Newtonovi zakoni

Newton je obravnaval le 1 ali 2 telesi. Osnovni pojmi

- Koordinatni sistem (ki miruje ali pa se giblje s konstantno hitrostjo)
- Lega telesa
- Čas in iz tega potem hitrost in pospešek
- Masa (definirana kot množina snovi, ki jo primerjamo s tehtanjem)
- Koncept točkastega telesa (običajno pomembno, da ima lego in maso?
- Sila

Prvi Newtonov zakon

Točkasto telo se giblje enakomerno, če nanj ne deluje nobena sila

Drugi Newtonov zakon

$$\vec{F} = m\vec{a}$$

Tretji Newtonov zakon

$$\vec{F}_{12} = -\vec{F}_{21}$$

- Velja za električno in gravitacijsko silo/polje. **Za magnetno ne velja** [Proti primeri v zvezku]
- **Ne velja** za dinamične procese.
- Učitelji pogosto narobe učijo, da je 3. NZ to, da je otrok, ki sedi na klopci v ravnovesju ker potiska na klopco toliko kot klopca potiska nanj. To ne velja, ker ne gre za točke. Ravnovesje pride **iz statičnega ravnovesja** (ukrivljenost klopce pod tezo otroka da prozno silo) in ne iz 3. NZ. Velja pa 3.NZ med delci v klopci oz. na meji med klopco in otrokom.

Opis gibanja enega delca

Descartes je bil tisti, ki je propagiral, da »svet je dobro naoljen urni mehanizem«.:

$$\vec{r}(t) = \big(x(t), y(t), z(t)\big) \qquad \vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z}) \qquad \vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = \dot{\vec{v}} = (\ddot{x}, \ddot{y}, \ddot{z})$$

$$\vec{F} = m\vec{a}; \;\; \vec{F}\big(\vec{r}, \dot{\vec{r}}\,\big) = \dot{\vec{p}}; \quad m\vec{v} = m\dot{\vec{r}} = \vec{p}$$

Galilejeve transformacije

Obstaja 10 neodvisnih transformacij med inercialnimi sistemi. Te transformacije predstavljajo **grupo Galilejevih transformacij**, ki je poseben primer Poincarejeve grupe ($c \to \infty$).

- **3 rotacije:** $\underline{\underline{O}}$ je 3x3 ortogonalna matrika/transformacija
$$\vec{r}\,' = \underline{\underline{O}}\vec{r}$$
- **3 translacije**: $\vec{R}$ je konstantni vektor $\frac{d^2\vec{R}}{dt^2} = 0$
$$\vec{r}\,' = \vec{r} + \vec{R}$$
- **3 premiki hitrosti:** $\vec{u}$ je konstanten $\frac{d\vec{u}}{dt} = 0$
$$F' = \vec{r} + \vec{u}t$$
- **1 premik časa:**
$$t' = t + t_0$$

Ortonormirana baza

V fiksnem: $\vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta}$; $\alpha, \beta = \{x, y, z\}$; $\frac{d\vec{e}_\alpha}{dt} = 0$
V gibajočem: $\vec{e}'_\alpha \cdot \vec{e}'_\beta = \delta_{\alpha\beta}$

Neinercialni koordinatni sistemi (glej sliko)

Pripravimo vse potrebno(poglej sliko kaj pomeni kaj):

$$\vec{v} = \vec{V} + \vec{v}'_{neinerc} + \vec{\omega}' \times \vec{r}'$$

$$\vec{R}(t) = (R_x, R_y, R_z) \quad \vec{r} = \vec{R} + \vec{r}'$$

To lahko zapišemo po bazi v fiksnem sistemu (brez črtice) in v gibajočem (z črtico), ki ima časovno odvisne bazne vektorje:

$$\vec{R} = \sum_{\alpha=1}^{3} x_\alpha \vec{e}_\alpha \qquad \vec{r}' = \sum_{\alpha=1}^{3} x'_\alpha \vec{e}'_\alpha$$

Izrazimo hitrost $\vec{v}'$:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}'}{dt} = \vec{v}_R(t) + \frac{d\vec{r}'}{dt} = \sum_\alpha \dot{x}_\alpha \vec{e}_\alpha$$

$$\frac{d\vec{e}'_\alpha}{dt} = \sum_{\alpha=1}^{3} a_{\alpha\beta}\, \vec{e}'_\beta$$

To pomnožimo z vsakim od baznih vektorjev $\vec{e}'_\gamma$ in upoštevamo ortonormiranost

$$\frac{d\vec{e}'_\alpha}{dt} \cdot \vec{e}'_\gamma = a_{\alpha\gamma}$$

Tako smo dobili predpis za element antisimetricne matrike $\underline{\underline{a}}$:

$$a_{\alpha\gamma} = \frac{d\vec{e}'_\alpha}{dt} \cdot \vec{e}'_\gamma = \frac{d}{dt}(\vec{e}'_\alpha \cdot \vec{e}'_\gamma) - \vec{e}'_\alpha \cdot \frac{d\vec{e}'_\gamma}{dt} = \frac{d}{dt}(\delta) - \vec{e}'_\alpha \cdot \frac{d\vec{e}'_\gamma}{dt} = -a_{\gamma\alpha}$$

$$\underline{\underline{a}} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega'_3 & -\omega'_2 \\ -\omega'_3 & 0 & \omega'_1 \\ \omega'_2 & -\omega'_1 & 0 \end{bmatrix}; \quad \vec{\omega}' = \sum_\alpha \omega'_\alpha \vec{e}_\alpha = (\omega'_1, \omega'_2, \omega'_3)$$

Tako recimo v primeru $\alpha = 1$ dobimo:

$$\frac{d\vec{e}'_1}{dt} = \sum_\beta a_{1\beta} \vec{e}'_\beta = a_{12}\vec{e}'_2 + a_{13}\vec{e}'_3 = \omega'_3\vec{e}'_2 - \omega'_2\vec{e}'_3 = \omega' \times \vec{e}_1$$

S cikličnimi permutacijami ugotovimo se ostale komponente:

$$\frac{d\vec{e}'_\alpha}{dt} = \vec{\omega}' \times \vec{e}'_\alpha$$

Tako lahko zdaj izračunamo prej neznano hitrost:

$$\frac{d\vec{r}'}{dt} = \frac{d}{dt}\sum_{\alpha} x'_\alpha \vec{e}'_\alpha = \sum_{\alpha}\left[\frac{dx'_\alpha}{dt}\vec{e}'_\alpha + x'_\alpha \frac{d\vec{e}'_\alpha}{dt}\right] = \left(\frac{d\vec{r}'}{dt}\right)_{neinerc} + \sum_{\alpha} x'_\alpha (\vec{\omega}' \times \vec{e}'_\alpha)$$

Končno lahko izrazimo hitrost:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}'}{dt} = \frac{d\vec{R}}{dt} + \left(\frac{d\vec{r}'}{dt}\right)_{neinerc} + \vec{\omega}' \times \vec{r}'$$

Lahko se preverimo:

$$\vec{\omega}' = \sum_{\alpha} \omega'_\alpha \vec{e}'_\alpha \quad \Rightarrow \quad \frac{d\vec{\omega}'}{dt} = \sum_{\alpha} \dot{\omega}'_\alpha e'_\alpha + \vec{\omega}' \times \vec{\omega}' = \left(\frac{d\vec{\omega}'}{dt}\right)_{neinerc}$$

Izrazimo pospešek

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{R}}{dt^2} + \frac{d^2\vec{r}'}{dt^2} = \sum_{\alpha} \ddot{x}_\alpha \vec{e}_\alpha$$

$$\frac{d^2\vec{r}'}{dt} = \frac{d}{dt}\left(\left(\frac{d\vec{r}'}{dt}\right)_{neiner} + \vec{\omega}' \times \vec{r}'\right) = \frac{d}{dt}\left(\sum_{\alpha} \dot{x}'_\alpha \vec{e}'_\alpha + \vec{\omega}' \times \sum_{\alpha} x'_\alpha \vec{e}'_\alpha\right) =$$

$$= \sum_{\alpha} \ddot{x}'_\alpha \vec{e}'_\alpha + \sum_{\alpha} \dot{x}'_\alpha \dot{\vec{e}}'_\alpha + \dot{\vec{\omega}}' \times \vec{r}' + \vec{\omega}' \times \sum_{\alpha} \dot{x}'_\alpha \vec{e}'_\alpha + \vec{\omega}' \times \sum_{\alpha} x'_\alpha \dot{\vec{e}}'_\alpha =$$

$$= \left(\frac{d^2\vec{r}'}{dt^2}\right)_{neinerc} + \vec{\omega}' \times \sum_{\alpha} \dot{x}'_\alpha \vec{e}'_\alpha + \dot{\vec{\omega}}' \times \vec{r}' + \vec{\omega}' \times \vec{v}' + \vec{\omega}' \times (\vec{\omega}' \times \vec{r}')$$

Torej:

$$\frac{d^2\vec{r}'}{dt^2} = \left(\frac{d^2\vec{r}'}{dt^2}\right)_{neinerc} + 2\vec{\omega}' \times \left(\frac{d\vec{r}'}{dt}\right)_{neinerc} + \dot{\vec{\omega}}' \times \vec{r}' + \vec{\omega}' \times (\vec{\omega}' \times \vec{r}')$$

Sistemske sile

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} = m\left(\frac{d^2\vec{R}}{dt^2}\right) + m\left(\frac{d^2\vec{r}'}{dt^2}\right) = m\vec{a}$$

$$\begin{aligned}\vec{a} &= \vec{a}' + 2(\vec{\omega}' \times \vec{v}') + \vec{\omega}' \times (\vec{\omega}' \times \vec{r}') + \dot{\vec{\omega}}' \times \vec{r}' \\ &= \vec{a}' + 2(\vec{\omega}' \times \vec{v}') + m\omega'^2\vec{r}' - m(\vec{\omega}' \cdot \vec{r})\vec{\omega}' + \vec{a}_R + \dot{\vec{\omega}}' \times \vec{r}'\end{aligned}$$

Definiramo:

$$m\left(\frac{d^2\vec{r}'}{dt^2}\right)_{neinerc} \stackrel{\text{def}}{=} \vec{F}_{sist} + \vec{F} = m\vec{a}$$

$$\vec{F}_{sist} = -m\vec{a} + m\vec{a}' = m(\vec{a}' - \vec{a}) = -2m\,\vec{\omega}' \times \vec{v}' + m\omega'^2\vec{r}' - m(\vec{\omega}' \cdot \vec{r}')\vec{\omega}' - m\dot{\vec{\omega}}' \times \vec{r}' - m\vec{a}_R$$

Coriolisova sila, centrifugalna sila, sila v smeri osi vrtenja (nima imena), …, pospešek izhodišča.

Osnovni pojem (za 1 delec)

Gibalna količina

$$\vec{p} = m\vec{v} \qquad \frac{d\vec{p}}{dt} = m\vec{a} = \vec{F}$$

Vrtilna količina

$$\vec{l} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \qquad \frac{d\vec{l}}{dt} = m\dot{\vec{r}} \times \vec{p} = m\,\vec{r} \times \dot{\vec{p}} = \vec{r} \times \vec{F} = \vec{M}$$

Energija

- **Kinetična**:

$$T = \frac{1}{2}m\,|\vec{v}|^2 = \frac{1}{2}mv^2 = \frac{1}{2}m(\vec{v}\cdot\vec{v})$$

- **Delo sile:** $\vec{F} = \vec{F}(\vec{r})$; $P = \vec{F}\cdot\vec{v}$ je moč.

$$A = \int_{pot} \vec{F}\cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = \int_{t_1}^{t_2} P(t)dt$$

$$A = \int_{pot} m\vec{a}\cdot d\vec{r} = \int_{t_1}^{t_2} m\,\dot{\vec{v}}\cdot\vec{v}dt = (*)$$

$$\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt} = 2\vec{v}\frac{d\vec{v}}{dt} = \frac{d}{dt}\vec{v}$$

$$\Rightarrow (*) = \frac{1}{2}\int_{t_1}^{t_2} mv^2dt = \frac{1}{2}mv^2(t_2) - \frac{1}{2}mv^2(t_1)$$

$$A = T(\vec{r}_2) - T(\vec{r}_1) = T\big(\vec{r}(t_2)\big) - T\big(\vec{r}(t_1)\big)$$

- **Potencialna energija:**

$$A_I = \int_{pot\ I} \vec{F}(\vec{r})d\vec{r} \qquad A_{II} = \int_{pot\ II} \vec{F}(\vec{r})d\vec{r}$$

Ce velja $A_I = A_{II}$, velja $\oint \vec{F}\cdot d\vec{r} = -A_I + A_{II} = 0$

$$\vec{F}(\vec{r}) = -\nabla u = -\left(\frac{\partial u(\vec{r})}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = -\frac{\partial u}{\partial r} = -\nabla_{\vec{r}}u$$

$$A = -\int_{pot} \nabla u\cdot d\vec{r} = -\big(u(\vec{r}_2) - u(\vec{r}_1)\big) = -u(\vec{r}_2) + u(\vec{r}_1)$$

- **Celotna energija:**

$$E = T + u \qquad \Delta E = E_2 - E_1 = (T_2 - T_1) + (u_2 - u_1) = A + (-A) = 0$$

Konzervativne sile

V 1D

Sila je odvisna samo od kraja in **<u>ne</u>** od časa:

$$u(x) = -\int_{x_0}^{x} F(s)ds\,; \quad -\frac{du}{dx} = F(x)$$

V 2D:

$$\vec{F}(x,y) = F_1(x,y)\vec{e}_1 + F_2(x,y)\vec{e}_2$$

ampak tudi:

$$\vec{F} = -\nabla u = -\frac{\partial u}{\partial x}\vec{e}_1 - \frac{\partial u}{\partial y}\vec{e}_2$$

$$\Rightarrow \ \frac{\partial F_1}{\partial y} = -\frac{\partial^2 u}{\partial x \partial y} \qquad \frac{\partial F_2}{\partial x} = -\frac{\partial^2 u}{\partial y \partial x}$$

Ce je mešani odvod zvezen potem je **obvezno potrebni pogoj, da je sila konzervativna**:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \Rightarrow \quad \vec{F} = -\nabla u$$

Nekonzervativne sile

- **Linearni zakon upora:**

$$\vec{F} = -k\vec{v};\ \ k \geq 0 \qquad A = \int \vec{F}d\vec{r} = -k\int_{t_1}^{t_2} \vec{v}\cdot\vec{v}\,dt = -k\int v^2 dt \leq 0$$

- **Trenje:**

$$F = -kF_N\, sgn(v_x) = -k|F_N|\frac{v}{|v|}; k \geq 0 \Rightarrow A \leq 0$$

- **Kvadratni zakon upora**:

$$\vec{F} = -kv^2\frac{\vec{v}}{v}; k \geq 0$$

Sistem N točkastih teles

Imamo sistem N točkastih teles.

