hydrogen is needed to detect 1000 solar neutrinos per year?

Lets assume a neutrino flux of $\phi = 6 \cdot 10^{14}\,\mathrm{m}^{-2}\mathrm{s}^{-1}$, an interaction cross-section of $\sigma_\nu = 10^{-48}\,\mathrm{m}^2$ and that the number of scattering centers is the same as the number of nucleons in the hydrogen molecule. Thus the number of detected neutrinos per year is given as:

$$N = t\phi\sigma \, \frac{mN_a}{A} \, . \tag{1.54}$$

If we want to detect 1000 neutrinos per year we can calculate the mass of hydrogen as:

$$\begin{aligned} m &= \frac{A}{N_a t\phi\sigma} \cdot 1000 \\ &= \frac{1\,\dfrac{\mathrm{g}}{\mathrm{mol}}}{\left[6.02 \cdot 10^{23}\,\dfrac{1}{\mathrm{mol}}\right] (3.15 \cdot 10^7\,\mathrm{s}) \left[6 \cdot 10^{14}\,\dfrac{1}{\mathrm{m}^2\mathrm{s}}\right] (10^{-48}\,\mathrm{m}^2)} \cdot 1000 \\ &= 87.9\,\mathrm{kg} \, . \end{aligned} \tag{1.55}$$

1.4.2 What is the estimate maximum energy transfer in one collision of a WIMP with a germanium nucleus?

WIMP's (Weakly Interacting Massive Particles) are hypothetical particles that are thought to make up dark matter. Let's say that we are trying to detect them through elastic scattering with germanium nuclei. We take the mass of a WIMP to be 100 GeV and presume that they are stationary in intergalactic space. Our solar system is moving through intergalactic space at a velocity of $2.2 \cdot 10^5\,\mathrm{m/s}$. We first need to estimate the maximum energy transfer in one collision with a germanium nucleus from which we can then calculate the needed mass of germanium in the detector to get one event per year. The estimated cross-section for WIMP-nucleus scattering is $\sigma_{\text{WIMP}} = 10^{-45}\,\text{cm}^2$. The estimated flux of WIMP's is $\phi = 10^5\,\text{cm}^{-2}\text{s}^{-1}$.

We can directly use equation (1.52) to calculate the mass of germanium needed in the detector:

$$\begin{aligned} m &= \frac{NA}{N_a t\phi\sigma} \\ &= \frac{1 \cdot 72.6\,\frac{\text{g}}{\text{mol}}}{\left[6.02 \cdot 10^{23}\,\frac{1}{\text{mol}}\right](3.15 \cdot 10^7\,\text{s})\left[10^5\,\frac{1}{\text{cm}^2\text{s}}\right](10^{-45}\,\text{cm}^2)} \\ &= 3.826 \cdot 10^7\,\text{t} = 38\,\text{kt}\,. \end{aligned} \tag{1.56}$$

where we used the atomic mass of germanium $A = 72.6\,\text{g/mol}$ and $N = 1$. To calculate the maximum energy transfer in one collision we'd need to calculate the kinematics of the problem along with taking into account conservation laws. This would be best done in the center of mass frame and then transformed back to the lab frame. This is a bit too much for this exercise so we can try to estimate the maximum energy transfer semi-classically. The kinetic energy the germanium nucleon receives is:

$$W_2 = \frac{4mM}{(m+M)^2}\,W_1\,, \tag{1.57}$$

where M is the mass of the WIMP, m is the mass of the germanium nucleus and W_1 is the kinetic energy of the WIMP before the collision. The germanium nucleus is made of 72 nucleons and the mass of a nucleon is $\sim 1\,\text{GeV}$. Thus the mass of the germanium nucleus is $m = 72\,\text{GeV}$. We can calculate β for the WIMP as:

$$\beta = \frac{v}{c} = \frac{2.2 \cdot 10^5\,\frac{\text{m}}{\text{s}}}{3 \cdot 10^8\,\frac{\text{m}}{\text{s}}} = 0.733 \cdot 10^{-3}\,. \tag{1.58}$$

From this we can calculate the initial kinetic energy of the WIMP as:

$$W_1 = \frac{1}{2}Mv^2 = \frac{1}{2}Mc^2\beta^2 = 0.5 \cdot 100 \cdot 10^9\,\frac{\text{eV}}{c^2}\,c^2 \cdot (0.733 \cdot 10^{-3})^2 = 27.0\,\text{keV}\,. \tag{1.59}$$

Thus the maximum energy transfer in one collision is:

$$\begin{aligned} W_2 &= \frac{4 \cdot 72 \cdot 100}{(72+100)^2}\,27.0\,\text{keV} \\ &= 0.9735 \cdot 27.0\,\text{keV} \\ &= 26.3\,\text{keV}\,. \end{aligned} \tag{1.60}$$

2 Detectors

2.1 Semiconductor Detectors

Semiconductor detectors are most commonly placed close to the interaction point of a collider or experiment since their purpose is to measure the positions of various output particles without disturbing them. The most common semiconductor detectors are silicon and germanium detectors. They are made of a p-n junction which is reverse biased. When a charged particle passes through the detector it creates electron-hole pairs which are then separated by the electric field of the reverse biased p-n junction. The electrons and holes are then collected at the electrodes of the detector. The average number of detected electron-hole pairs is given as:

$$\langle N \rangle = \frac{\Delta E}{w}\,, \tag{2.1}$$

where ΔE is the energy deposited in the detector and w is the energy needed to create an electron-hole pair in the semiconductor. Some common values for w are:

$$\begin{aligned} w_{\text{Si}}(300\,\text{K}) &= 3.6\,\text{eV}\,, \\ w_{\text{Si}}(77\,\text{K}) &= 3.7\,\text{eV}\,, \\ w_{\text{Ge}}(77\,\text{K}) &= 2.9\,\text{eV}\,, \end{aligned}$$

The energy resolution of a semiconductor detector is given as:

$$R = \frac{\sigma_N}{\langle N \rangle} = \frac{\sqrt{F \cdot \langle N \rangle}}{\langle N \rangle} \,, \tag{2.2}$$

where F is the Fano factor which is a measure of the fluctuations in the number of detected electron-hole pairs. The Fano factor is usually around:

$$F = \begin{cases} 0.086 \;-\; 0.16 & \text{for silicon} \,, \\ 0.06 \;-\; 0.13 & \text{for germanium} \,. \end{cases} \tag{2.3}$$

Using the equation (2.1) we can calculate the energy resolution directly as:

$$R = \sqrt{\frac{F \cdot w}{\Delta E}} = \frac{R_{\text{FWHM}}}{2.35} \,, \tag{2.4}$$

where R_{FWHM} is the full width at half maximum of the energy resolution.

2.1.1 What is the energy resolution of a silicon detector at 300 K?

Consider a detector with a 1 mm thick silicon layer at 300 K. In the case of a perpendicularly crossing kaon with a rest mass of 0.493 GeV and momentum of 4 GeV what is the energy resolution? Silicon has the following properties:

$$\begin{aligned} w_{\text{Si}}(300\,\text{K}) &= 3.6\,\text{eV} \,, \\ \rho &= 2.32\,\text{g/cm}^3 \,, \\ Z &= 14 \,, \\ A &= 28\,\text{g/mol} \,. \end{aligned}$$

We can calculate the energy loss of the kaon in the silicon detector using the Landau distribution (1.26). Before that we need to calculate the velocity β and the Lorentz factor γ for the kaon. Using (1.5) we get:

$$\beta = \frac{4\,\frac{\text{GeV}}{c}\,c}{\sqrt{\left(4\,\frac{\text{GeV}}{c}\,c\right)^2 + \left(0.493\,\frac{\text{GeV}}{c^2}\,c^2\right)^2}} \approx 0.99249 \,, \tag{2.5}$$

which gives us a Lorentz factor of:

$$\gamma = \frac{1}{\sqrt{1 - (0.99249)^2}} \approx 8.175 \,. \tag{2.6}$$

Next we need to calculate the mean excitation energy I using (1.3) for $Z \geq 13$:

$$I = 14\left(9.76 + 58.8 \cdot 14^{-1.19}\right)\,\text{eV} = 172.1\,\text{eV} \,, \tag{2.7}$$

and the maximum energy transfer in a single collision W_{max} using (1.4):

$$W_{\text{max}} = 2 \cdot 0.511\,\frac{\text{MeV}}{c^2}\,c^2 \cdot (0.99249)^2 (8.175)^2 = 67.3\,\text{MeV} \,. \tag{2.8}$$

We also need to calculate ξ using (1.27):

$$\begin{aligned} \xi &= \frac{0.3\,\frac{\text{MeV cm}^2}{\text{g}}}{2}\,\frac{14}{28}\,\frac{0.1\,\text{cm} \cdot 2.32\,\frac{\text{g}}{\text{cm}^3}}{(0.99249)^2} \\ &= 0.0178\,\text{MeV} \,. \end{aligned} \tag{2.9}$$

Now all that is left to get the most probable energy loss is to use the Landau distribution (1.26):

$$\begin{aligned} \Delta_p &= 0.0178\,\text{MeV}\left[\ln\frac{67.3 \cdot 10^6\,\text{eV}}{172.1\,\text{eV}} + \ln\frac{0.0178 \cdot 10^6\,\text{eV}}{172.1\,\text{eV}} + 0.2 - (0.99249)^2\right] \\ &= 0.298\,\text{MeV} \,. \end{aligned} \tag{2.10}$$

From this we can calculate the energy resolution of the detector using the formula (2.4):

$$R = \sqrt{\frac{0.086 \cdot 3.6\,\text{eV}}{0.298 \cdot 10^6\,\text{eV}}} = 0.11\%\,, \tag{2.11}$$

$$R_{\text{FWHM}} = 2.35 \cdot 0.11\% = 0.26\%\,. \tag{2.12}$$

For the purpose of education if we were to repeat the entire calculation while taking values for germanium we'd get:

$$I = 342\,\text{eV}\,,$$
$$\xi = 35.77\,\text{keV}\,,$$
$$\Delta_p = 573\,\text{keV}\,,$$

which would give us an energy resolution of $R_{\text{Ge}} = 0.072\%$ and $R_{\text{FWHM}_{\text{Ge}}} = 0.17\%$.

2.1.2 What voltage is needed to get a 1 mm thick depletion region in the detector from the previous exercise?

Let's say that the impurity concentration is $N_D = 6 \cdot 10^{14}\,\text{cm}^{-3}$ and the dielectric constant of silicon is $\varepsilon = 11.7$. The equation for the depletion region width is:

$$d = \sqrt{\frac{2\varepsilon\varepsilon_0 U}{e_0 N_D}}\,. \tag{2.13}$$

We can use this equation to find the voltage U needed to get a 1 mm thick depletion region:

$$U = \frac{d e_0 N_D}{2\varepsilon\varepsilon_0} = \frac{(0.1\,\text{cm})^2\left[1.6 \cdot 10^{-19}\,\text{As}\right]\left(6 \cdot 10^{14}\frac{1}{\text{cm}^3}\right)}{2 \cdot 11.7\left(8.85 \cdot 10^{-12} \cdot 10^2\,\frac{\text{As}}{\text{V cm}}\right)} = 45.4\,\text{V}\,. \tag{2.14}$$

2.2 Ionization Detectors

Ionization Chambers Gas ionization chambers are detectors that are filled with a gas and have an electric field applied to them. When a charged particle passes through the gas it ionizes the gas atoms and the electrons and ions are collected at the electrodes of the detector. Without multiplication of the signal the energy resolution is given as:

$$R = \sqrt{\frac{F \cdot w}{\Delta E}}\,, \tag{2.15}$$

where F is the Fano factor which is around $F = 0.2$ for gas ionization chambers and where we already took into account the number of created electron-ion pairs and that they are Poisson distributed:

$$N = \frac{\Delta E}{w}\,, \tag{2.16}$$

$$\sigma_N = \sqrt{F \cdot N}\,. \tag{2.17}$$

The Fano factor is needed to correct the fact that subsequent electron-ion pairs are not entirely statistically independent. w is the energy needed to create an electron-ion pair in the gas.

Proportional Counters Proportional counters are gas ionization chambers with a multiplication factor. Since the created electron-ion pairs are hard to detect we use the Townsend avalanche effect to amplify the signal, which gives us a cascade of electron-ion pairs. For argon Ar it is $w_{\text{Ar}} = 26\,\text{eV}$. The amount of charge we collect is:

$$Q = N \cdot e \qquad \text{without multiplication}\,, \tag{2.18}$$

$$Q_{\text{mult}} = N \cdot e \cdot M \qquad \text{with multiplication}\,, \tag{2.19}$$

where M is the multiplication factor. The energy resolution of a gas ionization chamber with multiplication is **lower** than without multiplication. The energy resolution with multiplication is given as:

$$R_{\mathrm{mult}} = \sqrt{\frac{w\,(F+b)}{\Delta E}}\,, \tag{2.20}$$

where b is a constant that depends on the detector, usually between 0.4 and 0.7.

2.2.1 What is the energy resolution of a gas ionization chamber?

What is the energy resolution of an ionization chamber with a thickness of $d = 10\,\mathrm{cm}$, for a MIP particle if the gas used is the so-called *magic gas* which has the properties:

$$\begin{aligned}
&Q_{\mathrm{Ar}} = 75\%\,,\\
&Q_{\mathrm{C_4H_{10}}} = 25\%\,,\\
&\rho_{\mathrm{Ar}} = 1.66\,\mathrm{g/L}\,,\\
&\rho_{\mathrm{C_4H_{10}}} = 2.5\,\mathrm{g/L}\,,\\
&w_{\mathrm{Ar}} = 26\,\mathrm{eV}\,,\\
&w_{\mathrm{C_4H_{10}}} = 23\,\mathrm{eV}\,,\\
&F_{\mathrm{Ar}} = 0.2\,,\\
&F_{\mathrm{C_4H_{10}}} \approx 0.2\,,
\end{aligned}$$

where C_4H_{10} is isobutane and Q represents percentage by volume in the gas mixture. Since we're dealing with a MIP (Minimum Ionizing Particle) we can assume that the energy deposited is:

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = 2\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}} \tag{2.21}$$

We need to calculate the deposited energy by the individual gas components. So the energy deposited in the argon is:

$$\begin{aligned}
\Delta E_{\mathrm{Ar}} &= Q_{\mathrm{Ar}} \cdot \rho_{\mathrm{Ar}} \cdot d \left(-\frac{\mathrm{d}E}{\mathrm{d}x}\right)\\
&= 0.75 \cdot 1.66 \cdot 10^{-3}\,\frac{\mathrm{g}}{\mathrm{cm^3}} \cdot 10\,\mathrm{cm} \cdot 2\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}}\\
&= 0.0249\,\mathrm{MeV}\,.
\end{aligned} \tag{2.22}$$

Likewise for isobutane we get:

$$\begin{aligned}
\Delta E_{\mathrm{C_4H_{10}}} &= Q_{\mathrm{C_4H_{10}}} \cdot \rho_{\mathrm{C_4H_{10}}} \cdot d \left(-\frac{\mathrm{d}E}{\mathrm{d}x}\right)\\
&= 0.25 \cdot 2.5 \cdot 10^{-3}\,\frac{\mathrm{g}}{\mathrm{cm^3}} \cdot 10\,\mathrm{cm} \cdot 2\,\frac{\mathrm{MeV\,cm^2}}{\mathrm{g}}\\
&= 0.014\,\mathrm{MeV}\,.
\end{aligned} \tag{2.23}$$