Gibalna količina

$$m_1\vec{a}_1 = \vec{F}_1^z + \vec{F}_{12} + \sum_{j>2}\vec{F}_{1j}$$
$$m_2\vec{a}_2 = \vec{F}_2^z + \vec{F}_{21} + \sum_{j\neq 2}\vec{F}_{2j} \ldots$$

$$\Rightarrow \vec{F}^Z \stackrel{\text{def}}{=} m\vec{a}^* = \sum_{i=1}^{N} m_i\vec{a}_i = \sum_{i=1}^{N}\vec{F}_i + \vec{F}_{21} + \vec{F}_{12} + \sum_{j=1, j\neq i, i=1}^{N}\vec{F}_{ij}$$

Tu izrazimo zadnjo vsoto kot in upoštevamo 3. NZ:

$$\sum_i\sum_{i<j}\left(\vec{F}_{ij} + \vec{F}_{ji}\right) = 0$$

Tako dobimo **pospešek težišča**:

$$\Rightarrow \vec{F}^Z = m\vec{a}^* = \sum_{i=1}^{N} m_i \vec{a}_i$$

Sedaj se lahko vrnemo h gibalni:

$$\vec{P} = \sum_{i=1}^{N} m_i \vec{v}_i = m\vec{v}^*$$

$$\vec{r}^* = \frac{\sum_i^N m_i \vec{r}_i}{\sum_i^N m_i} = \vec{r}_T \qquad \vec{v}^* = \dot{\vec{r}}^* = \frac{1}{m}\sum_i m_i \vec{v}_i = \frac{1}{m}\sum_i \vec{p}_i$$

Tako dobimo res **Izrek o gibalni količini za sistem N točkastih teles**:

$$\vec{P} = m\vec{v}^*$$

Pomemben je se **Izrek o gibanju težišča**:

$$\vec{a}^* = \frac{d}{dt}\vec{v}^* \quad \Rightarrow \frac{d\vec{P}}{dt} = m\vec{a}^*$$

Vrtilna količina

$$\vec{L} = \sum_i \vec{l}_i = \sum_i \vec{r}_i \times \vec{p}_i \quad \dot{\vec{L}} = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i) = \sum_i \vec{r}_i \times m\,\vec{a}_i = \sum_i \vec{r}_i \times \vec{F}_i$$

Upoštevati moramo navor notranjih in zunanjih sil:

$$\Rightarrow \dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^{zun} + \sum_{i,j\neq i} \vec{r}_i \times \vec{F}_{ij}^{not} = \vec{M}^z + \vec{M}^n = \vec{M}$$

Ce podrobneje pogledamo samo notranje sile:

$$\sum_{i=1;i\neq j}^{N} \vec{r}_i \times \vec{F}_{ij} = \sum_{i,j<i} (\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}) = \sum_{i,j<i} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} = \vec{M}^{not}$$

V primeru, da so notranje sile oblike:

$$\vec{F}_{ij} = F_{ij}\frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|} \Rightarrow \vec{M}^{not} = 0$$

$$\dot{\vec{L}} = \vec{M}^{zun}; \vec{M}^{zun} = 0 \quad \Rightarrow \vec{L} = konst.$$

Energija

- **Delo sile:**

$$A = \sum_i \int_{pot} \vec{F}_i(\vec{r}_i)d\vec{r}_i = A^{zun} + A^{not}$$

Delo zunanjih sil in notranjih sil sta potem tako:

$$A^{zun} = \sum_i \int_{pot_i} \vec{F}_i^z(\vec{r}_i) d\vec{r}_i \qquad A^{not} = \sum_{i,j\neq i} \int_{pot_i} \vec{F}_{ij} d\vec{r}_i$$

$$A = \sum_i \int_{pot_i} m_i \dot{\vec{v}}_i \cdot \vec{v}_i dt = T(t_2) - T(t_1) = \Delta T$$

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \sum_i T_i$$

Kinetična energija je prikladna, ker je vedno pozitivna za vsak delec. Hitrost i-tega delca lahko zapišemo kot hitrost težišča sistema in njegovo hitrost glede na težišče:

$$\vec{v}_i = \vec{v}^* + \vec{v}_i'$$

$$\Rightarrow T = \frac{1}{2}\sum_i m_i(\vec{v}^* + \vec{v}_i') \cdot (\vec{v}^* + \vec{v}_i') = \frac{1}{2}\sum_i m_i\left(v^{*2} + v_i'^2 + 2\vec{v}_i' \cdot \vec{v}^*\right)$$

$$\sum_i (m_i \vec{v}_i{}') = 0 \;\; \Rightarrow \sum_i (\vec{v}_i - \vec{v}^*) m_i = \sum_i m_i \vec{v}_i - \sum_i m_i \vec{v}^* = 0$$

Tako dobimo kinetično energijo kot kinetično energijo težišča in potem prištete se vse delce:

$$T = \frac{1}{2}\left(\sum_i m_i\right) v^{*2} + \frac{1}{2}\sum_i m_i v_i'^2$$

Konzervativne sile (potencialna energija)

$$\vec{F}_i^z = -\nabla_i U_i^z(\vec{r}_i); \quad \nabla_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right)$$

$$A^z = \sum_i \int_{pot_i} \vec{F}_i^z d\vec{r}_i = -U^z(t_2) + U^z(t_1) = -\Delta U^z$$

U je lahko odvisen od kraja kot kako nehomogeno polje in lahko tudi časovno spremenljiv. Pogosto je V **centralni potencial**:

$$V_{ij} = V_{ij}(|\vec{r}_{ij}|); \quad \vec{r}_{ij} = \vec{r}_i - \vec{r}_j$$

Takrat je delo notranjih sil po poti i in po poti j:

$$A^{not} = \sum_{i,j<i}\left(\int_{pot_i} \vec{F}_{ij} d\vec{r}_i + \int_{pot_j} \vec{F}_{ji} d\vec{r}_j\right) = \sum_{i,j<i} \int_{pot(i,j)} \vec{F}_{ij} d\vec{r}_{ij}$$

Kdaj je delo notranjih sil nič? $A^{not} = 0$

- $\vec{F} \perp d\vec{r}$
- $\vec{F} = 0$ (neodvisni delci)
- $d\vec{r} = 0$ (toga telesa)

$$\vec{F}_{ij} = -\nabla_{ij} V_{ij}(\vec{r}_{ij}); \quad \nabla_{ij} = \left(\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{ij}}, \frac{\partial}{\partial z_{ij}}\right)$$

$$V = V(|\vec{r}|) = V(r); \;\; r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla V(r) = \left(\frac{\partial V}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial V}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial V}{\partial r}\frac{\partial r}{\partial z}\right) = \frac{\partial V}{\partial r}\nabla r$$

$$\Rightarrow \nabla V(r) = \frac{\partial V(r)}{\partial r}\frac{\vec{r}}{r} \qquad \vec{F}_{ij} = -\frac{\partial V_{ij}(\vec{r}_{ij})}{\partial r_{ij}}\frac{\vec{r}_{ij}}{r_{ij}}$$

Delo notranjih sil je potem:

$$A^{not} = -\sum_{i,j<i}\int \nabla_{ij}V_{ij}(r_{ij})dr_{ij} = -\sum_{i,j<i}\left(\frac{\partial}{\partial x_i}V_{ij}dx_i + \cdots\right) = -U^n(t_2) + U^n(t_1) = -\Delta U^n$$

$$U^n = \sum_{i,j<i} V_{ij}(r_{ij}) = \sum_{ij}\frac{1}{2}V_{ij}(r_{ij})$$

In dobimo zopet celotno energijo ki se ohranja, ker $\Delta T = A$ in $\Delta U = -A$:

$$E = T + U^z + U^n; \quad \Delta E = 0$$

Virialni teorem

Izpeljava:

$$V(\vec{r}) = \alpha r^n \qquad 2\langle T\rangle = n\langle V\rangle$$

Lagrangeev Formalizem

Newtonove enačbe povsem zadoščajo. Drugi formalizmi pa v določenih primerih olajšajo formulacijo in izračun. Lagrangeev formalizem uporablja posplošene koordinate in vezmi med koordinatami.

Holonomna vez je funkcija: $f(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) = 0$
Neholonomna vez je funkcija, kjer $\nexists f = 0$ npr. $f = x^2 + y^2 + z^2 - R^2 \geq 0$

Sistem holonomnih vezi

$$f_k(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t) = 0; \quad k = 1,2,\dots,K$$

Recimo, da imamo presek krogle z ravnino. Imamo torej, dve vezi, če se lahko nek delec giblje samo po robu preseka:

$$f_1 = x^2 + y^2 + z^2 - R^2 \;\; f_2 = y - d = 0; \;\; k = 2$$

Imamo 3 koordinate in 2 vezi. Torej imamo samo 1 posplošeno koordinato.

Posplošene (generalizirane) koordinate

N delcev opisemo z $3N$ koordinatami. Ce imamo k vezi nam ostane torej:

$$n = 3N - k$$

neodvisnih koordinat s katerimi lahko izrazimo originalne koordinate:

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, q_n, t); i = 1,2,\dots,N$$

$$q_j = q_j(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, t); \;\; j = 1,2,\dots,n \;\; n = 3N - k$$

d'Alembertov princip (poglej slikce)

Z roko opravimo delo, ampak se na točkah 1,2,3 ne pozna. Prejme jo vzmet.

$$\delta A = 0$$

Sile zapišemo kot:

$$\vec{F}_i = \vec{F}_i^{aktivne} + \vec{F}_i^{vezi}$$

Kjer aktivne sodelujejo pri premiku, sile vezi pa ne opravljajo dela pri premiku ampak jih rabimo za statiko. Ce zapišemo koordinate in raztezek vzmeti s:

$$y_2 = l\sin\phi \qquad \delta y_2 = l\cos\phi\ \delta\phi$$
$$y_3 = 2l\sin\phi \qquad \delta y_3 = 2l\cos\phi\,\delta\phi$$
$$s = y_3 - h = 2l\sin\phi - h\ \Rightarrow F_3 = ks$$

Tako lahko zapišemo delo kot:

$$\delta A = \sum_i \left(\vec{F}_i^a\ \delta\vec{r}_i + \vec{F}_i^v\ \delta\vec{r}_i\right) = \sum_i \vec{F}_i^a\ \delta\vec{r}_i = F_2\, dy_2 - F_3\delta y_3 = 0$$

Virtualno delo je 0. Masne točke od dela niso nič »profitirale«. So prijele delo in ga potem oddale. Torej v našem primeru:

$$F_2 l\cos\phi\ \ \delta\phi - k(2l\sin\phi - h)2l\ cos\phi\ \delta\phi = (F_2 l\cos\phi - k(2l\sin\phi - h)2l\ cos\phi)\delta\phi = Q\ \delta\phi = 0$$

Kjer je Q **posplošena sila** in $\delta\phi$ **posplošena koordinata**.
Vse ostale sile so vezi. Ne prispevajo k delu, ampak zagotavljajo, da so stvari pri miru.