Now we can calculate the number of created electron-ion pairs for argon and isobutane using the formula (2.16):

$$N_{\mathrm{Ar}} = \frac{0.0249 \cdot 10^6\,\mathrm{eV}}{26\,\mathrm{eV}} = 958\,, \tag{2.24}$$

$$N_{\mathrm{C_4H_{10}}} = \frac{0.014 \cdot 10^6\,\mathrm{eV}}{23\,\mathrm{eV}} = 609\,, \tag{2.25}$$

from which we can calculate the standard deviation using the formula (2.17):

$$\sigma_{N_{\mathrm{Ar}}} = \sqrt{0.2 \cdot 958} = 13.84\,, \tag{2.26}$$

$$\sigma_{N_{\mathrm{C_4H_{10}}}} = \sqrt{0.2 \cdot 609} = 11.03\,. \tag{2.27}$$

The trick here is how to combine the two deviations and number of created electron-ion pairs. We know from statistics that we can sum the squares of the deviations and then take the square root of the sum to get the total deviation. The number of pairs is simply the sum of the number of pairs created in argon and isobutane. From this we can calculate the energy resolution as:

$$
\begin{aligned}
R &= \frac{\sqrt{{\sigma_{N_{\mathrm{Ar}}}}^2 + {\sigma_{N_{\mathrm{C_4H_{10}}}}}^2}}{N_{\mathrm{Ar}} + N_{\mathrm{C_4H_{10}}}} \\
&= \frac{\sqrt{(13.84)^2 + (11.03)^2}}{958 + 609} \\
&= 1.1\% \,. \qquad (2.28)
\end{aligned}
$$

2.2.2 What is the energy resolution of a proportional counter?

For educational purposes let's calculate how the energy resolution worsens with multiplication in a proportional counter. Let's keep the rest of the data as in the previous exercise and assume that the multiplication factor is $M = 900$ and that the constant $b = 0.5$. To get the new energy resolution we can recycle our previous result and add an additional term to the resolution that is due to multiplication. The combined resolution is:

$$
R = \sqrt{R_N^2 + R_M^2} \,, \qquad (2.29)
$$

where R_N is the resolution without multiplication and R_M is the resolution decrease due to multiplication. The resolution decrease due to multiplication is given as:

$$
\begin{aligned}
R_M &= \sqrt{\frac{b}{N}} \\
&= \sqrt{\frac{0.5}{958 + 609}} \\
&= 1.8\% \,, \qquad (2.30)
\end{aligned}
$$

where we took the **primary number of created pairs**, not the number of pairs after multiplication. Thus the total resolution is:

$$
R = \sqrt{(0.011)^2 + (0.018)^2} = 2.9\% \,. \qquad (2.31)
$$

2.2.3 Why the cylindrical geometry? What voltage would be needed to achieve the same electric field in a parallel plate capacitor?

Let's imagine that the structure of our detector is analogous to a Geiger-Muller tube, so a cylinder with a wire in the middle. This is only an approximation since actual ionization chambers are more complex and are often made of thousands of parallel wires inside a cylindrical gas chamber. The reason for the cylindrical geometry is exactly the properties of the electric field. Taking the center wire thickness to be $a = 0.008\,\mathrm{cm}$ and the radius of the cylinder to be $b = 1\,\mathrm{cm}$, the electric field inside the cylinder is given as:

$$
E(r) = \frac{U}{r \ln \frac{b}{a}} \,, \qquad (2.32)
$$

which at $U = 2000\,\mathrm{V}$ yields an electric field at the center wire of the cylinder:

$$
\begin{aligned}
E(r = b) &= \frac{2000\ \mathrm{V}}{1\ \mathrm{cm}\ \ln \frac{1\ \mathrm{cm}}{0.008\ \mathrm{cm}}} \\
&= 5.2 \cdot 10^6\ \mathrm{V/m} \,. \qquad (2.33)
\end{aligned}
$$

In comparison, if we wanted to achieve the same electric field inside a planar capacitor with a distance of $d = 1\,\mathrm{cm}$ we'd need a voltage of:

$$
U = E \cdot d = 5.2 \cdot 10^6\ \mathrm{V/m} \cdot 1\ \mathrm{cm} = 52\ \mathrm{kV} \,. \qquad (2.34)
$$

A power supply that can provide 52 kV is much more expensive and harder to maintain than a power supply that can provide 2 kV, hence the practical choice of a cylindrical geometry.

2.3 Scintillation Detectors

In scintillation detectors the energy of a particle is converted into light. This happens when the particle interacts with the scintillator and excites the atoms in the scintillator. The excited atoms then de-excite through an intermediate state and emit light. The light is then collected by a photomultiplier tube (PMT) which converts the light into an electrical signal. The intermediate state is important since it allows for the scintillator to emit light out of the crystal. If it were not so the emitted photons would have enough energy to one again excite the atoms in the scintillator. We have two types of scintillators: organic and inorganic. Organic scintillators are made of organic compounds and are usually liquid or plastic. Inorganic scintillators are made of inorganic compounds and are usually crystals. We can the energy needed for the creation of one photon for different scintillators:

$$\text{Organic} \begin{cases} w_{\text{Anthracene}} = 60\ \text{eV}\ , \\ w_{\text{Plastic}} = 25\ \text{eV}\ , \end{cases}$$

$$\text{Inorganic} \begin{cases} w_{\text{NaI(Tl)}} = 100\ \text{eV}\ , \\ w_{\text{BGO}} = 300\ \text{eV}\ . \end{cases}$$

A more commonly used property of scintillators is their **light yield** which is the number of photons produced per unit energy, like so:

$$\text{Ly} = \frac{\text{N photons}}{\Delta\text{E energy deposited by particle}}\ . \tag{2.35}$$

The light yield is usually given in photons/MeV. The energy resolution of a scintillation detector is given as:

$$R = \frac{\sigma_N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}}\ , \tag{2.36}$$

where N is the number of scintillation photons, which is calculated simply by:

$$N = \text{Ly} \cdot \Delta\text{E} = \frac{\Delta\text{E}}{\text{w}}\ . \tag{2.37}$$

Sometimes we also se the use of the efficiency of the scintillator which is defined as:

$$\varepsilon = \frac{E_\gamma}{\Delta E} = \frac{N \cdot h\nu}{\Delta E}\ . \tag{2.38}$$

2.3.1 How many scintillation photons are produed in a CsI(Tl) scintillator?

Our electromagnetic calorimeter uses 30 cm long CsI(Tl) crystals. If a particle deposits 4 GeV of energy in the crystal how many scintillation photons are produced? The light yield of CsI(Tl) is 60000 photons/MeV. We get the number of produced photons using the formula (2.37):

$$N = 60000\ \frac{\text{photons}}{\text{MeV}} \cdot 4 \cdot 10^3\ \text{MeV} = 240 \cdot 10^6\ \text{photons}\ . \tag{2.39}$$

This theoretically gives us an energy resolution of:

$$R = \frac{1}{\sqrt{240 \cdot 10^6}} = 0.0065\%\ , \tag{2.40}$$

however the actual value is higher due to the quantum efficiency $\bar{\text{QE}}$ of the photomultiplier tube and the actual efficiency of collection of the scintillation photons $\varepsilon_{\text{coll}}$. We take an average value for the quantum efficiency as it is generally dependent on the wavelength of the light. If we take $\bar{\text{QE}} = 0.2$ and $\varepsilon_{\text{coll}} = 0.7$ we get the new corrected number of detected particles as:

$$N_{\text{det}} = N \cdot \bar{\text{QE}} \cdot \varepsilon_{\text{coll}} = 240 \cdot 10^6 \cdot 0.2 \cdot 0.7 = 33.6 \cdot 10^6\ . \tag{2.41}$$

$$R_{\text{eff}} = \frac{1}{\sqrt{240 \cdot 10^6 \cdot 0.2 \cdot 0.7}} = 0.017\%\ . \tag{2.42}$$

Use of Scintillators in Nuclear Medicine

2.3.2 What is the Resolution of a SPECT Camera?

A SPECT camera is a Single Photon Emission Computed Tomography camera. It is used in nuclear medicine to detect gamma rays. The camera uses a NaI(Tl) scintillator with a light yield of 40000 photons/MeV. The incoming gamma rays have an energy of 140.5 keV. We can calculate the number of produced scintillation photons using the formula (2.37):

$$N = 40000\,\frac{\text{photons}}{\text{MeV}} \cdot 140.5 \cdot 10^{-3}\,\text{MeV} = 5620\,\text{photons}\,. \tag{2.43}$$

The energy resolution of the camera is then:

$$R = \frac{1}{\sqrt{5620}} = 1.33\%\,. \tag{2.44}$$

2.3.3 What is the Resolution of a PET Camera with a BGO scintillator?

A PET camera is a Positron Emission Tomography camera. It is used in nuclear medicine to detect positron annihilation gamma rays. An interesting practical use of anti-matter by the way. Let's say we don't have the light yield of BGO but we know that the energy needed to create one photon is $w = 300\,\text{eV}$. The incoming gamma rays have an energy of 511 keV. We can calculate the number of produced scintillation photons using the formula (2.37):

$$N = \frac{511 \cdot 10^{3}\,\text{eV}}{300\,\text{eV}} = 1703\,\text{photons}\,, \tag{2.45}$$

from which we can simply use the formula (2.36) to get the energy resolution:

$$R = \frac{1}{\sqrt{1703}} = 2.4\%\,. \tag{2.46}$$

2.4 Particle Identification Detectors

We can separate particles based on their interactions and their mass. We almost always have a strong magnetic field present inside the detector/ This is so we can determine the momentum of the particle by measuring the curvature of the particle's trajectory. The momentum in the transverse plane is given as:

$$p_T = q \cdot B \cdot R\,, \tag{2.47}$$

where q is the charge of the particle, B is the magnetic field and R is the radius of curvature. To determine the particles mass we also need to measure its velocity. We can then use the equation for relativistic momentum:

$$p = \gamma m v\,, \tag{2.48}$$

to find the mass of the particle. If we consider typical values of $B = 1\,\text{T}$ and $R = 1\,\text{m}$ we can calculate the momentum of a proton to be:

$$\begin{aligned} p_T &= e_0 \cdot 1\,\text{T} \cdot 1\,\text{m} \cdot \frac{\text{c}}{\text{c}} \\ &= e_0 \cdot 1\,\frac{\text{V s}}{\text{m}^2} \cdot 1\,\text{m} \cdot \frac{3 \cdot 10^{8}\,\frac{\text{m}}{\text{s}}}{\text{c}} \\ &= 300\,\text{MeV/c}\,. \end{aligned} \tag{2.49}$$

We have a few types of detectors that can be used for particle identification:

- Time-of-Flight detectors
- Cherenkov detectors / Threshold counters
- Transition Radiation detectors (layers of materials with different refractive indices)
- Calorimeters ($\mathrm{d}E/\mathrm{d}x$ in a gas or a solid)

We will only be looking at the first two.

Time-of-Flight Detectors Time-of-Flight detectors use the simple principle of measuring the time it takes for a particle to travel a certain distance. If we measure the time t_1 when the particle enters the detector and the time t_2 when the particle exits the detector we can calculate the velocity of the particle as:

$$\mathrm{TOF} = \mathrm{t_2} - \mathrm{t_1} = \frac{\mathrm{L}}{\mathrm{v}} = \frac{\mathrm{L}}{\beta \mathrm{c}} , \tag{2.50}$$

where L is the length of the detector. Taking into account equation (1.5) we can derive that the time-of-flight is:

$$\mathrm{TOF} = \frac{\mathrm{L}}{\mathrm{c}} \sqrt{1 + \left(\frac{\mathrm{mc}^2}{\mathrm{pc}}\right)^2} . \tag{2.51}$$

Since we often want to use the time-of-flight to differentiate between particles it also makes sense to calculate the time-of-flight difference between two particles:

$$\Delta t = \mathrm{TOF_1} - \mathrm{TOF_2} = \frac{\mathrm{L}}{\beta_1 \mathrm{c}} - \frac{\mathrm{L}}{\beta_2 \mathrm{c}} , \tag{2.52}$$

where we can again use the equation (1.5) to get:

$$\Delta t = \frac{L}{c} \left[\sqrt{1 + \left(\frac{m_1 c^2}{p_1 c}\right)^2} - \sqrt{1 + \left(\frac{m_2 c^2}{p_2 c}\right)^2} \right] . \tag{2.53}$$

It is common to assume that $p_1 = p_2 = p$ in such cases.

2.4.1 Can we differentiate pions from kaons?

We are trying to differentiate pions with a rest mass of 0.1396 GeV and kaons with a rest mass of 0.4937 GeV using a time-of-flight detector, with a distance of $L = 2\,\mathrm{m}$. What is the time-of-flight difference between the two particles if they have a momentum of $p = 2\,\mathrm{GeV/c}$? We can directly use equation (2.53) to get:

$$\begin{aligned} \Delta t &= \frac{2\,\mathrm{m}}{3 \cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}} \left[\sqrt{1 + \left(\frac{0.1396\,\mathrm{GeV}}{2\,\mathrm{GeV}}\right)^2} - \sqrt{1 + \left(\frac{0.4937\,\mathrm{GeV}}{2\,\mathrm{GeV}}\right)^2} \right] \\ &= 183\,\mathrm{ps} . \end{aligned} \tag{2.54}$$

What if the momentum of the particles is $p = 4\,\mathrm{GeV/c}$? Replacing the momentum in the previous equation we get:

$$\Delta t_4 = 46\,\mathrm{ps} . \tag{2.55}$$

We see that the faster the particles are the harder it is to differentiate between them. We did not explicitly calculate the temporal resolution of the detector here, but we can assume that Δt_4 is much harder to measure than Δt.

Cherenkov Detectors/Threshold Counters Threshold counters are detectors that measure the Cherenkov radiation emitted by particles. They use the threshold velocity for the creation of Cherenkov photons as a way to differentiate between particles. We've already seen the formula for the Cherenkov angle (1.38) and the threshold velocity (1.39).

$$\begin{aligned} \beta < \beta_{\mathrm{thr}} &= \frac{1}{n} \qquad \text{no Cherenkov radiation} , \\ \beta > \beta_{\mathrm{thr}} &= \frac{1}{n} \qquad \text{Cherenkov radiation} . \end{aligned}$$

2.4.2 Can we differentiate pions from kaons using a Cherenkov detector?

Like before we are trying to differentiate pions with a rest mass of 0.1396 GeV and kaons with a rest mass of 0.4937 GeV using a Cherenkov detector. We are using an aerogel radiator with a refractive index of $n = 1.01$. If the momentum of the incoming particles is $p = 1.5\,\mathrm{GeV/c}$, can we separate the two particles?

We first need to calculate the velocities of the particles. Using (1.5) as we've done many times before we get:

$$\beta_\pi = \frac{1.5\,\frac{GeV}{c}\,c}{\sqrt{\left(1.5\,\frac{GeV}{c}\,c\right)^2 + \left(0.1396\,\frac{GeV}{c^2}c^2\right)^2}} = 0.09957\,,$$

$$\beta_K = \frac{1.5\,\frac{GeV}{c}\,c}{\sqrt{\left(1.5\,\frac{GeV}{c}\,c\right)^2 + \left(0.4937\,\frac{GeV}{c^2}c^2\right)^2}} = 0.94987\,.$$

The threshold velocity for the creation of Cherenkov radiation is:

$$\beta_{\rm thr} = \frac{1}{1.01} = 0.9901\,, \tag{2.56}$$

which means we will be able to differentiate between the two since kaons will not emit Cherenkov radiation while pions will. Lets repeat this calculation for $p = 4\,\mathrm{GeV/c}$:

$$\beta_\pi = 0.9994\,, \tag{2.57}$$

$$\beta_K = 0.9925\,. \tag{2.58}$$

Now both particles produce Cherenkov radiation which means we can no longer differentiate between them using this method.