Princip virtualnega dela (glej slike)

$$i = 1, \dots, N; 3N$$

Statika:

$$\dot{\vec{r}}_i = 0 \Rightarrow \ddot{\vec{r}}_i = 0 \Rightarrow \vec{F}_i = 0$$

kjer je $\vec{F}_i$ vsota vseh sil na i-ti delec. Naredimo **virtualni premik**. $\delta\vec{r}_i$ so povezani med sabo.

$$\vec{F}_1 \cdot \delta\vec{r}_i = 0 \qquad \sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = 0$$

Razdelimo silo na sile, ki opravljajo delo in na sile vezi.

$$\vec{F}_i = \vec{F}_i^a + \vec{F}_i^v$$

Torej imamo:

$$\sum_i \vec{F}_i^a \delta\vec{r}_i + \sum_i \vec{F}_i^v \delta\vec{r}_i = 0$$

Upoštevamo, da sile vezi ne opravljajo dela in dobimo **pogoj statike oz. princip virtualnega dela**:

$$\sum_i \vec{F}_i^a \cdot \delta\vec{r}_i = 0 \qquad \delta A = 0$$

Dinamika:

Bernoulli in d'Alambert sta posplošila enačbo:

$$\vec{F}_i = m\ddot{\vec{r}}_i \quad \to \vec{F}_i - m_i\ddot{\vec{r}}_i = 0$$

$$\Rightarrow \sum_{i=1}^{N} (\vec{F}_i - m_i\ddot{\vec{r}}_i)\delta\vec{r}_i = 0$$

Kjer lahko spet sile razbijemo na aktivne in na sile vezi:

$$\sum_{i=1}^{N} (\vec{F}_i^a - m_i\ddot{\vec{r}}_i)\delta\vec{r}_i + \sum_{i=1}^{N} \vec{F}_i^v \delta\vec{r}_i = 0$$

Tu spet upoštevamo, da sile vezi ne opravljajo dela in dobimo **d'Alembertov princip**:

$$\sum_{i=1}^{N} (\vec{F}_i^a - m_i\ddot{\vec{r}}_i)\delta\vec{r}_i = 0$$

Spomnimo se še, da so $\boldsymbol{\delta\vec{r}_i}$ **med sabo povezani!** [Primer v zvezku]

Lagrangeeva enačba

Izhajamo iz d'Alembertovega principa:

$$\sum_i (\vec{F}_i^a - m_i\ddot{\vec{r}}_i)\delta\vec{r}_i = 0$$

Vpeljemo posplošene koordinate q_j s katerimi izrazimo stare $\vec{r}_i = \vec{r}_i(q_1, q_2, \dots, q_n, t)$

$$d\vec{r}_i = \sum_{j=1}^{n} \frac{\partial\vec{r}_i}{\partial q_j} dq_j + \frac{\partial\vec{r}_i}{\partial t} dt$$

Ce si sedaj pogledamo hitrost:

$$\vec{v}_i = \dot{\vec{r}}_i = \sum_{j=1}^{n} \frac{\partial\vec{r}_i}{\partial q_j}\, \dot{q}_j + \frac{\partial\vec{r}_i}{\partial t}$$

Poglejmo do ob casu $t; \delta t = 0$:

$$d\vec{r}_i = \sum_{j=1}^{n} \frac{\partial\vec{r}_i}{\partial q_j}\, dq_j$$

Statika:

$$\delta A = \sum_{i=1}^{N} \vec{F}_i^a d\vec{r}_i = 0$$

Prepišemo ta izraz s posplošenimi koordinatami:

$$\delta A = \sum_{i=1}^{N} \vec{F}_i^a \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \left(\sum_i \vec{F}_i^a \frac{\partial \vec{r}_i}{\partial q_j} \right) dq_j$$

$$\Rightarrow \sum_{j-1}^{n} Q_j \delta q_j = 0; \quad \forall q_j \ \Rightarrow Q_j = 0$$

Kjer je Q **Generalizirana/posplošena sila**:

$$Q = \sum_{i=1}^{N} \vec{F}_i^a \frac{\partial \vec{r}_i}{\partial q_j}$$

V sistemih, ki imajo navore so to lahko navori, ni pa to nujno. Tako smo dobili 2. pogoj za statiko (**vsota vseh navorov je 0**).

Dinamika

Izhajamo iz izraza v katerega vpeljemo posplošene koordinate:

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \delta \vec{r}_i = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} \left[\frac{d}{dt} \left(m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \dot{\vec{r}}_i \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} \right] \delta q_j$$

Za izpeljavo bomo rabili oba parcialna odvoda:

$$\frac{\partial \vec{r}_i}{\partial t} = \vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t} \quad \Rightarrow \quad \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}; \quad \frac{\partial \vec{r}_i}{\partial \dot{q}_j} = 0$$

$$\frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \sum_{k=1}^{n} \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} = \frac{\partial}{\partial q_k} \left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right) = \frac{\partial}{\partial q_j} \frac{d\vec{r}_i}{dt}$$

Tu je vredno omeniti, da je razlika med $\frac{\partial \vec{r}_i}{\partial t}$ in $\frac{d\vec{r}_i}{dt}$

$$\frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial \vec{r}_i}{\partial t}$$

Te odvode sedaj lahko vstavimo v prvoten izraz:

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \delta \vec{r}_i = \sum_{i=1}^{N} \left(\frac{d}{dt} \left(m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial \dot{q}_j} \right) - m_i \vec{v}_i \frac{\partial \vec{v}_i}{\partial q_j} \right) = (*)$$

$$\frac{\partial T_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \vec{v}_i \cdot \vec{v}_i \right)$$

$$\Rightarrow (*) = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}$$

To lahko vstavimo v d'Alembertov princip, kjer imamo generalizirano silo:

$$\sum_{j=1}^{n}\left(Q_j - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} + \frac{\partial T}{\partial q_j}\right)\delta q_j = 0 \quad \forall q_j$$

Da to velja mora biti del v oklepajih načelen in tako dobimo izraz:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$$

Ta izraz velja povsem splošno, tudi za nekonzervativne sile. Privzemimo zdaj, da so sile konzervativne:

$$\vec{F}_i^a = -\nabla_i V(\vec{r}_i)$$

$$Q_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_{i=1}^{N} \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} == -\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \cdots\right) = -\frac{\partial V}{\partial q_j}$$

Ce to vstavimo v enačbo od prej, dobimo:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Tu za popolnost se uvedemo **Lagrangeevo funkcijo** in dobimo **Euler-Lagrangeeve enačbe**:

$$L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j) - V(q_j, t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0; \quad j = 1,2,\dots,n$$

Komentar

Recimo, da imamo $V(q_j, \dot{q}_j)$ in da najdemo tako funkcijo, da velja:

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j} \leftrightarrow -\left(\nabla_{q_j} - \frac{d}{dt}\nabla_{\dot{q}_j}\right)V$$

Se vedno velja $L = T - V$ in Lagrangeeva enačba. Primer take sile je Lorentzeva sila:

$$\vec{Q} = \vec{F} = e\vec{v} \times \vec{B} = e\,\vec{\dot{q}} \times \vec{B}; \quad \vec{q} = \vec{r}$$

[Poglej primere v zvezku]

Lastnosti Lagrangiana

Nedoločenost do časovnega odvoda

Lagrangeeva funkcija je **nedoločena** do časovnega odvoda poljubne funkcije:

$$L(\underline{q}, \underline{\dot{q}}, t); \quad q_k = 1,2,\dots,n$$

Vzamemo lahko poljubno funkcijo:

$$F(q_1, q_2, \dots, q_n, t) = F(\underline{q}, t) \quad \rightarrow \quad L' = L + \frac{d}{dt}F$$

Dokaz z odvajanjem

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} - \frac{\partial L'}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left[\sum_i \frac{\partial F}{\partial q_i}\,\dot{q}_i + \frac{\partial F}{\partial t}\right] - \frac{\partial}{\partial q_j}\frac{d}{dt}F =$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{\partial F}{\partial q_i}\,\dot{q}_i + \frac{\partial F}{\partial t}\right) - \frac{\partial}{\partial q_j}\frac{dF}{dt} = 0$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{\partial F}{\partial q_i}\,\dot{q}_i + \frac{\partial F}{\partial t}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\frac{dF}{dt} = \frac{\partial}{\partial q_j}\frac{dF}{dt}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_j} - \frac{\partial L'}{\partial q_j} = 0 \;\Leftrightarrow\; \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Mehanska podobnost

$$L' = cL; \;\; c \in \mathbb{R}, c = konst.$$

Skalirne enačbe

Naj bo potencialna energija homogena funkcija koordinat. Naredimo razteg/skrčitev (zamenjamo dolžinsko enoto):

$$\vec{r}_i \rightarrow \alpha \vec{r}_i$$

Potem:

$$\begin{aligned}
&V(\alpha\vec{r}_1, \dots, \alpha\vec{r}_n) = \alpha^k V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N); \;\; k \in \mathbb{R} \\
&\vec{r}_i \rightarrow \alpha\vec{r}_i \\
&t \rightarrow \beta t \\
&\vec{v}_i \rightarrow \frac{\alpha}{\beta}\vec{v}_i \\
&T \rightarrow \left(\frac{\alpha}{\beta}\right)^2 T \\
&V \rightarrow \alpha^k V
\end{aligned}$$

Kjer naj velja, da uskladimo skaliranje:

$$\left(\frac{\alpha}{\beta}\right)^2 = \alpha^k \;\Rightarrow \beta = \alpha^{1-\frac{1}{2}k}$$

$$\Rightarrow L = T - V \rightarrow \alpha^k L = cL$$

Trajektorije v novih enotah, so si geometrijsko podobne. Napišemo si lahko **skalirne enačbe**:

$$\left(\frac{t'}{t}\right) = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}k} \qquad \left(\frac{v'}{v}\right) = \left(\frac{l'}{l}\right)^{\frac{1}{2}k} \qquad \left(\frac{E'}{E}\right) = \left(\frac{l'}{l}\right)^{k}$$

Primeri skalirnih enacb:

1. **Harmonska sila** $V(x) \propto x^2; k = 2$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}2} = \frac{l'}{l}$$

   Nihajni čas ni odvisen od amplitude

2. **Homogeno polje** $V(x) \propto x; k = 1$

$$\frac{t'}{t} = \sqrt{\frac{l'}{l}}$$

   Kot prosti pad, kjer $t = \sqrt{2gh}$

3. **Gravitacija, Coulomb** $V(x) \propto \frac{1}{x};\ \ k = -1$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{\frac{3}{2}} \to \left(\frac{t'}{t}\right)^2 = \left(\frac{l'}{l}\right)^3$$

   Kot 3. Keplerjev zakon.

4. **Virialni teorem**:

Ciklične koordinate, konstante gibanja, ohranjene količine

Konstanta gibanja

$$F(\underline{q}, \underline{\dot{q}}, t) \to F(t);\ \ \underline{q}(t)$$

Ce velja:

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\, \dot{q}_i + \sum_{i=1}^{n} \frac{\partial F}{\partial \dot{q}_i}\, \ddot{q}_i + \frac{\partial F}{\partial t} = 0$$

potem je F **konstanta gibanja**.

Npr. če velja $V \neq V(q_j)$ za neki q_j in enako $T \neq T(q_j)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_j} = konst. = F$$

Ciklične koordinate

q_j ne nastopa v L $\ \ L \neq L(q_j)$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \quad \Rightarrow p_j = \frac{\partial L}{\partial \dot{q}_j} = konst.$$

Potem imenujemo p_j **posplošeni moment**.

Hamiltonova funkcija

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j} - L$$

$$\frac{dH}{dt} = \sum_j^{n} \left(\frac{\partial L}{\partial \dot{q}_j}\, \ddot{q}_j + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j - \frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial \dot{q}}\ddot{q}_j\right) - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$$

Zapišimo kinetično energijo s posplošenimi koordinatami:

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^2 \; ; \;\; \vec{r}_i = \vec{r}_i(q_1, \dots, q_n)$$

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \sum_{j,k=1}^{n} \left(\frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j \cdot \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \right) = \sum_{j,k=1}^{n} \frac{1}{2} w_{jk}(q_1, \dots, q_n) \dot{q}_j \dot{q}_k$$

$$w_{jk} = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = w_{kj}$$

Tako je potem:

$$H = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = 2T - T + V$$

$$\Rightarrow H = T + V = konst.; \;\; \frac{\partial L}{\partial t} = 0$$

Emmy Noether izrek

Pravi, da ima vsaka odvedljiva simetrija akcije nekega fizikalnega sistema z konservativnimi silami ustrezen ohranitveni zakon. Recimo, ce je Lagrangian nekega sistema neodvisen od spremembe izhodisca koordinatnega sistema potem bo sistem ohranjal (linear) gibalno kolicino.

$$q_i(t) \to Q_i(s, t); \quad Q_j \to q_j \; ko \; s \to 0$$

Lagrangeeva funkcija naj ne bo odvisna od s. V 1D:

$$x \to X = x + sa; \;\; a = konst. \;\; s \in \mathbb{R}$$
$$q(t) \to Q(s, t)$$

Naredili smo enoparametricno transformacijo koordinat (tu v bistvu samo premik izhodišča)

$$\frac{dL}{ds} = \frac{d}{ds} L\left(\underline{Q_i}(s,t), \underline{\dot{Q}_i}(s,t), t\right) = 0$$

Izrazimo to Lagrangeevo funkcijo v novih koordinatah:

$$\frac{dL}{ds} = \sum_i \frac{\partial L}{\partial Q_i} \frac{\partial Q_i}{\partial s} + \sum_i \frac{\partial L}{\partial \dot{Q}_i} \frac{\partial \dot{Q}_i}{\partial s} =$$

Vzamemo, ko gre $s \to 0$

$$= \sum_i \frac{\partial L}{\partial q_i} \frac{\partial Q_i}{\partial s} \Big|_{s \to 0} + \sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \dot{Q}_i}{\partial s} \Big|_{s \to 0} =$$

Sedaj uporabimo E-L enačbo:

$$= \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial Q_i}{\partial s}\ |_{s\to 0} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\left(\frac{d}{dt}\frac{\partial Q_i}{\partial s}\right)\ |_{s\to 0} = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial Q_i}{\partial s}\ |_{s\to 0}$$

Prepoznamo posplošen moment in dobimo to, kar pravi izrek:

$$\sum_i p_i \frac{\partial Q_i}{\partial s}\ |_{s\to 0} = konst.$$

Hamiltonov princip minimalne akcije

Fermatov princip v optiki

Princip pravi, da zarek svetlobe med dvema točkama vzame tako pot, da jo opraviti v minimalnem času.

$$T = \int_{od\ A\ do\ B} dt = min. = \int \frac{1}{c}\frac{c}{v}\frac{ds}{dt}dt = \frac{1}{c}\int n(s)ds = min.$$

Vprašanje je če lahko podobno naredimo za delce. Hamilton si je izmislil **akcijo**, ki je funkcional:

$$S = S[\{\underline{q}\},\{\dot{\underline{q}}\}] = \int_{t_1}^{t_2} L\big(\underline{q}(t),\dot{\underline{q}}(t),t\big)dt$$

Vzemimo:

$$t_1\colon q_i(t_1);\ \ \dot{q}_i(t_1) = v_i$$

Obstaja <u>ena sama</u> rešitev Lagrangeeve enačbe, ki bo prišla v točko 2 v času t_2. S je funkcija poti (podobno kot problem brahistohrone). Iščemo krivuljo pri kateri je $S = min.$

Izpeljava Euler-Lagrangeevih enačb (obvezno glej slikco)

Naredimo variacijo koordinate

$$L = L\big(\underline{q},\dot{\underline{q}},t\big)\ \ q_i(t) \to q_i(t) + \delta q_i(t)$$

$$S = \int_{t_1}^{t_2} L dt$$

$$\delta S = \int_{t_1}^{t_2}\left(\sum_i \frac{\partial L}{\partial q_i}\ \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ \delta \dot{q}_i\right)dt = (*)$$

Zadnji člen izračunamo preko per partesa:

$$\frac{\partial L}{\partial \dot{q}_i}\ \delta\dot{q}_i = \frac{\partial L}{\partial \dot{q}_i}\delta q_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\ dt = -\int_{t_1}^{t_2}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\ dt$$

Tako dobimo:

$$\Rightarrow (*) = \int_{t_1}^{t_2}\left(\sum_i\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\right)\right)\delta q_i dt = \sum_i \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\right)\delta q_i dt = 0\ \ \forall q_i$$

To je lahko izpolnjeno le takrat, ko je integrand ničeln. Tako iz Hamiltonovega principa minimalne akcije dobimo **Euler-Lagrangeeve enačbe**:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q} = 0$$

Dokaz, da je Lagrangian nedoločen do časovnega odvoda

$$L' = L + \frac{d}{dt}F(\underline{q}, t)$$

To lahko preverimo preko Hamiltonovega principa. Vpeljemo akcijo:

$$S' = \int_{t_1}^{t_2} L' dt = \int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2} \frac{d}{dt} F\, dt = S + F(\underline{q}(t), t)\Big|_{t_1}^{t_2}$$

Poglejmo sedaj koliko se integral spremeni, če pot malo spremenimo od prave rešitve, ki jo da E-L:

$$\delta S' = \delta S + \delta F\Big|_{t_1}^{t_2}$$

$$\delta F\Big|_{t_1}^{t_2} = \sum_i \frac{\partial F}{\partial q_i}\, \delta q_i(t)\Big|_{t_1}^{t_2} = 0$$

Tu pa upoštevamo, da je premik δq_i v začetni točki ob t_1 in v končni točki ob t_2 enak nič. Tako smo dokazali, da lahko dodamo tako funkcijo, ker velja:

$$\delta S' = \delta S$$

Enodimenzionalni problem

$q(t)$ ena dimenzija, en delec

$$L = \frac{1}{2}w(q)\dot{q}^2 - V(q) \quad E = konst. = H = T + V = \frac{1}{2}w(q)\dot{q}^2 + V(q)$$

$$\dot{q} = \frac{dq}{dt} = \pm\sqrt{\frac{2(E - V(q))}{w(q)}}$$

$$\Rightarrow \int_{t_1}^{t_2} dt = \pm \int_{q_1}^{q_2} \sqrt{\frac{w(q)}{2(E - V(q))}}\, dq$$

Ta enačba je praktično vedno rešljiva za $t(q)$. Za dobiti $q(t)$ pa ni nujno vedno res lahko in se $\pm$.

Problem dveh teles (poglej slikco)

$$T = \frac{1}{2}m|\dot{\vec{r}}_1|^2 + \frac{1}{2}m_2|\dot{\vec{r}}_2|^2$$

$$M = m_1 + m_2; m = \frac{m_1 m_2}{m_1 + m_2}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \;\; \vec{r}_T = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)$$

$$\Rightarrow \vec{r}_1 = \vec{r}_T - \frac{m_2}{M}\vec{r} \qquad \vec{r}_2 = \vec{r}_T + \frac{m_1}{M}\vec{r}$$

Torej je kinetična energija:

$$T = \frac{1}{2}M\left|\dot{\vec{r}}_T\right|^2 + \frac{1}{2}m\left|\dot{\vec{r}}\right|^2$$

Tako lahko zapišemo Lagrangian:

$$L = \frac{1}{2}M\left|\dot{\vec{r}}_T\right|^2 + \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 - U\left(\vec{r}, \dot{\vec{r}}\right)$$

Zanima nas $\vec{r}_{1,2}(t)$. Poznamo $\vec{r}_{1,2}(0)$ in $\dot{\vec{r}}_{1,2}(0)$ kar pomeni, da imamo $6 + 6 = 12$ začetnih pogojev. Poiščimo najprej konstante gibanja:

$\vec{r}_T$ je ciklična koordinata

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}_T} = 0 \;\Rightarrow\; M\dot{\vec{r}}_T = P_T = konst.$$

Posplošen moment je gibalna količina težišča.

$$\vec{r}_T(t) = \vec{r}_T(0) + \dot{\vec{r}}_T(0)t$$

Tu smo porabili že $3 + 3$ začetnih pogojev.

$$L = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 - U\left(\vec{r}, \dot{\vec{r}}\right)$$

Energija je konstantna

$$H = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 + U\left(\vec{r}, \dot{\vec{r}}\right) = konst.$$

Predpostavimo, da je potencial centralen:

$$U\left(\vec{r}, \dot{\vec{r}}\right) = V(\vec{r}) = V(r)$$

Ohranitev vrtilne količine

$$\vec{F} = -\nabla V = -\frac{dV}{dr}\frac{\vec{r}}{r} \Rightarrow \dot{\vec{L}} = \vec{M} = \vec{r} \times \vec{F} = 0 \;\Rightarrow \vec{L} = m\vec{r} \times \dot{\vec{r}} = konst.$$

Delec se bo lahko gibal le v ravnini, ker:

$$\vec{r} \cdot \vec{L} = \vec{r}(\vec{r} \times m\dot{\vec{r}}) = 0$$

V ravnini je dobro/ugodno delati z (r, ϕ) zato L pretvorimo v:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

Izračunamo posplošeni moment za ϕ:

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = l = l_0 = konst.$$

Ce zapišemo spremembo ploščine dobimo ploščinsko hitrost oz. **2. Keplerjev zakon** (zaenkrat za posplošen $V(r)$)

$$\Delta S = \frac{1}{2} r r \Delta\phi = \frac{1}{2} r^2 \dot{\phi} \Delta t \quad \Rightarrow \frac{dS}{dt} = \frac{1}{2} r^2 \dot{\phi} = konst.$$

Keplerjev problem

Torej kot vemo od problema dveh teles imamo ohranjene količine:

$$\vec{p}_T, E, \vec{l}, \vec{r} \cdot \vec{l} = 0$$

Izkaze se pa tudi, da v posebnem primeru gravitacijskega potenciala imamo ohranitev **Laplace-Runge-Lenzovega vektorja**:

$$\vec{A} = \vec{p} \times \vec{L} - mk \frac{\vec{r}}{r} = konst.$$

Dokažimo:

$$m\ddot{\vec{r}} = \vec{F}(\vec{r}) = f(r)\frac{\vec{r}}{r}; \quad f(r) = -\frac{\partial V(r)}{\partial r}$$

$$\dot{\vec{p}} = m\ddot{\vec{r}}$$

$$\dot{\vec{p}} \times \vec{L} = \frac{mf(r)}{r}\left(\vec{r} \times \left(\vec{r} \times \dot{\vec{r}}\right)\right) = \frac{mf(r)}{r}\left((\vec{r} \cdot \dot{\vec{r}})\vec{r} - \dot{\vec{r}} r^2\right)$$

Naredimo pomožni račun:

$$\frac{d}{dt} r^2 = \frac{d}{dt}(\vec{r} \cdot \vec{r}) = 2\vec{r} \cdot \dot{\vec{r}} - 2r\dot{r} \qquad \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}\vec{r}}{r^2}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\vec{p} \times \vec{L}\right) = \dot{\vec{p}} \times \vec{L} + \vec{p} \times \dot{\vec{L}} = \dot{\vec{p}} \times \vec{L} \qquad ; \vec{L} = konst.$$

$$\frac{d}{dt}\left(\vec{p} \times \vec{L}\right) = \dot{\vec{p}} \times \vec{L} = \frac{mf(r)}{r}\left((\vec{r} \cdot \dot{\vec{r}})\vec{r} - \dot{\vec{r}} r^2\right) = -mf(r)r^2\left(\frac{\dot{\vec{r}}}{r} - \frac{\dot{r}\vec{r}}{r^2}\right) = -mf(r)r^2 \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$
$$= \frac{d}{dt}\left(-mk\frac{\vec{r}}{r}\right)$$

Izraz $-f(r)r^2 = k = konst.$ je konstanten, če imamo **Gravitacijsko** ali **Coulombsko silo**:

$$\vec{F} = f(r)\frac{\vec{r}}{r}; \; f(r) = \frac{mM}{r^2} G$$

Tako smo dobili izraz:

$$\frac{d}{dt}\left[\vec{p} \times \vec{L} - mk\frac{\vec{r}}{r}\right] = 0 = \frac{d}{dt}\vec{A} \quad \Rightarrow \vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r} = konst.$$

Preverimo lahko se, da je LRL vektor res v ravnini:

$$\vec{A}\cdot\vec{L} = \left(\vec{p}\times\vec{L}\right)\cdot\vec{L} - \frac{mk}{r}\left(\vec{r}\cdot\vec{L}\right) = 0$$

Za pogoje pri Keplerjevem problemu:

$$\vec{A} = \vec{A}_0 = \vec{A}(0);\ \ \vec{p}(0), \vec{r}(0)$$

imamo potem:

$$3\left[\vec{L}\right] + 3\left[\vec{A}\right] + 3\left[\vec{p}_T\right] + 1[E] = 10$$

konstant gibanja, ki pa sicer niso neodvisne. Sedaj lahko se izpeljemo obliko orbit:

$$\vec{A}\cdot\vec{r} = A_0 r\cos\phi = \vec{r}\cdot\left(\vec{p}\times\vec{L}\right) - mkr$$

$$\vec{r}\cdot\left(\vec{p}\times\vec{L}\right) = \vec{L}\cdot(\vec{r}\times\vec{p}) = \vec{L}\cdot\vec{L} = l_0^2 = konst.$$

$$\Rightarrow r(A_0\cos\phi + mk) = l_o$$

$$\Rightarrow r = \frac{l_0^2}{mk\left(1 + \frac{A_0}{mk}\cos\phi\right)};\ \ \epsilon = \frac{A_0}{mk}\quad r_0 = \frac{l_0^2}{mk}$$

$$\Rightarrow r(\phi) = \frac{r_0}{1 + \epsilon\cos\phi}$$

Klasifikacija orbit pri Keplerjevem problemu (nujno glej sliko)

$$\vec{A}(0) = \vec{A}_0 = m\vec{v}\times(\vec{r}\times m\vec{v}) - m\,mMG\frac{\vec{r}}{r};\quad v(0) = v_0, r(0) = R$$

$$\Rightarrow (\pm)\left|\vec{A}_0\right| = m^2Rv_0^2 - m^2MG;\ \ \epsilon = \frac{A_0}{mk}$$

$$\epsilon = \frac{m^2Rv_0^2 - m^2MG}{m^2MG} = \frac{Rv_0^2}{MG} - 1$$

Vezane orbite pri Keplerjevem problemu

1. **Prosti pad:** $\epsilon = -1; v_0 = 0$
2. **Krožnica:** $\epsilon = 0; v_0 = v_1 = \sqrt{\frac{MG}{R}}$ (hitrost je prva kozmična hitrost)
3. **Elipsa:** $-1 < \epsilon < 0$ (manjša od krožnice)
4. **Parabola**: $\epsilon = 1; v_0 = v_2 = \sqrt{\frac{2MG}{R}}$ (hitrost je druga kozmična hitrost/escape velocity)
5. **Elipsa**: $0 < \epsilon < 1$ (večja od krožnice)

Nevezane orbite pri Keplerjevem problemu (sipanje)

6. **Hiperbola**: $\epsilon > 1;\ \ v_0 > v_2$
7. $\epsilon \to \infty; v_0 \to \infty; |A_0| \to \infty$ gre sipalni kot proti $\phi_0 = \frac{\pi}{2}$ ker $\cos\phi_0 = \frac{1}{\epsilon}$

Energija

$$E = \frac{1}{2}mv^2(t) - \frac{mMG}{r(t)} = konst. = E_0$$

V primeru parabole:

$$E_0 > 0$$

V primeru hiperbole:

$$E_0 < 0$$

Keplerjev problem s splošnim $V(r)$

$V(r)$ je sedaj se splošen. Napišimo Lagrangian v polarnih koordinatah:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = l_0 = konst. \quad \Rightarrow \quad \dot{\phi} = \frac{l_0}{mr^2} = V_{cfg}(r)$$