Ring Imaging Cherenkov Detectors (RICH) RICH detectors use a photodection system in combination with a Cherenkov radiator to detect a ring of Cherenkov photons emitted by a particle. The radius of the ring is proportional to the Cherenkov angle (1.38) which is proportional to the particles velocity.

2.4.3 Can we differentiate pions from kaons using a RICH detector?

Again let's try to differentiate pions with a rest mass of 0.1396 GeV and kaons with a rest mass of 0.4937 GeV using a RICH detector made of aerogel with a refractive index of $n = 1.05$. What are the Cherenkov angles of the two particles if their momentum is 4 GeV/c?

We first need to calculate the velocities of the particles using (1.5). We will reuse the values of the velocities from the previous exercise (2.57, 2.58). From this we can directly use equation (1.38) to get the Cherenkov angles for the particles:

$$\theta_\pi = \arccos\frac{1}{1.05\cdot 0.9994} = 17.64^\circ = 307.9\,\mathrm{mRad}\,,$$

$$\theta_K = \arccos\frac{1}{1.05\cdot 0.9925} = 16.34^\circ = 285.3\,\mathrm{mRad}\,.$$

If we can resolve these two angles we can differentiate between the two particles but that depends on the experimental setup greatly.

3 Data Analysis

The second part of the course is dedicated to data analysis. This would usually be done via computers with the help of a software package like `ROOT`. In this course however we will stick to the basics and do everything by hand.

3.1 Resolving Power

It is important for us to differentiate a couple of terms that are often used in the context of data analysis of data from detectors. Since I took this course in Slovene originally I will also add the Slovene terms.

- **Resolving Power** (*Ločljivost*) is a measure of how well we can differentiate between two peaks in a spectrum. It is defined as:

$$D = \frac{|\overline{x}_2 - \overline{x}_1|}{\sigma} , \tag{3.1}$$

  where $\sigma = \sigma_1 = \sigma_2$ or $\sigma = \max(\sigma_1, \sigma_2)$ and $\overline{x}_1$ and $\overline{x}_2$ are the means of the two peaks.

- **Efficiency** (*Učinkovitost*) is a measure of how well we can detect a particle. It is the ratio of the number of detected particles to the number of particles that passed through the detector.

- **False Events** (Lažni dogodki) are events that are detected by the detector but are not actual events. They can be due to noise or other factors. This is the same as a false positive in statistics.

- **False Negative** (*Lažni negativ*) is an event that is not detected by the detector but is an actual event.

- **False Positive** (*Lažni pozitiv*) is an event that is detected by the detector but is not an actual event.

We can assume all our measurements are Gaussian distributed. Where the Gaussian distribution is given as:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) , \tag{3.2}$$

where μ is the mean of the distribution and σ is the standard deviation. For convenience of calculations we have been provided a table of the values of the integral of the Gaussian distribution. $g(x_0)$ represents the symmetric integral of the Gaussian distribution from $-x_0$ to x_0, while $f(x_0)$ represents the integral of the Gaussian distribution from $-\infty$ to x_0. In the context of data analysis $g(x)$ gives us the false positive rate while $f(x)$ gives us the efficiency of the detector. The table is given in Table 1.

x_0	$g(x_{\text{thr}})$ [%]	$f(x_{\text{thr}})$ [%]
σ	68.3	84.1
2σ	95.5	97.7
3σ	99.73	99.865
4σ	99.994	99.997
5σ	99.9994	99.99997
$1.282 \cdot \sigma$	80	90
$1.645 \cdot \sigma$	90	95
$2.327 \cdot \sigma$	98	99

Table 1: Table of the integral of the Gaussian distribution.

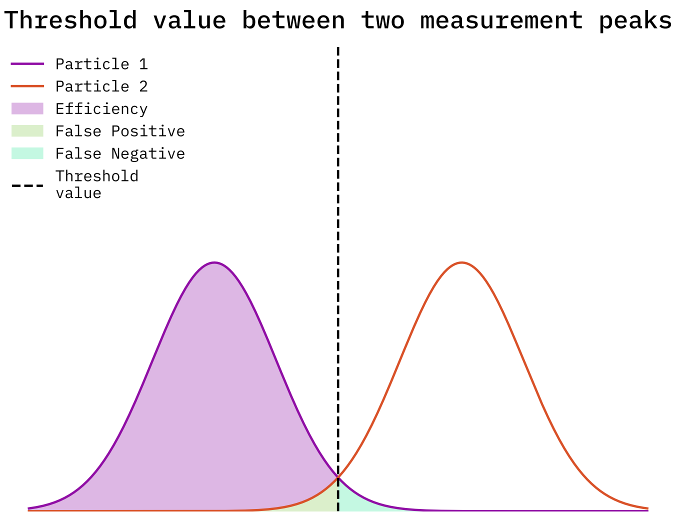

Figure 1: Visual representation of the Gaussian distributions and the threshold value.

Say we wanted to set the threshold value x_{thr} so that we can differentiate the first particle from the second with an efficiency of 99%. We read the width of the Gaussian distribution from the table where $f(x_{\text{thr}}) = 99\%$ and set the threshold value to:

$$x_{\text{thr}} = x_1 + 2.327 \cdot \sigma , \tag{3.3}$$

where x_1 is the mean of the distribution of the first particle. We can also calculate the percentage of false events F for this threshold value as:

$$x_0' = x_2 - x_{\text{thr}} , \tag{3.4}$$

where x_2 is the mean of the distribution of the second particle. The percentage of false events is then:

$$F = 1 - g(x_0') . \tag{3.5}$$

If we want to set a threshold for a given rate of false events F we can use the same formula as before where we replace the width of the Gaussian distribution for the threshold value x_{thr} with the value of x'_0 from the table where $g(x'_0) = 1 - F$.

3.1.1 What is the Resolving Power of the Detector in 2.4.1?

From 2.4.1 we calculated the time-of-flight difference between pions and kaons to be $\Delta t = 183\,\text{ps}$ for $p = 2\,\text{GeV/c}$ and $\Delta t = 46\,\text{ps}$ for $p = 4\,\text{GeV/c}$. We can calculate the resolving power of the detector using the formula for the resolving power:

$$D = \frac{|\Delta t|}{\sigma} \,, \tag{3.6}$$

where the temporal resolution of the detector is given as $\sigma = 60\,\text{ps}$. The resolving power for $p = 2\,\text{GeV/c}$ is:

$$D = \frac{183\,\text{ps}}{60\,\text{ps}} = 3.05 \,, \tag{3.7}$$

while for $p = 4\,\text{GeV/c}$ it is:

$$D = \frac{46\,\text{ps}}{60\,\text{ps}} = 0.77 \,. \tag{3.8}$$

We can see that the resolving power of the detector is much better for $p = 2\,\text{GeV/c}$ than for $p = 4\,\text{GeV/c}$. Where do we have to set the threshold for the time-of-flight difference to be able to differentiate between pions and kaons with an efficiency of 95%? 95% corresponds to $1.645 \cdot \sigma$. We can use the following:

$$\text{TOF}_{\text{thr}} = \text{TOF}_{\pi} + 1.645 \cdot \sigma \,, \tag{3.9}$$

Using Equation (2.51) we can calculate the time-of-flight for pions and kaons as:

$$\text{TOF}_{\pi} = \frac{2\,\text{m}}{3 \cdot 10^8\,\frac{\text{m}}{\text{s}}} \sqrt{1 + \left(\frac{0.1396\,\text{GeV}}{2\,\text{GeV}}\right)^2} = 6.68\,\text{ns} \,,$$

$$\text{TOF}_{\text{K}} = \frac{2\,\text{m}}{3 \cdot 10^8\,\frac{\text{m}}{\text{s}}} \sqrt{1 + \left(\frac{0.4937\,\text{GeV}}{2\,\text{GeV}}\right)^2} = 6.87\,\text{ns} \,.$$

From this we can calculate the threshold value for the time-of-flight value:

$$\text{TOF}_{\text{thr}} = 6.68\,\text{ns} + 1.645 \cdot 60\,\text{ps} = 6.68\,\text{ns} + 98.7\,\text{ps} = 6.78\,\text{ns} \,. \tag{3.10}$$

We can also calculate the percentage of false events F for this threshold value as:

$$x'_0 = \text{TOF}_{\text{K}} - \text{TOF}_{\text{thr}} = 6.87\,\text{ns} - 6.78\,\text{ns} = 84\,\frac{\sigma}{60\text{ps}}\text{ps} = 1.4 \cdot \sigma \,, \tag{3.11}$$

$$F = 1 - f(x'_0) = 1 - 92\% = 8\% \,.$$

3.1.2 What is the Resolving Power of the Detector in 2.4.3?

Lets assume that the measurement for a singular event has a resolution of:

$$\sigma_{\text{track}} = \frac{\sigma_0}{\sqrt{N}} \,, \tag{3.12}$$

where the resolution for a single photon is $\sigma_0 = 14\,\text{mRad}$ and $N = 10$ is the number of detected photons, thus $\sigma_{\text{track}} = 4.427\,\text{mRad}$. We can calculate the resolving power of the detector using Equation (3.1):

$$\begin{aligned} D &= \frac{|\theta_\pi - \theta_K|}{\sigma_{\text{track}}} \\ &= \frac{307.9\,\text{mRad} - 285.3\,\text{mRad}}{4.427\,\text{mRad}} \\ &= 5.1 \,. \end{aligned} \tag{3.13}$$

Let's calculate the efficiency of the detection of kaons and the fraction of false events due to pions if we take the threshold value for the Cherenkov angle to be directly in the middle of the two Cherenkov angles (or rather, the peaks of their Gaussian distributions). The threshold value for the Cherenkov angle is then:

$$\theta_{\text{thr}} = \frac{|\theta_\pi - \theta_K|}{2} = \frac{307.9\,\text{mRad} - 285.3\,\text{mRad}}{2} = 11.3\,\text{mRad}\,. \tag{3.14}$$

Transforming this into units of σ we get:

$$x_0' = \frac{11.3\,\text{mRad}}{4.427\,\text{mRad}} = 2.55\,. \tag{3.15}$$

which corresponds to 99.4% of the Gaussian distribution (from Table 1). This is directly the efficiency of the detection of kaons. The fraction of false events due to pions is then:

$$F = 1 - 99.4\% = 0.6\%\,. \tag{3.16}$$

3.2 Measurement of Momentum

We know we can measure the momentum of a particle using a magnetic field and the radius of curvature of the particle's trajectory. The transverse momentum is given as:

$$p_t = eBR\,, \tag{3.17}$$

where e is the charge of the particle, B is the magnetic field and R is the radius of curvature. Since the charge and the magnetic field are generally known all we have to do is to determine the radius of curvature. During exercises we briefly went over the derivation of how this is done in a planar case where we have three measurements. We came to some intermediate results like:

$$s = x_2 - \frac{x_1 + x_3}{2}\,, \tag{3.18}$$

$$s = \frac{eBL^2}{8p_t}\,, \tag{3.19}$$

$$\sigma_s^2 = \sigma_2^2 + \frac{\sigma_1^2}{4} + \frac{\sigma_3^2}{4} \Rightarrow \frac{6}{4}\sigma_x^2\,. \tag{3.20}$$

These without a sketch of the geometry are relatively useless. To make things short, for three points, the uncertainty due to tracking is:

$$\left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{Track}} = \frac{8p_t}{eBL^2}\sigma_x\sqrt{\frac{3}{2}}\,, \tag{3.21}$$

where σ_x is the uncertainty of the measurement of the position of the particle. We see that the uncertainty is proportional to the momentum of the particle. More general for a set of N measurements we have:

$$\left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{Track}} = \frac{p_t}{eBL^2}\sigma_x\sqrt{\frac{720}{N+4}}\,. \tag{3.22}$$

We must often also correct this due to multiple scattering. It is important to differentiate that L in the context of tracking means the distance of flight, while L in the context of multiple scattering means the thickness of the material. In the event of a gas detector we can say that $d = L$. Thus the correction due to multiple scattering in a **semiconductor detector** is:

$$\left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{MSC}} = \frac{13.6\,\text{MeV}}{c_0 eBL}\sqrt{\frac{d}{X_0}}\,, \tag{3.23}$$

where X_0 is the radiation length of the material. In the event of a **gas detector** the correction is:

$$\left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{MSC}} = \frac{13.6\,\text{MeV}}{c_0 eB\sqrt{X_0 L}}\,. \tag{3.24}$$

Kind reminder, that we sum up the errors as:

$$\left(\frac{\sigma_{p_t}}{p_t}\right)^2 = \left(\frac{\sigma_{p_t}}{p_t}\right)^2_{\text{Track}} + \left(\frac{\sigma_{p_t}}{p_t}\right)^2_{\text{MSC}}\,. \tag{3.25}$$

3.2.1 For a given semiconductor detector, which p_t can we measure with a 10% uncertainty?

Lets imagine a semiconductor detector which uses silicon detectors with a thickness of $d = 300\,\mu$m, which allow a resolution of $\sigma_x = 10\,\mu$m. The detector is made of 4 such layers with a $L = 7.3$ cm gap in between each silicon layer. The radiation length of silicon is $X_0 = 9.37$ cm, while the magnetic field is $B = 2$ T.

It is important for us to consider that we have 4 layers of silicon detectors of thickness d which give us only 3 gaps of size L. This also gives us $N = 4$ measurements of position. Let us find the momentum p_0 for which the total uncertainty is at most 10%. We can first calculate the uncertainty due to multiple scattering using Equation (3.23), since the right-hand side of the equation is independent of momentum:

$$\begin{aligned}\left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{MSC}} &= \frac{13.6\,\text{MeV}}{eB3L}\sqrt{\frac{4d}{X_0}} \\ &= \frac{13.6\cdot 10^6\,\text{eV}}{3\cdot 10^8\,\frac{\text{m}}{\text{s}}\cdot \text{e}\cdot 2\,\text{T}\cdot 3\cdot 7.3\cdot 10^{-2}\,\text{m}}\sqrt{\frac{4\cdot 300\cdot 10^{-6}\,\text{m}}{9.37\cdot 10^{-2}\,\text{m}}} \\ &= 0.0117 \approx 1.2\%\,,\end{aligned} \tag{3.26}$$

where we've taken into account that $\text{T} = \text{Vs}/\text{m}^2$ and added a c to the denominator to get the units right (so we can use eV). From here we can use Equation (3.25) to calculate the uncertainty due to tracking:

$$\begin{aligned}\left(\frac{\sigma_{p_t}}{p_t}\right)^2_{\text{Track}} &= \left(\frac{\sigma_{p_t}}{p_t}\right)^2 - \left(\frac{\sigma_{p_t}}{p_t}\right)^2_{\text{MSC}} = 0.1^2 - 0.012^2 \\ \Rightarrow \left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{Track}} &= 9.9\%\,.\end{aligned} \tag{3.27}$$

From this we can now determine the momentum p_0 for which the total uncertainty is at most 10%:

$$\begin{aligned}p_0 &= \left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{Track}} \frac{eB\cdot(3L)^2}{\sigma_x}\frac{1}{\sqrt{\frac{720}{N+4}}}\cdot c_0 \\ &= 0.099\cdot\frac{\text{e}\cdot 2\,\text{T}\cdot\left(3\cdot 7.3\cdot 10^{-2}\,\text{m}\right)^2}{10\cdot 10^{-6}\,\text{m}}\frac{1}{\sqrt{\frac{720}{4+4}}}\cdot 3\cdot 10^8\,\frac{\text{m}}{\text{s}} \\ &= 30\,\text{GeV/c}\,.\end{aligned} \tag{3.28}$$

If we wanted to find the value of the momentum where both the uncertainty due to tracking and multiple scattering are equal we would have to set the two equal to each other and solve for p_t:

$$\begin{aligned}\frac{\sigma_x p_t}{c_0 eBL^2}\sqrt{\frac{720}{N+4}} &= 0.0117 \\ \Rightarrow p_t &= 3.6\,\text{GeV/c}\,.\end{aligned} \tag{3.29}$$

3.2.2 For the Central Drift Chamber used in the Belle experiment, what is the uncertainty of the momentum measurement for a 0.5 GeV/c particle?