To zgleda kot neke sorte centrifugalen potencial. Skupaj lahko sestavimo efektivni potencial:

$$V_{ef}(r) = V(r) + V_{cf}(r)$$

V_{cfg} je rotacijska (kinetična) energija delca, pri konstantni vrtilni količini l. Ce označimo z d **impact parameter.**

$$\vec{L} = \vec{r} \times \vec{p} \Rightarrow |\vec{L}| = l = mv_T d = konst. \Rightarrow v_T = \frac{l}{md}$$

$$V_{cf} = \frac{1}{2}mv_T^2 = \frac{l^2}{2md^2}$$

Tako lahko zapišemo efektiven potencial:

$$V_{eff} = V + \frac{l^2}{2md^2}$$

Naprej pa postopamo kot že znamo (v 1D):

$$E = \frac{1}{2}m\dot{r}^2 + V + \frac{l^2}{2mr^2}$$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - V_{eff}}$$

$$\int_{t_0}^{t} dt = \pm\sqrt{\frac{m}{2}}\int_{r_0}^{r(t)} \frac{dr}{\sqrt{E - V_{eff(r)}}} \Rightarrow t = f(r)$$

Tako dobimo $r(t) = f^{-1}(t)$. Radi bi prisli do orbit $r(\phi)$:

$$mr^2\dot{\phi} = l \quad \Rightarrow \frac{d\phi}{dt} = \frac{l}{mr^2} \; oz. \;\; dt = \frac{mr^2}{l}d\phi$$

$$\Rightarrow \int_{\phi_0}^{\phi} d\phi = \pm\sqrt{\frac{1}{2m}} \int_{r_0}^{r(t)} \frac{l dr}{r^2\sqrt{E - V_{eff}}} \Rightarrow \phi = g(r)$$

In če to znamo potem obrniti dobimo orbite $r(\phi) = g^{-1}(\phi)$

Klasifikacije orbit (poglej pac slike)

Fiksirani konstanti E in $l \neq 0$, ker ce bi bil $l = 0$ bi imel samo prosti pad.

Kepler $V(r) = -\frac{|\alpha|}{r}$

$V(r) = -\frac{|\alpha|}{r^{\beta}}$

Ce je $0 < \beta < 2$, je kvalitiativno isto kot $\beta = 1$
Ce je $\beta > 2$ je pri $r \to 0;\ V_{ef} < 0$

Odbojni potencial $V(r) = |\alpha| r^{\beta}; \beta > 0$

Gibanje je vedno omejeno za $l > 0$. Za $l = 0; \beta = 2$ dobimo sinusno nihanje. Za $l \neq 0$ in $\beta = 2$ (parabolični potencial) dobimo spet elipse.

Binetova zveza

Pogosto je ugodno računati z $u(\phi) = \frac{1}{r}$. Enačba gibanja iz E-L je potem takem:

$$m\ddot{r} - mr^2\dot{\phi}^2 = f(r);\ \ \dot{\phi} = \frac{l}{mr^2}$$

Izrazimo $\ddot{r}$ in r in dobimo **Binetovo zvezo/enačbo**:

$$\frac{d^2}{d\phi^2}u + u = F(u); \quad F(u) = -\frac{m}{l^2}\frac{1}{u^2} f\left(\frac{1}{u}\right)$$

Za Keplerjev problem $V = -\frac{\alpha}{r};\ f\left(\frac{1}{u}\right) = -\alpha u^2$ dobimo torej:

$$\frac{d^2}{d\phi^2}u + u = \frac{\alpha m}{l^2}$$

Od koder sledi enačba za stožnice:

$$u = \frac{1}{r(\phi)} = A\cos\phi + B\sin\phi + C$$

Opis lege togega telesa

Togo telo ima lastnost, da je razdalja med vsakima dvema točkama konstanta:

$$|\vec{r}_i - \vec{r}_j| = c_{ij} = konst.$$

Skupaj lahko togo telo opišemo s 6 koordinatami:
3 koordinate od $\vec{r}_1$
2 koordinati (polarni in azimutalni kot) od $\vec{r}_{21}$
1 koordinata (rotacija okoli osi $\vec{r}_{21}$) od $\vec{r}_{13}$

Običajno vpeljemo gibajoči se koordinatni sistem, kjer je $\vec{R}$ lega izhodišča in $\vec{r}'$ relativna koordinata

$$\vec{r}_i = \vec{R} + \vec{r}_i'$$

V splosnem gre za neinercialni sistem. Med koordinatami $\vec{r}_i$ in $\vec{r}_i'$ transformiramo z matriko/linearno preslikavo.

Rotacija koordinatnega sistema/telesa

Pasivna rotacija (rotacija koordinatnega sistema)

$$\begin{bmatrix} \cos\phi \\ -\sin\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & +\sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \vec{r}' = R_3(\phi)\vec{r}$$

kjer je R_3 pasivna rotacija.

Aktivna rotacija (rotacija telesa)

$$\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ +\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \widetilde{\vec{r}'} = \widetilde{R_3}(\phi)\vec{r} = R_3(-\phi)\vec{r}$$

kjer je $\widetilde{R_3}$ aktivna rotacija.

Rotacija je ortogonalna transformacija

$$\vec{a}' \cdot \vec{b}' = R\vec{a} \cdot R\vec{b} = \vec{a}R^T R\vec{b} = \vec{a} \cdot \vec{b}$$

$$R = R_2 R_1, \qquad R^{-1} = R_1^{-1} R_2^{-1} = R_1^T R_2^T = R^T$$

Produkt ortogonalnih je ortogonalna.

Nekatere lastnosti determinante (pri ortogonalnih matrikah)

$$\det AB = \det A \det B \quad \det R^T = \det R$$
$$\det R^T R = \det R \det R^T = \det I = 1 \Rightarrow \det R = \pm 1$$
$$\det R^T = \frac{1}{\det R}$$

Eulerjevi koti

Precesija (rotacija za kot ϕ)

Rotacija okoli osi x za kot ϕ

$$\vec{r} \to \vec{r}'' \quad T(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Nutacija (rotacija za kot θ)

Rotacija okoli nove osi x, ki je zdaj x'', za kot θ

$$\vec{r}'' \to \vec{r}''' \quad U(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Rotacija (rotacija za kot ψ)

Rotacija okoli nove osi z, ki je zdaj z''', za kot ψ

$$\vec{r}''' \to \vec{r}' \; V(\psi) = \begin{bmatrix} \cos\psi & \psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Celotna transformacija je torej:

$$R = V(\psi)U(\theta)T(\phi); \;\; R^T = T^T U^T V^T = R^{-1}$$

in je **pasivna**. Obratna transformacija pa je aktivna.

Eulerjev izrek o rotacijah

Naj bo R rotacijska oz. ortogonalna matrika, ki preslika $\vec{r} \to \vec{r}'$:

$$R\vec{r} = \vec{r}'$$

Tedaj $\forall R\ \exists \vec{n} \neq 0$: $R\vec{n} = \vec{n}$ (kot $\lambda = 1$). Izrek pravi, da lahko te tri rotacije pravzaprav nadomestimo z eno rotacijo okoli neznane osi $\vec{n}$ za neznani kot.

Dokaz

Rotacijske matrike so ortogonalne, torej velja: $R^{-1} = R^T$ oz. $RR^T = R^T R = I$
Za determinanto vemo, da je ± 1:

$$1 = \det RR^T = \det R^T \det R = (\det R)^2 \Rightarrow \det R = \pm 1$$

Ce je $\det R = -1$ pravimo temu **neprava rotacija**, ki je sestavljena iz zrcaljenja in rotacije. Dokazati želimo, da $\exists \vec{n}$:

$$R\vec{n} = \vec{n} = I\vec{n} \;\Rightarrow (R - I)\vec{n} = 0 \Rightarrow \det(R - I) = 0$$

Uporabimo znane zveze za determinanto:

$$\det(-A) = \det\left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} A\right] = \det(-I)\det A = (-1)^3 \det A = -\det A$$

Iz tega sledi, da je $\det R^{-1} = 1$ in potem je:

$$\begin{aligned}\det(R - I) = \det(R - I)^T = \det(R^T - I\,) = \det(R^{-1} - R^{-1}R) - \det\big(R^{-1}(I - R)\big) \\ = \det R^{-1} \det\big(-(R - I)\big) = -\det(R - I)\end{aligned}$$

To lahko velja samo, če je $\det(R - I) = 0$ iz cesa sledi, da je $\lambda = 1$ lastna vrednost za R.($\lambda_{2,3} = e^{\pm i\phi}$).
Torej smo uspešno dokazali:

$$(R - I)\vec{n} = 0 \Leftrightarrow R\vec{n} = \vec{n}$$

Togo telo z eno nepremično (fiksno) točko

Zanima nas vrtilna količina oz. radi bi prišli do vztrajnostnega momenta:

$$\vec{\omega}(t) = \omega(t)\vec{n}(t)$$

$$\frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)_{nein} + \vec{\omega} \times \vec{r} \qquad \vec{v}(t) = \vec{\omega} \times \vec{r}$$

Poglejmo sedaj vrtilno količino za togo telo:

$$\vec{L} = \sum_{i=1}^{N} \vec{l}_i = \sum_{i=1}^{N} m_i(\vec{r}_i \times \vec{v}_i) = \sum_{i=1}^{N} m_i\big(\vec{r}_i \times (\vec{\omega}_i \times \vec{r}_i)\big) = \sum_{i=1}^{N} m_i\big[\big(r_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i\big] = \underline{J}\omega$$

Ker pa vsot po točkah ne znamo vedno izračunati, prevedemo celotno zadevo v integral:

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

$$M = \sum_{i=1}^{N} m_i \to \int \rho \, d^3 r$$

$$J_{\alpha\beta} = \int \rho(\vec{r})\left[r^2 \delta_{\alpha\beta} - x_\alpha x_\beta\right] d^3 r\,; \;\; \beta, \alpha \in \{x, y, z\}$$

$$\Rightarrow \underline{J} = \int \rho(\vec{r}) \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yx & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} d^3 r$$

Dokažemo se, da gre res za tenzor. Naredimo rotacijo:

$$\vec{L}' = R\vec{L} \quad \vec{\omega}' = R\vec{\omega}$$

Želimo ohraniti zvezo po rotaciji:

$$\vec{L} = \underline{J}\omega \Rightarrow \vec{L}' = \underline{J}'\vec{\omega}'$$

$$\vec{L}' = R\vec{L} = R\underline{J}\vec{\omega} = RJR^{-1}R\vec{\omega} = R\underline{J}R^{-1}\vec{\omega}'$$

$$\Rightarrow \vec{L}' = R\underline{J}R^{-1}\vec{\omega}' = \underline{J}'\vec{\omega}'$$

Ce se nek objekt tako transformira potem je res **tenzor**.

Lastne vrednosti in lastne vektorji vztrajnostnega tenzorja

$$\underline{J}\hat{e}_\alpha = \lambda_\alpha \hat{e}_\alpha;\;\; \lambda_\alpha = J_{1,2,3} \geq 0 \;\; \alpha \in \{x, y, z\}\; oz. \{1,2,3\}$$
$$J_\alpha \neq J_\beta \Rightarrow \hat{e}_\alpha \cdot \hat{e}_\beta = 0$$

Energija vrtečega se togega telesa

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_1 (\vec{\omega} \times \vec{r}_i) \cdot \vec{v}_i = \frac{1}{2}\sum_{i=1}^{N} m_i \, \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) = \frac{1}{2}\vec{\omega}\vec{L} = \frac{1}{2}\vec{\omega} \cdot \underline{J}\vec{\omega}$$

V lastnem sistemu:

$$\underline{J} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix};\;\; \vec{\omega} = \sum_{\alpha=1}^{3} \omega_\alpha \vec{e}_\alpha\,; \;\; J_\alpha = J_{\alpha\alpha}$$

$$\underline{J}\vec{\omega} = \sum_{\alpha=1}^{3} J_{\alpha\alpha}\omega_\alpha \vec{e}_\alpha \;\; \Rightarrow \frac{1}{2}\vec{\omega}\,\underline{J}\vec{\omega} = \frac{1}{2}\sum_{\alpha=1}^{3} J_{\alpha\alpha}\omega_\alpha^2 \geq \;\; \forall\vec{\omega}$$

Iz tega sledi, da je $\underline{J}$ **pozitivno definitna matrika**:

$$J_{\alpha\alpha} \geq 0$$

Frisbee (Kako dobiti Eulerjeve enačbe v lastnem sistemu)

Napišimo enačbe gibanja za frisbee (2.NZ za inercialni sistem) in zapišemo $\vec{L}$ v lastnem sistemu:

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{L}}{dt}\right)_{nein} + \vec{\omega} \times \vec{L} = \vec{M}$$

$$\vec{L} = \underline{J}\vec{\omega} = \sum_{\alpha=1}^{3} J_{\alpha\alpha}\omega_\alpha \vec{e}_\alpha = (J_1\omega_1, J_2\omega_2, J_3\omega_3)$$

$$\left(\frac{d\vec{L}}{dt}\right)_{nein} = \sum_{\alpha=1}^{3} J_{\alpha\alpha}\dot{\omega}_\alpha \vec{e}_\alpha$$

$$\vec{\omega} \times \vec{L} = \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} \times \begin{pmatrix}J_1\omega_1\\ J_2\omega_2\\ J_3\omega_3\end{pmatrix} = \big((J_3 - J_2)\omega_2\omega_3, (J_1 - J_3)\omega_1\omega_3, (J_2 - J_1)\omega_1\omega_2\big)$$

To lahko razpišemo po komponentah in dobimo **Eulerjeve enačbe v lastnem sistemu $\vec{\omega}(t)$:**

$$\begin{aligned} J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 &= M_1 \\ J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 &= M_2 \\ J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 &= M_3 \end{aligned}$$

To so točne enačbe.