The Central Drift Chamber used in the Belle experiment is a gas detector which has 50 layers of anode wires in a space of $L = 70$ cm. The gas used is a mixture of 50% helium and 50% butane by volume, which has a radiation length of $X_0 = 640$ m. The magnetic field is $B = 1.5$ T. The spatial resolution of the detector is $\sigma_x = 130\,\mu$m.

Since we are dealing with a gas detector we can use Equation (3.24) to calculate the uncertainty due to multiple scattering, where we've already assumed that $d = L$:

$$\begin{aligned}\left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{MSC}} &= \frac{13.6\,\text{MeV}}{c_0 eB\sqrt{X_0 L}} \\ &= \frac{13.6\cdot 10^6\,\text{eV}}{3\cdot 10^8\,\frac{\text{m}}{\text{s}}\cdot \text{e}\cdot 1.5\,\text{T}\cdot\sqrt{640\,\text{m}\cdot 70\cdot 10^{-2}\,\text{m}}} \\ &= 0.00143 \approx 0.14\%\,.\end{aligned} \tag{3.30}$$

From here we can use Equation (3.22) to calculate the uncertainty due to tracking:

$$\begin{aligned}\left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{Track}} &= \frac{p_t}{eBL^2}\sigma_x\sqrt{\frac{720}{N+4}}\\ &= \frac{0.5\cdot 10^9\,\text{eV}}{3\cdot 10^8\,\frac{\text{m}}{\text{s}}\cdot \text{e}\cdot 1.5\,\text{T}\cdot (70\cdot 10^{-2}\,\text{m})^2}\cdot 130\cdot 10^{-6}\,\text{m}\sqrt{\frac{720}{50+4}}\\ &= 0.00107 \approx 0.11\%\,. \end{aligned} \tag{3.31}$$

From this we can calculate the total uncertainty of the momentum measurement using Equation (3.25):

$$\begin{aligned}\left(\frac{\sigma_{p_t}}{p_t}\right)^2 &= \left(\frac{\sigma_{p_t}}{p_t}\right)^2_{\text{Track}} + \left(\frac{\sigma_{p_t}}{p_t}\right)^2_{\text{MSC}} = (0.00107)^2 + (0.00143)^2 = 0.000003189\\ \Rightarrow \left(\frac{\sigma_{p_t}}{p_t}\right) &= 0.18\%\,. \end{aligned} \tag{3.32}$$

3.3 Particle Tracking

At this point during exercises we sketched how a particle would travel through 3 dimensional space and which parameters of the track we'd have to determine to reconstruct the track. The general parametrization is given as:

$$x = x_0 + R\left(\sin\phi - \sin\phi_0\right)\,, \tag{3.33}$$

$$y = y_0 - R\left(\cos\phi - \cos\phi_0\right)\,, \tag{3.34}$$

$$z = z_0 - R\cot\theta\left(\phi - \phi_0\right)\,, \tag{3.35}$$

where R is the radius of curvature, θ is the azimuthal angle, θ_0 is the azimuthal angle at the point of closest approach to the origin, ψ is the polar angle, ψ_0 is the polar angle at the point of closest approach to the origin, and x_0, y_0, z_0 are the coordinates of the point of closest approach to the origin. From this we see that we have to determine 5 parameters (the sixth can be determined from the others $x_0 = y_0/\tan\psi_0$). I was planning on adding an image of this but it is somewhat irrelevant since all practical exercises we did are in 1 dimension.

In practice we fit these parameters to our measurements using various optimization methods. We had a look at two different methods:

- **Least Squares Method** (*Metoda najmanjših kvadratov*) where we minimize the sum of the squares of the residuals. This is a **global method** since we use all the data points at once.
- **Kalman Filter** (*Kalmanov filter*) which is a **progressive method** where we update our estimate of the parameters as we get new data points.

Let's now imagine that we want to fit a linear model to a set of measurements. The scenario is presented in Figure 2.

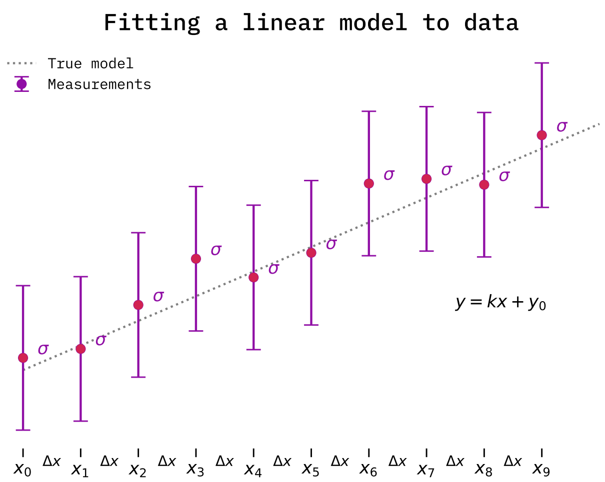

Figure 2: Fitting a linear model to a set of measurements.

We can write the model iteratively as:

$$y_{n+1} = y_n + k_n \cdot \Delta x \,, \tag{3.36}$$

where k_n is the slope of the line after n measurements. We will denote the uncertainty of the measurement with σ and the uncertainty of the estimated parameter k_n with $\sigma(k)$. Presented below are tables of how to calculate $k_n\Delta x$ and $\sigma(k)\Delta x$ for both of the methods.

Kalman Filter

n	$k_n\Delta x$	$\sigma(k)\Delta x$ if $\sigma_i = \sigma$
2	$y_2 - y_1$	$\sqrt{2}\,\sigma$
3	$\frac{1}{5}(3y_3 - y_2 - 2y_1)$	$\sqrt{\frac{14}{25}}\,\sigma = 0.748 \cdot \sigma$
4	$\frac{1}{70}(30y_4 - y_3 - 18y_2 - 11y_1)$	$0.524 \cdot \sigma$

Table 2: Table for Kalman Filter.

Least Squares Method

n	$k_n\Delta x$	$\sigma(k)\Delta x$ if $\sigma_i = \sigma$
2	$y_2 - y_1$	$\sqrt{2}\,\sigma$
3	$\frac{1}{2}(y_3 - y_1)$	$\frac{1}{\sqrt{2}}\sigma = 0.707 \cdot \sigma$
4	$\frac{1}{10}(3y_4 + y_3 - y_2 - 3y_1)$	$\frac{1}{\sqrt{5}}\sigma = 0.447 \cdot \sigma$

Table 3: Table for Least Squares Method.

The professor claims this is all that is needed for the exam, however one could also derive these from the general formulas for the least squares method and the Kalman filter.

3.3.1 For given measurements what is the relative error of the slope using the Kalman Filter and the Least Squares Method?