Poglejmo nekaj primerov

Najenostavnejši primer $\vec{M} = 0$:

$$\frac{d\vec{L}}{dt} = 0 \;\Rightarrow\; \vec{L} = konst.$$

Naprej pa ne gre, ker so Eulerjeve enačbe potem nelinearne.

Vzemimo osno simetrično telo (kot Frisbee):

a) $J_1 = J_2 = J_3 = J$ kot recimo krogla ali kocka

$$\Rightarrow \dot{\omega}_i = 0 \Rightarrow \vec{\omega} = \vec{\omega}_0 = konst. \Rightarrow \vec{L} = \underline{J}\vec{\omega} = J\vec{\omega}$$

Tako telo se ne trese (nima wobble).

b) $J_1 = J_2 \neq J_3$ kot recimo Frisbee ali cigara

$$J_3\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \omega_0 = konst.$$

Iz preostalih dveh pa dobimo dejansko linearni enačbi:

$$\begin{aligned} J_1\dot{\omega}_1 - (J_1 - J_3)\omega_2\omega_0 = 0 \;&\rightarrow\; J_1\dot{\omega}_1 = (J_1 - J_3)\omega_2\omega_0 \\ J_1\dot{\omega}_2 - (J_3 - J_1)\omega_1\omega_0 = 0 \;&\rightarrow\; J_1\dot{\omega}_2 = (J_3 - J_1)\omega_1\omega_0 \end{aligned}$$

Enačbi med sabo delimo:

$$\frac{\dot{\omega}_1}{\dot{\omega}_2} = -\frac{\omega_2}{\omega_1} \Rightarrow \dot{\omega}_1 = -\Omega\omega_2 \;\; \dot{\omega}_2 = \Omega\omega_1; \;\; \Omega = \frac{J_3 - J_1}{J_1}\omega_0$$

$$\Rightarrow \ddot{\omega}_1 = -\Omega\dot{\omega}_2$$

In iz tega dobimo enačbo nihala:

$$\ddot{\omega}_1 + \Omega^2 \omega_1 = 0$$

In za posamični komponenti velja:

$$\begin{aligned} \omega_1(t) &= A\cos(\Omega t + \delta) \\ \omega_2(t) &= B\sin(\Omega t + \delta) \\ \omega_1^2 + \omega_2^2 &= A^2 \end{aligned}$$

$$\vec{\omega} = \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_0 \end{pmatrix}; |\vec{\omega}|^2 = A^2 + \omega_0^2$$

V primeru frisbee-ja lahko izračunamo Ω. Vzamemo, da je disk torej $J_3 = \frac{1}{2} mR^2$ in $J_1 = J_2 = \frac{1}{4} mR^2$. V **lastnem sistemu** dobimo:

$$\Omega = \frac{J_3 - J_1}{J_1}\omega_0 = \frac{\left(\frac{1}{2} - \frac{1}{4}\right) mR^2}{\frac{1}{4} mR^2}\omega_0 = \omega_0$$

Medtem ko bi v **mirujočem sistemu** dobili:

$$\Omega = 2\omega_0$$

Ω predstavlja frekvenco »wobble« torej precesijo z osi oz. vektorja $\hat{e}_3$ okoli vrtilne količine $\vec{L}$. V primeru frisbee-ja se wobble dogaja z 2x frekvenco kot se vrti.

Wobble zemlje

Zemlja je geoid, tako da velja $J_3 \neq J_1 = J_2$

$$\frac{J_3 - J_1}{J_1} = 0.00327;\ \omega_0 = \frac{2\pi}{dan} \quad \rightarrow \quad \Omega = \frac{2\pi}{300} dni$$

Eksperimentalno pa je izmerjen kot:

$$\Omega_0 = 433\ dni$$

To je zaradi nehomogenosti zemlje (npr. tudi tekoča je pod površjem).

Stabilnost vrtavke (pojav Dzhanibekova)

Vzemimo nesimetricno telo $J_1 \neq J_2 \neq J_3$

$$\begin{aligned} J_1\dot{\omega}_1 - (J_2 - J_3)\omega_2\omega_3 &= 0 \\ J_2\dot{\omega}_2 - (J_3 - J_1)\omega_3\omega_1 &= 0 \\ J_3\dot{\omega}_3 - (J_1 - J_2)\omega_1\omega_2 &= 0 \end{aligned}$$

Naj se vrti okoli lastne osi s konstantno frekvenco. Torej:

$$\omega_1 = \omega_2 = 0;\ \ \omega_3 = \Omega;\ \ \dot{\Omega} = 0$$

Kaj se zgodi, če to malo zmotimo (recimo da bi frcnili vrtavko)?

$$\vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \to \vec{\omega} + \vec{\eta}; |\vec{\eta}| \ll |\vec{\omega}| \ \Rightarrow \begin{pmatrix} \eta_1 \\ \eta_2 \\ \Omega + \eta_3 \end{pmatrix}$$

Potem se Eulerjeve enačbe glasijo:

$$\begin{aligned} J_1\dot{\eta}_1 - (J_2 - J_3)\eta_2(\Omega + \eta_3) = 0 \\ J_2\dot{\eta}_2 - (J_3 - J_1)\eta_1(\Omega + \eta_3) = 0 \\ J_3(\Omega + \eta_3)' - (J_1 - J_3)\eta_1\eta_2 = 0 \end{aligned}$$

Upoštevamo, da je η majhen. Torej zanemarimo kvadratni red $\mathcal{O}(\eta_i\eta_j)$:

$$\begin{aligned} J_1\dot{\eta}_1 &= (J_2 - J_3)\Omega\eta_2 + \mathcal{O}(\eta_2\eta_3) \\ J_e\dot{\eta}_2 &= (J_3 - J_1)\Omega\eta_1 + \mathcal{O}(\eta_1\eta_3) \\ J_3\dot{\eta}_3 &= \mathcal{O}(\eta_1\eta_2) + 0 \quad \Rightarrow \eta_3 = konst. \end{aligned}$$

$$\Rightarrow J_1\ddot{\eta}_1 = \frac{(J_2 - J_3)(J_3 - J_1)}{J_2}\Omega^2\eta_1$$

$$\ddot{\eta}_1 - c\eta_1 = 0; \ \ c = (J_2 - J_3)(J_3 - J_1)\frac{\Omega^2}{J_1J_2}$$

Rešitve

- $c < 0; c = -\omega_c^2$

$$\eta_1 = A\cos\left(\sqrt{|c|}t + \delta\right)$$
$$\eta_2 = B\sin\left(\sqrt{|c|}t + \delta\right)$$
$$-\frac{B}{A} = \sqrt{\frac{J_1}{2}\left(\frac{J_3 - J_1}{J_2 - J_3}\right)}$$

- $c > 0;$
  Ce se vrti okoli osi s srednjim vztrajnostnim momentom $J_1 < J_3 < J_2$ ali $J_2 < J_3 < J_1$ je **nestabilna!**
  Eksponent nekaj časa rabi, da »prime« in obrne os, potem se obrne vrtavka v drugo stabilno lego in tam spet traja preden eksponent obrne nazaj in to se ponavlja.

  Sicer pa se mi zdi, da je stabilna.

Formalna obravnava vrtavke

(ϕ, θ, ψ) so odvisni od časa. Zanima nas $\vec{\omega}$ v mirujočem sistemu.

$$\vec{\omega} = \dot{\phi}\hat{e}_3 + \dot{\theta}\vec{e}_1^{\,\prime\prime} + \dot{\psi}\hat{e}_3{}'$$

Vse hočemo izraziti v bazi (x', y', z'). Izrazimo prvo $\hat{e}_3 = (0,0,1)$:

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\ \sin\theta\\ \cos\theta\end{pmatrix}$$

$$\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi\ 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ \sin\theta\\ \cos\theta\end{pmatrix}=\begin{pmatrix}\sin\theta\sin\psi\\ \sin\theta\cos\psi\\ \cos\theta\end{pmatrix}$$

Potem izrazimo se $\vec{e}_1'' = (1,0,0)$ iz baze '' v bazo ':

$$\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi\ 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}\cos\psi\\ -\sin\psi\\ 0\end{pmatrix}$$

Zadnji vektor pa je že v pravilni bazi. Tako lahko $\vec{\omega}$ prepišemo v črticno bazo (lastni sistem):

$$\vec{\omega}=\begin{pmatrix}\sin\theta\sin\psi\,\dot{\phi}+\cos\psi\,\dot{\theta}\\ \sin\theta\cos\psi\,\dot{\phi}+\sin\psi\,\dot{\theta}\\ \cos\theta\,\dot{\phi}+\dot{\psi}\end{pmatrix}$$

Radi bi napisali Lagrangian $L(\phi,\theta,\psi,\dot{\phi},\dot{\theta},\dot{\psi},t)$ =?. Vzamemo da je oddaljenost po osi z' enaka l. Gravitacijski potencial je potem:

$$V = mgl\cos\theta$$

Dodatno privzamemo, da je $J_1 = J_2$ sicer je pretežko. Prepisati moramo kinetično energijo:

$$T=\frac{1}{2}(J_1\omega_1^2+J_2\omega_2^2)+\frac{1}{2}J_3\omega_3^2=\frac{1}{2}J_1(\omega_1^2+\omega_2^2)+\frac{1}{2}J_3\omega_3^2$$

v črticno bazo. Začnemo:

$$\omega_1^2+\omega_2^2=\sin^2\theta\sin^2\psi\,\dot{\phi}^2+\sin^2\theta\cos^2\psi\,\dot{\phi}^2+(\cos^2\psi+\sin^2\psi)\dot{\theta}^2 \\ +(\sin\theta\sin\psi\cos\psi-\sin\theta\cos\psi\sin\psi)\dot{\phi}\dot{\theta}=\sin^2\theta\,\dot{\phi}^2+\dot{\theta}^2$$

$$\omega_3^2=\left(\dot{\psi}+\cos\theta\,\dot{\phi}\right)^2$$

Tako dobimo Lagrangian:

$$L=L(\theta,\dot{\phi},\dot{\theta},\dot{\psi})=\frac{1}{2}J_1\left(\sin^2\theta\,\dot{\phi}^2+\dot{\theta}^2\right)+\frac{1}{2}J_3\left(\dot{\psi}+\cos\theta\,\dot{\phi}\right)^2-mgl\cos\theta$$

Opazimo, da sta ϕ in ψ ciklični koordinati. Tako da poiščimo konstante gibanja.

Ohranitev energije

$$i)\ E=\frac{1}{2}J_1\left(\sin^2\theta\,\dot{\phi}^2+\dot{\theta}^2\right)+\frac{1}{2}J_3\left(\dot{\psi}+\cos\theta\,\dot{\phi}\right)^2+mgl\cos\theta=konst.$$

ψ je ciklična

$$ii)\ \ p_\psi = \frac{\partial L}{d\dot{\psi}} = J_3(\dot{\psi} + \cos\theta\,\dot{\phi}) = J_3\omega_3 = J_1 a \Rightarrow \omega_3 = konst.\ \ a = \frac{J_3}{J_1}\omega_3$$

ϕ je ciklicna

$$iii)\ \ p_\phi = \frac{\partial L}{\partial\dot{\phi}} = J_1\sin^2\theta\,\dot{\phi} + J_3\cos\theta\,(\dot{\psi} + \cos\theta\,\dot{\phi}) = konst. =$$
$$= (J_1\sin^2\theta + J_3\cos^2\theta)\dot{\phi} + J_3\cos\theta\,\dot{\psi} = J_1 b$$

Sedaj povežemo enačbi $ii)$ in $iii)$:

$$ii) \Rightarrow J_3\dot{\psi} = J_1 a - J_3\cos\theta\,\dot{\phi}$$

To vstavimo v $iii)$

$$(J_1\sin^2\theta + J_3\cos^2\theta)\dot{\phi} + \cos\theta\,(J_1 a - J_3\cos\theta\,\dot{\phi}) = J_1 b$$
$$\Rightarrow \dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

Uporabimo se ohranitev energije:

$$E = \frac{1}{2}J_1\left(\frac{(b - a\cos\theta)^2}{\sin^2\theta} + \dot{\theta}^2\right) + \frac{1}{2}J_3\omega_3^2 + mgl\cos\theta = konst. = \tilde{E}(\theta, \dot{\theta}) + \frac{1}{2}J_3\omega_3^2$$

$$\tilde{E} = \tilde{E}_0 = \frac{1}{2}J_1\dot{\theta}^2 + \tilde{V}(\theta); \tilde{V}(\theta) = \frac{1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta$$

To je pa enodimenzionalno! Vstavimo spremenljivko $u = \cos\theta$.

$$\dot{\theta}^2 = \frac{2(\tilde{E}_0 - \tilde{V})}{J_1} \rightarrow \frac{d\theta}{dt} = \pm\sqrt{\frac{2}{J_1(\tilde{E}_0 - \tilde{V})}}$$

$$\Rightarrow t = \sqrt{\frac{J_1}{2}}\int_{\theta(0)}^{\theta(t)}\frac{d\theta}{\sqrt{\tilde{E}_0 - \tilde{V}(\theta)}}$$

Tu obvezo poglej slike rezultatov in graf funkcije $f(u)$! Dobimo rešitev

$$f(u) = (1 - u^2)(\alpha - \beta u) - (b - au)^2;\ \ \alpha = \frac{2\tilde{E}_0}{J_1}\ \ \ \beta = \frac{2mgl}{J_1}$$

Kjer α predstavlja celo energijo in β predstavlja maksimalno potencialno energijo. Obe sta normirani na J_1.

Enakomerna precesija (poglej slikce) -> Primer hitre vrtavke

$\dot{\theta} = 0$ in $\dot{\phi} = konst.$

$$\Rightarrow f(u_0) = (1 - u_0^2)(\alpha - \beta u_0) - (b - au_0)^2 = 0$$

$$\frac{df}{du}\Big|_{u=u_0} = -2u_0(\alpha - \beta u_0) - \beta(1 - u_0^2) + 2a(b - au_0) = 0$$

Prej smo izpeljali:

$$\dot{\phi} = \frac{b - au}{1 - u^2}$$

$$0 = f(u_0) = (1 - u_0^2)(\alpha - \beta u_0) - (1 - u_0^2)^2 \dot{\phi}^2 = 0$$

$$\frac{df}{du_0} = 0 = -2u_0 \dot{\phi}^2 (1 - u_0^2) - \beta(1 - u_0^2) + 2a\dot{\phi}(1 - u_0^2)$$

$$\Rightarrow \frac{\beta}{2} = a\dot{\phi} - \dot{\phi}^2 u_o$$

Za dani u_0 moramo dodati $\dot{\phi}$, da ne bo nutacije: $u_0 \to \dot{\phi}$

$$\frac{mgl}{J_1} = \dot{\phi}\frac{J_3\omega_3}{J_1} - \dot{\phi}^2 u_0$$

Dobimo kvadratno enačbo za $\dot{\phi}$:

$$A\dot{\phi}^2 + B\dot{\phi} + C = 0; \;\; A = -J_1 \cos\theta_0 \;\; B = J_3\omega_3 \; C = -mgl$$

To je rešljivo, kot je $B^2 \geq 4AC$. Tako dobimo pogoj za **hitro vrtavko**:

$$J_3^2\omega_3^2 \geq 4J_1 mgl \cos\theta_0$$

Speča vrtavka (glej spet slikce)

$\theta = 0$ oz. $u = 1$, $\dot{\theta} = 0$ in $f(1) = 0$ in $a = b$ in $\alpha = \beta$

$$f(u) = (1 - u^2)\big(\alpha(1 - u)\big) - a^2(1 - u)^2 = 0$$

$$u_{1,2} = 1$$

$$u_3\!: \;\; (1 - u)^2(\alpha(1 + u) - a^2) = 0 \; \to \; \alpha(1 + u_3) = a^2$$

$$u_3 = \frac{a^2}{\alpha} - 1 \geq 1 \;\; \Rightarrow a^2 \geq 2\alpha$$

Za DN/Vaje je bilo treba dobiti:

$$\Rightarrow T \geq 2\frac{J_1}{J_3}V$$

Oz. tako dobimo pogoj, da mora biti rotacijska energija večja od maksimalne potencialne:

$$\frac{1}{2}J_3\omega_3^2 \geq 2\frac{J_1}{J_3}mgl$$

Mala nihanja (poglej slikco)

Napišimo Lagrangian:

$$T = \frac{1}{2}m_1\big(\dot{l_1} + l_1^2\dot{\phi}_1^2\big) + \frac{1}{2}m_2\big(\dot{l}_2^2 + l_2^2\dot{\phi}_2^2\big)$$

$$V = m_1 g l_1(1-\cos\phi_1) + m_2 g l_2(1-\cos\phi_2) + \frac{1}{2}k_1(l_1 - l)^2 + \frac{1}{2}k_2(l_2 - l)^2 + [sklopitvena\ vzmet]$$

Koraki za rešitev

- Najdi mirovno lego/ravnovesje
- Majhne amplitude okoli ravnovesja

$$L = T - V = \frac{1}{2}\sum_{ij} w_{ij}(\underline{q})\dot{q}_i\dot{q}_j - V(\underline{q});\ \ w_{ij} = w_{ji}$$

Stabilna oz. labilna lega

$$q_i(t) = q_i^0\,\big|_{t=0};\ \ \dot{q}_i(t) = 0\,\big|_{t=0}$$

$$\frac{\partial V}{\partial q_i}\,\Big|_{q_i^0} = 0;\ \ \underline{q}^0 = (q_1^0, \dots, q_n^0)$$

Dovolj je če najdemo lokalni minimum, $V(\underline{q}) \geq V(\underline{q}^0)$. Razvijemo po Taylorju potem glede na mirovno lego. Linearni člen bo enak 0 v ravnovesju in $V(\underline{q}^0) = 0$.

$$V(\underline{q}) = V(\underline{q}^0) + \sum_i \frac{\partial V}{\partial q_i}\,\Big|_{\underline{q}_i^0}\eta_i + \frac{1}{2}\sum_{ij}\frac{\partial^2 V}{\partial q_i \partial q_j}\,\Big|_{\underline{q}_i^0}\eta_i\eta_j + \cdots$$

Za večje rede razvojev bi dobili na koncu nelinearne enačbe gibanja. Tu η predstavlja odmik od mirovne lege:

$$\eta_i = q_i - q_i^0$$

Razvijemo w_{ij} do najnižjega reda:

$$w_{ij}(\underline{q}) = w_{ij}(\underline{q}^0) + \sum_l \frac{\partial w_{ij}}{\partial q_l}\eta_l + \cdots$$

To zato, ker bomo v resnici rabili $w_{ij}\dot{q}_i\dot{q}_j \to \eta_l\dot{\eta}_i\dot{\eta}_j = \mathcal{O}(\eta^3)$ kar je višji red kot želimo, zato bomo zanemarili.

Zapišimo vse matrično:

$$T_{ij} = w_{ij}(\underline{q}^0);\quad V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$$

$$\tilde{L} = \frac{1}{2}\sum_{ij}\left(T_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j\right)$$

Tako dobimo sklopljen sistem linearnih enačb drugega reda:

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{\eta}_i} - \frac{\partial\tilde{L}}{\partial\eta_i} = 0;\quad i = 1,2,\dots,n$$

$$\sum_j T_{ij}\ddot{\eta}_j + \sum_j V_{ij}\eta_j = 0;\ \ i = 1,2,\dots,n$$

To lahko zapišemo tudi matrično:

$$\tilde{L} = \frac{1}{2}\underline{\dot{\eta}}^T\underline{\underline{T}}\underline{\dot{\eta}} - \frac{1}{2}\underline{\eta}^T\underline{\underline{V}}\underline{\eta}$$

Pri našem šolskem primeru, če vzamemo $l_2 = l_1$ in $m_1 = m_2$:

$$\phi_1, \phi_2, \ \ \widetilde{\phi_1} = \phi_1 + \phi_2 \quad \widetilde{\phi_2} = \phi_1 - \phi_2$$

$$\ddot{\tilde{\phi}}_1 + \omega_1^2\tilde{\phi}_1 = 0 \Rightarrow \ \omega_1 = \omega_0$$
$$\ddot{\tilde{\phi}}_2 + \omega_2^2\tilde{\phi}_2 = 0 \Rightarrow \omega_2 > \omega_0$$

Lastna načina bi bila $a_{sim} = (1,1)$ in $a_{asim} = (1,-1)$

Lastna nihanja

Za stabilno ravnovesno lego vzamemo nastavek:

$$\eta_i(t) = \alpha a_i e^{i\omega t}$$

(za labilno bi pa vzeli eksponent $e^{\pm|\omega|t}$). Vstavimo ta nastavek:

$$\sum_j V_{ij}a_j - \omega^2\sum_j T_{ij}a_j = 0$$

$$\underline{\underline{V}}\underline{a} = \omega^2\underline{\underline{T}}\underline{a} = \lambda\underline{\underline{T}}\underline{a}; \ \ \omega \in \mathbb{R}, \lambda = \omega^2 \geq 0$$

Imamo torej primer **posplošene diagonalizacije**. Lahko si ogledamo v posebnem, ko je $\underline{\underline{T}}$ diagonalna in dobimo **običajen problem lastnih vrednosti**:

$$(\underline{\underline{V}} - \lambda\underline{\underline{T}})\underline{a} = 0 \rightarrow (\underline{\underline{V}} - \lambda T\underline{\underline{I}})\underline{a} = 0 \rightarrow (\underline{\underline{V}} - \tilde{\lambda}\underline{\underline{I}}) = 0$$

$$\forall k, l\text{:} \ \ (\underline{\underline{V}} - \tilde{\lambda}_k\underline{\underline{I}})a_k = 0 \qquad a_l^T(\underline{\underline{V}} - \tilde{\lambda}_l\,\underline{\underline{I}}) = 0$$

Tako dobimo:

$$\underline{\underline{V}}\underline{a}_k = \tilde{\lambda}_k\underline{a}_k \qquad \Big| \cdot \underline{a}_l^T$$

Podobno naredimo se za ta drugo enačbo:

$$\underline{a}_l^T\underline{\underline{V}}\underline{a}_k = \tilde{\lambda}_k\underline{a}_l^T\underline{a}_k$$
$$\underline{a}_l^T\underline{\underline{V}}\underline{a}_k = \tilde{\lambda}_l\underline{a}_l^T\underline{a}_k$$

Enačbi med sabo odštejemo in dobimo:

$$0 = (\tilde{\lambda}_k - \tilde{\lambda}_l)\underline{a}_l^T\underline{a}_k$$

Ce $\tilde{\lambda}_l \neq \tilde{\lambda}_k$ in smo normirali vektorje potem velja:

$$\underline{a}_l^T\underline{a}_k = \delta_{lk}$$

Potem smo problem diagonalizirali:

$$\underline{\underline{A}} = (\underline{a}_1, \dots, \underline{a}_n) \Rightarrow \underline{\underline{A}}^T \underline{\underline{V}}\,\underline{\underline{A}} - \underline{\underline{A}}^T \underline{\underline{A}}\,\underline{\underline{\tilde{\Lambda}}} = 0; \;\; \underline{\underline{\tilde{\Lambda}}} = \begin{pmatrix} \tilde{\lambda}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tilde{\lambda}_n \end{pmatrix}$$

$$\underline{\underline{A}}^T \underline{\underline{V}}\,\underline{\underline{A}} = \underline{\underline{\tilde{\Lambda}}}$$

Splošen primer

V splošnem $\underline{\underline{T}}$ ni diagonalna matrika

$$\underline{\underline{V}}\,\underline{a} = \omega^2 \underline{\underline{T}}\,\underline{a} = \lambda \underline{\underline{T}}\,\underline{a}$$

$$\begin{aligned} \underline{\underline{V}}\,\underline{a}_k &= \lambda_k \underline{\underline{T}}\,\underline{a}_k \quad &\big|\, a_l^T \cdot \\ \underline{a}_l^T \underline{\underline{V}} &= \lambda_l \underline{a}_l^T \underline{\underline{T}} \quad &\big|\, \cdot a_k \end{aligned}$$

Po množenju dobimo dve enačbi, ki ju med sabo odštejemo:

$$\begin{aligned} \underline{a}_l^T \underline{\underline{V}}\,\underline{a}_k &= \lambda_k \underline{a}_l^T \underline{\underline{T}}\,\underline{a}_k \\ \underline{a}_l^T \underline{\underline{V}}\,\underline{a}_k &= \lambda_l \underline{a}_l^T \underline{\underline{T}}\,\underline{a}_k \end{aligned}$$

Tako kot prej, če imamo normirane vektorje in $\lambda_k \neq \lambda_l \Rightarrow \underline{a}_l^T \underline{\underline{T}}\,\underline{a}_k = \delta_{lk}$

$$\Rightarrow \underline{\underline{A}}^T \underline{\underline{T}}\,\underline{\underline{A}} = \underline{\underline{I}}$$

$$\underline{\underline{A}}^T \underline{\underline{V}}\,\underline{\underline{A}} = \underline{\underline{A}}^T \underline{\underline{T}}\,\underline{\underline{A}}\,\underline{\underline{\Lambda}} = \underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$$

Normalne koordinate

$$\tilde{L} = \frac{1}{2} \underline{\dot{\eta}}^T \underline{\underline{T}}\,\underline{\dot{\eta}} - \frac{1}{2} \underline{\eta}^T \underline{\underline{V}}\,\underline{\eta}$$

$$\eta_i(t) = \sum_k \alpha_k(t) a_{ki} = \sum a_{ki} \alpha_k(t)$$

Kjer je:

$$\underline{\eta} = \underline{\underline{A}}\,\underline{\alpha} \quad \underline{\eta}^T = \underline{\alpha}^T \underline{\underline{A}}^T$$