We measured the position of a particle in three different planes which are seperated by a 10 cm gap. The measurements are:

$$\begin{aligned} y_1 &= 2.35\,\text{mm} \quad \sigma_1 = 0.05\,\text{mm}\,, \\ y_2 &= 3.35\,\text{mm} \quad \sigma_2 = 0.05\,\text{mm}\,, \\ y_3 &= 4.15\,\text{mm} \quad \sigma_3 = 0.1\,\text{mm}\,. \end{aligned}$$

It is important to note here that the **uncertainty of the measurements** is **NOT the same** for all measurements. We will have to manually calculate the errors of the estimated slope for both methods. We can start by calculating $k_3\Delta x$ for the Kalman Filter:

$$\begin{aligned} k_3 &= \frac{1}{5\Delta x}(3y_3 - y_2 - 2y_1) \\ &= \frac{1}{5 \cdot 100\,\text{mm}}(3 \cdot 4.15 - 3.35 - 2 \cdot 2.35)\,\text{mm} \\ &= 8.8 \cdot 10^{-3}\,. \end{aligned} \tag{3.37}$$

(3.38)

We can calculate the error by taking the square of each term with its respective error. This means:

$$k_n \Delta x = \frac{1}{\alpha}\left(\beta_n y_n + \cdots + \beta_1 y_1\right) \quad \Rightarrow \quad \left(\sigma(k_n)\Delta x\right)^2 = \left(\frac{\beta_n}{\alpha}\sigma_n\right)^2 + \cdots + \left(\frac{\beta_1}{\alpha}\sigma_1\right)^2 . \tag{3.39}$$

So in our case we have:

$$\left(\sigma(k_3)\Delta x\right)^2 = \left(\frac{3}{5}\sigma_3\right)^2 + \left(\frac{-1}{5}\sigma_2\right)^2 + \left(\frac{-2}{5\cdot}\sigma_1\right)^2$$
$$\Rightarrow \sigma(k_3) = 6.4 \cdot 10^{-4} . \tag{3.40}$$

From this we can find the relative of the slope for the Kalman Filter:

$$\frac{\sigma(k_3)}{k_3} = \frac{6.4 \cdot 10^{-4}}{8.8 \cdot 10^{-3}} = 7.27\% . \tag{3.41}$$

Analogue to this we can calculate the slope and its error for the Least Squares Method:

$$\begin{aligned} k_3 &= \frac{1}{2\Delta x}\left(y_3 - y_1\right) \\ &= \frac{1}{2 \cdot 100\,\text{mm}}\left(4.15 - 2.35\right)\ \text{mm} \\ &= 9 \cdot 10^{-3} . \end{aligned} \tag{3.42}$$
$$\tag{3.43}$$

Likewise for the error of the slope:

$$\left(\sigma(k_3)\Delta x\right)^2 = \left(\frac{1}{2}\sigma_3\right)^2 + \left(\frac{-1}{2}\sigma_1\right)^2$$
$$\Rightarrow \sigma(k_3) = 5.59 \cdot 10^{-4} . \tag{3.44}$$

From which we can get the relative error:

$$\frac{\sigma(k_3)}{k_3} = \frac{5.59 \cdot 10^{-4}}{9 \cdot 10^{-3}} = 6.21\% . \tag{3.45}$$

From this we can see that different optimization methods yield different results. The Kalman Filter gives us a relative error of 7.27% while the Least Squares Method gives us a relative error of 6.21%.

3.4 Bonus Exercises

3.4.1 Matching uncertainties in a Central Drift Chamber*

Consider that we are measuring the deposited energy and transverse momentum of protons using a Central Drift Chamber. The chamber is filled with Helium under normal conditions where it has the following properties:

$$\begin{aligned} \rho &= 0.17\,\text{g/L} = 0.00017\,\text{g/cm}^3 , \\ A &= 4\,\text{g/mol} , \\ Z &= 2 , \\ F &= 0.24 , \\ w &= 41.3\,\text{eV} . \end{aligned}$$

The position detector system is made out of 25 layers of anode wires. The dimension of this setup is $L = 1\,\text{m}$. The magnetic field inside is $B = 2\,\text{T}$. What is the uncertainty of the spatial measurements in a Central Drift Chamber such that the uncertainty of the momentum measurement is the same as the uncertainty of the measured deposited energy if we are measuring a 5 GeV/c proton?

We first have to calculate the energy loss of the proton in such a system. Despite the fact that we have a meter of gas, this still counts as a thin absorber so we must use the Landau distribution from Equation

(1.26) to calculate the energy loss. We can check if this is alright later on. For this we first need to calculate the velocity of the particle using Equation (1.5):

$$\beta = \frac{pc}{\sqrt{(pc)^2 + (Mc^2)^2}} = \frac{5\,\frac{\text{GeV}}{c}\,c}{\sqrt{\left(5\,\frac{\text{GeV}}{c}\,c\right)^2 + \left(0.938\,\frac{\text{GeV}}{c^2}\,c^2\right)^2}} \approx 0.98285\,, \tag{3.46}$$

and the γ factor:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 5.4228\,. \tag{3.47}$$

With this we can calculate ξ which is given by Equation (1.27):

$$\xi = \frac{K}{2}\left\langle\frac{Z}{A}\right\rangle\frac{L\rho}{\beta^2} = \frac{0.3\,\frac{\text{MeVcm}^2}{\text{g}}}{2}\cdot\frac{2}{4}\cdot\frac{100\,\text{cm}\cdot 0.00017\,\frac{\text{g}}{\text{cm}^3}}{(0.98285)^2} = 0.0013\,\text{MeV}\,, \tag{3.48}$$

and the mean excitation energy I using Equation (1.3) for $Z < 13$:

$$I = Z\left(12 + \frac{7}{Z}\right)\,\text{eV} = 2\left(12 + \frac{7}{2}\right)\,\text{eV} = 31\,\text{eV}\,. \tag{3.49}$$

We can also calculate our approximation for the maximum energy transfer in a single collision W_{max} using Equation (1.4):

$$W_{\text{max}} = 2m_ec^2\beta^2\gamma^2 = 2\cdot 0.511\,\text{MeV}\cdot(0.98285)^2\cdot(5.4228)^2 \approx 29\,\text{MeV}\,. \tag{3.50}$$

Now we can calculate the energy loss using the Landau distribution:

$$\begin{aligned}\Delta_p &= \xi\left[\ln\frac{2m_ec^2\beta^2\gamma^2}{I} + \ln\frac{\xi}{I} + j - \beta^2\right]\\ &= 0.0013\,\text{MeV}\left[\ln\frac{29\cdot 10^6\,\text{eV}}{31\,\text{eV}} + \ln\frac{0.0013\cdot 10^6\,\text{eV}}{31\,\text{eV}} + 0.2 - (0.98285)^2\right]\\ &= 0.0013\,\text{MeV}\cdot 16.719\\ &= 0.022\,\text{MeV}\,.\end{aligned} \tag{3.51}$$

From this we can calculate the number of produced electron-ion pairs N using Equation (2.16):

$$N = \frac{\Delta_p}{w} = \frac{0.022\cdot 10^6\,\text{eV}}{41.3\,\text{eV}} \approx 532\,. \tag{3.52}$$

Thus we can then use Equation (2.15) to calculate the resolution for the deposited energy:

$$R = \frac{\sqrt{F\cdot N}}{N} = \frac{\sqrt{0.24\cdot 532}}{532} \approx 2.1\%\,. \tag{3.53}$$

Now we can use the equation for tracking resolution Equation (3.22) to calculate the required spatial measurement uncertainty:

$$\begin{aligned}
\sigma_x &= \frac{eBL^2 \left(\frac{\sigma_{p_t}}{p_t}\right)_{\text{Track}}}{p_t} \cdot \left[\frac{720}{N+4}\right]^{-\frac{1}{2}} \\
&= \frac{3 \cdot 10^8 \, \frac{\mathrm{m}}{\mathrm{s}} \cdot \mathrm{e} \cdot 2\,\mathrm{T} \cdot (1\,\mathrm{m})^2 \cdot 0.021}{5 \cdot 10^9\,\mathrm{eV}} \cdot \left[\frac{720}{25+4}\right]^{-\frac{1}{2}} \\
&= 0.0048\,\mathrm{m} = 4.8\,\mathrm{mm}\,. \qquad (3.54)
\end{aligned}$$