Tako dobimo v Lagrangianu:

$$\tilde{L} = \frac{1}{2} \underline{\dot{\alpha}}^T \underline{\underline{A}}^T \underline{\underline{T}}\,\underline{\underline{A}}\,\underline{\dot{\alpha}} - \frac{1}{2} \underline{\alpha}^T \underline{\underline{A}}^T \underline{\underline{V}}\,\underline{\underline{A}}\,\underline{\alpha}$$

$$\Rightarrow \tilde{L} = \frac{1}{2} \underline{\dot{\alpha}}^T \underline{\dot{\alpha}} - \frac{1}{2} \underline{\alpha}^T \underline{\underline{\Lambda}}\,\underline{\alpha}$$

Oz. če to napišemo po komponentah dobimo:

$$\tilde{L} = \frac{1}{2} \sum_k \left(\dot{\alpha}_k^2 - \omega_k^2 \alpha_k^2 \right) \qquad \tilde{H} = \frac{1}{2} \sum_k \left(\dot{\alpha}_k^2 + \omega_k^2 \alpha_k^2 \right) = \sum_k H_k\,; \;\; k = 1, 2, \dots, n$$

To je iste strukture kot $\dot{x}^2 + \omega^2 x \;\Rightarrow \ddot{\alpha}_k + \omega_k^2 \alpha_k = 0 \;\forall k = 1, 2 \dots, n$

$$\alpha_k(t) = a_{k_0} \cos(\omega_k t + \delta_k)$$

Hamiltonov formalizem

$$L = T - V; \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\Rightarrow \frac{dp_i}{dt} = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

Tako lahko definiramo:

$$H(\underline{q}_i, \underline{p}_i, t) = \sum_i p_i \dot{q}_i - L = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_i p_i \dot{q}_i - L = T + V$$

Da pridemo do Hamiltonovih enačb napišimo dH na dva načina. Naredimo spremembo in upoštevamo prej izračunana odvoda Lagrangiana:

$$q_i \to q_i + dq_i \quad p_i \to p_i + dp_i$$

$$dH = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) - \frac{\partial L}{\partial t} dt = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt$$

Naredimo se totalni odvod direktno:

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt$$

Izenačimo ta dva izraza in dobimo **Hamiltonove enačbe** ki povezujejo posplošene koordinate in impulze.

$$\Rightarrow \; -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad i = 1,2, \dots, N$$

$$\frac{dt}{dH} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt}\right) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

Poglej si v zvezku primer za en delec.

Izpeljava Hamiltonovih enačb iz Hamiltonovega principa

Zapišemo akcijo in izrazimo Lagrangian z Hamiltonianom:

$$S = \int_{t_1}^{t_2} L(\underline{q}_i, \underline{\dot{q}}_i, t) dt\,; \;\; L = \sum_i p_i \dot{q}_i - H$$

Naredimo variacijo:

$$\delta S = \int_{t_1}^{t_2} \delta \left[\sum_i (p_i \dot{q}_i) - H\right] dt = \int_{t_1}^{t_2} \sum_i \left(\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right) dt = (*)$$

En vmesni člen je treba integrirati per partes:

$$\int_{t_1}^{t_2} p_i \delta \dot{q}_i \, dt = p\delta q_i \,\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_i \delta q_i dt = -\int_{t_1}^{t_2} \dot{p}_i \delta q_i dt$$

Pri temu per partesu je prvi člen ničelne, ker je variacija δq_i na zacetku in koncu enaka nic. Za δp_i takega pogoja **ne** rabimo. Torej vidimo, da nimamo popolne simetrije med q_i in p_i. Tako dobimo:

$$\Rightarrow (*) = \int_{t_1}^{t_2} \sum_i \left[\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i + \left(-\dot{p}_i - \frac{\partial H}{\partial q_i}\right)\delta q_i\right] dt\,; \delta S = 0$$

Da bo to nič, morata biti oba notranja člena ničelna iz česa sledijo $\forall \delta q_i, \delta p_i$ **Hamiltonove enačbe**:

$$\Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Nabit delec v magnetnem polju (Lagrangeev in Hamiltonov formalizem)

Imam silo:

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

kar zapišemo kot:

$$F_i = -\frac{\partial V}{\partial q_i} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_i}; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Poglejmo si obe vektorski polji. Magnetno polje lahko zapišemo kot rotor **vektorskega potenciala $\vec{A}$**:

$$\nabla \cdot \vec{B} = 0 \quad \vec{B} = \nabla \times \vec{A}$$

Električno polje pa kot minus gradient **električnega potenciala ϕ**:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \nabla \times \vec{E} = -\nabla \times (\nabla\phi) - \nabla \times \frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right) = -\frac{\partial \vec{B}}{\partial t}$$

S tem lahko potem silo prepišemo v sledečo obliko:

$$\vec{F} = e\left[-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \left(\nabla \times \vec{A}\right)\right]$$

Substancialni odvod

Kot da bi se usedli na delček mase in gledali, kako se stvari (recimo vektorski potencial $\vec{A}$) spreminjajo, ko se peljemo s tem delckom mase (namesto da bi gledali nek kontrolni volumen).

$$\vec{r}(t + dt) = \vec{r}(t) + \vec{v}dt + O(dt^2); \quad \vec{A}(\vec{r}, t); \qquad dx = v_x dt, dy = v_y dt, \ldots$$

Poglejmo eno komponento:

$$dA_x = A_x(\vec{r} + \vec{v}dt, t + dt) - A_x(\vec{r}, t) = \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz + \frac{\partial A_x}{\partial t}dt =$$

$$= \left[v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_z}{\partial z}\right]dt + \frac{\partial A_x}{\partial t}dt \;\Rightarrow \frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + (\vec{v} \cdot \nabla)A_x$$

To bi lahko naredili se za ostali dve komponenti in bi dobili:

$$\frac{D\vec{A}}{Dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \nabla)\vec{A}$$

Velja pravilo:

$$\vec{v}\times(\nabla\times\vec{A})=\nabla(\vec{v}\cdot\vec{A})-(\vec{v}\cdot\nabla)\vec{A}\Rightarrow(\vec{v}\cdot\nabla)\vec{A}=\nabla(\vec{v}\cdot\vec{A})-\vec{v}\times(\nabla\times\vec{A})$$

Ce spet pogledamo samo eno komponento tega dvojnega vektorskega:

$$v_y(\nabla\times\vec{A})_z-v_z(\nabla\times\vec{A})_y=v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-v_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)=$$
$$=\cdots=\frac{\partial}{\partial x}(v_xA_x+v_yA_y+v_zA_z)-\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)A_x$$

To bomo rabili zdaj, ko vstavimo prejšnji izraz za dvojni vektorski v izraz za silo:

$$\frac{D\vec{A}}{Dt}=\frac{\partial\vec{A}}{\partial t}+\nabla(\vec{v}\cdot\vec{A})-\vec{v}\times(\nabla\times\vec{A})$$

$$\Rightarrow\vec{F}=e\left[-\nabla(\phi-\vec{v}\cdot\vec{A})-\frac{D\vec{A}}{Dt}\right]=e\left[-\nabla(\phi-\vec{v}\cdot\vec{A})+\frac{d}{dt}\left(-\nabla_{\vec{v}}(\vec{v}\cdot\vec{A})\right)\right]$$

Kjer je $\nabla_{\vec{v}}=\left(\frac{\partial}{\partial v_x},\ldots\right)$ in je $\nabla_{\vec{v}}(\vec{v}\cdot\vec{A})=\vec{A}$. Definiramo tu:

$$U\stackrel{\text{def}}{=}e\phi-e(\vec{v}\cdot\vec{A})=U(\vec{r},\vec{v},t)$$

Tako dobimo končen izraz za silo:

$$\vec{F}=-\nabla_{\vec{r}}U+\frac{d}{dt}\nabla_{\vec{v}}U$$

Končno lahko napišemo tudi Lagrangian:

$$L=T-U=\frac{1}{2}mv^2-e\phi+e(\vec{v}\cdot\vec{A})\quad\Rightarrow m\ddot{x}-e\frac{\partial}{\partial x}\phi+e(\vec{v}\times\vec{B})_x$$

Pogledamo lahko tudi Hamiltonian $H(\vec{r},\vec{p},t)$

$$p_i=\frac{\partial L}{\partial\dot{q}_i}\Rightarrow\vec{p}=\nabla_{\vec{v}}L=m\vec{v}+e\vec{A}$$

$$H=\vec{p}\cdot\vec{v}-L=mv^2+e(\vec{v}\cdot\vec{A})-\frac{1}{2}mv^2+e\phi 0e(\vec{v}\cdot\vec{A})=\frac{1}{2}mv^2+e\phi$$

Ampak Hamiltonian ni funkcija v ampak momenta p:

$$\vec{v}=\frac{\vec{p}-e\vec{A}}{m}$$

$$\Rightarrow H=\frac{(\vec{p}-e\vec{A})^2}{2m}+e\phi$$

$$\dot{\vec{r}}=\frac{\partial H}{\partial\vec{p}}\quad\dot{\vec{p}}=-\frac{\partial H}{\partial\vec{r}}\quad\Rightarrow m\ddot{r}=\vec{F}$$

Possionovi oklepaji

Imamo funkcijo:

$$f(\underline{q}, \underline{p}, t); \;\; q_i(t) \;\; p_i(t) = \frac{\partial L}{\partial \dot{q}_i}$$

Izračunajmo odvod po času in upoštevamo Hamiltonove enačbe:

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t} \stackrel{\text{def}}{=} \{f, H\} + \frac{\partial f}{\partial t}$$

Torej je definicija **Poissonovega oklepaja**:

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Torej je časovni odvod taksne funkcije:

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Oz. če f ni direktno odvisna od t:

$$\frac{df}{dt} = \{f, H\}$$

Ce je $\{f, H\} = 0 \Rightarrow \frac{df}{dt} = 0 \Rightarrow f = konst.$

Lastnosti Poissonovih oklepajev

Linearnost

$$\{f, \lambda g + \mu h\} = \lambda\{f, g\} + \mu\{f, h\}$$

Antisimetricnost

$$\{f, g\} = -\{g, f\}$$

Produkt

$$\{f, gh\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial gh}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial gh}{\partial q_i} \right) = \sum_i \left[\frac{\partial f}{\partial q_i} \left(\frac{\partial g}{\partial p_i} h + g \frac{\partial h}{\partial p_i} \right) - \frac{\partial f}{\partial p_i} \left(\frac{\partial g}{\partial q_i} h + g \frac{\partial h}{\partial q_i} \right) \right]$$

$$\Rightarrow \{f, gh\} = g\{f, h\} + h\{f, g\}$$

Jacobijeva identiteta (cikličnost)

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

Primeri Poissonovih oklepajev

1.

$$\{q_i, q_j\} = \sum_l \left(\frac{\partial q_i}{\partial q_l} \frac{\partial q_j}{\partial p_l} - \frac{\partial q_i}{\partial p_l} \frac{\partial q_j}{\partial q_l} \right) = \sum_l (\delta_{il} \, 0 - 0 \, \delta_{jl}) = 0$$

2.

$$\{p_i, p_j\} = 0$$

3.

$$\{q_i, p_j\} = \sum_l \left(\frac{\partial q_i}{\partial l} \frac{\partial p_j}{\partial p_l} - \frac{\partial q_i}{\partial p_l} \frac{\partial p_j}{\partial q_l} \right) = \sum_l (\delta_{il}\delta_{jl} - 0) = \delta_{ij}$$

4.

$$\dot{q}_i = \{q_i, H\} \quad \dot{p}_i = \{p_i, H\}$$

5. Vrtilna količina za en delec

$$\vec{L} = \vec{r} \times \vec{p}$$

$$l_x = yp_z - zp_y$$
$$l_y = zp_x - xp_z$$
$$l_z = xp_y - yp_x$$

$$\{l_x, l_y\} = \frac{\partial l_x}{\partial x}\frac{\partial l_y}{\partial p_x} + \frac{\partial l_x}{\partial y}\frac{\partial l_y}{\partial p_y} + \frac{\partial l_x}{\partial z}\frac{\partial l_y}{\partial p_z} - \frac{\partial l_x}{\partial p_x}\frac{\partial l_y}{\partial x} - \frac{\partial l_x}{\partial p_y}\frac{\partial l_y}{\partial y} - \frac{\partial l_x}{\partial p_z}\frac{\partial l_y}{\partial z} =$$
$$= 0 + 0 + (-p_y)(-x) - 0 - 0 - (y)p_x = xp_y - yp_x = l_z$$

$$\{l_i, l_j\} = \epsilon_{ijk} l_k$$

$$\{l^2, l_x\} = \{l_x^2 + l_y^2 + l_z^2, l_x\} = 0 + \{l_y^2, l_x\} + \{l_z^2, l_x\} = 0 + l_y\{l_y, l_x\} + l_y\{l_y, l_x\} + \cdots = -2l_y l_z + 2l_z l_y$$
$$= 0$$

6. Laplace-Runge-Lenzov vektor

$$\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r} \;\; \rightarrow \{l_i, A_j\} = \epsilon_{ijk} A_k$$

7.

$$\{l_i, q_j\} = \epsilon_{ijk} q_k$$
$$\{l_i, p_j\} = \epsilon_{ijk} p_k$$
$$\{\vec{p}, \vec{n} \cdot \vec{l}\} = \vec{n} \times \vec{p}$$

Recimo za $\vec{B}(\underline{q}, \underline{p}) \rightarrow \{l_i, B_j\} = \epsilon_{ijk} B_k$

8.

$$H = \frac{p^2}{2m} - \frac{km}{r}$$

$$\{\vec{A}, H\} = 0$$
$$\{\vec{l}, H\} = 0$$
$$\{l^2, H\} = 0$$