Standardni diferencialni operatorji v poljubnih ortogonalnih koordinatah

Kako izrazimo diferencialne operatorje v novih koordinatah?

Vzemimo koordinate $\xi = (\xi_1, \xi_2, \xi_3)$ v $\mathbb{R}^3$. Naj bo $\vec{r} = \vec{r}(\xi)$ parametrizacija prostora $\mathbb{R}^3$, torej izražava kartezičnih koordinat z ξ_1, ξ_2, ξ_3. (Npr. cilindrične $x = r\cos\phi, y = r\sin\phi, z = z$)

Predpostavimo, da so ξ ortogonalne za vsak nabor ξ ter:

$$\vec{r_j} = \vec{r_j}(\xi) = \frac{\partial \vec{r}}{\partial \xi_j}$$

In zahtevamo: $\langle \vec{r_j}, \vec{r_k} \rangle = 0 \quad za\ j \neq k$

Laméjevi koeficienti

$$H_j = \sqrt{\langle \vec{r_j}, \vec{r_j} \rangle} = |\vec{r_j}|; \quad j = 1,2,3 \qquad\qquad H = H_1 H_2 H_3$$

Izražava gradienta

Označimo $\vec{\eta_j} = \frac{\vec{r_j}}{|\vec{r_j}|}$. Podarimo, da so $\vec{r_j}, H_j, \vec{\eta_j}$ funkcije koordinat ξ. Naj bodo $x = (x_1, x_2, x_3)$ standardne kartezične koordinate v $\mathbb{R}^3$. Naj bo $u = u(x): \mathbb{R}^3 \to \mathbb{R}$ **gladka** funkcija in definiramo njeno izražavo U v koordinatah ξ, torej:

$$u(x) = U(\xi) = u\big(\vec{r}(\xi)\big)$$

Izražava gradienta u v kartezicnih koordinatah z gradientom U v koordiantah ξ:

$$(\nabla_x u)(\vec{r}) = \left\langle \nabla_\xi U, \begin{bmatrix} \vec{\eta_1}/H_1 \\ \vec{\eta_2}/H_2 \\ \vec{\eta_3}/H_3 \end{bmatrix} \right\rangle$$

Dokaz

Iz definicije $U = u \circ \vec{r}$ dobimo, s pomočjo verižnega pravila dobimo (za $\vec{r} = (r_1, r_2, r_3)$):

$$\frac{\partial U}{\partial \xi_j}(\xi) = \frac{\partial u}{\partial x}\big(\vec{r}(\xi)\big) \cdot \frac{\partial r_1}{\partial \xi_j} + \frac{\partial u}{\partial y}\big(\vec{r}(\xi)\big) \cdot \frac{\partial r_2}{\partial \xi_j} + \frac{\partial u}{\partial z}\big(\vec{r}(\xi)\big) \cdot \frac{\partial r_3}{\partial \xi_j} = \left\langle (\nabla_x u)(\xi)\ , \frac{\partial \vec{r}}{\partial \xi_j}(\xi) \right\rangle_{\mathbb{R}^3} =*$$

Od prej vemo, da velja: $\frac{\partial \vec{r}}{\partial \xi_j}(\xi) = \big(H_j \vec{\eta_j}\big)(\xi)$. Iz tega sledi:

$$*= H_j(\xi) \big\langle (\nabla_x u)\big(\vec{r}(\xi)\big)\ , \vec{\eta_j}(\xi) \big\rangle_{\mathbb{R}^3}$$

Kar lahko poenostavljeno zapišemo kot:

$$\frac{\partial U}{\partial \xi_j} = H_j \langle (\nabla_x u) \cdot \vec{r}, \vec{\eta_j} \rangle \quad za\ j = 1,2,3 \qquad (\lozenge)$$

Privzetki za $\forall \xi$ so, da je $\{\vec{\eta_1}(\xi), \vec{\eta_2}(\xi), \vec{\eta_3}(\xi)\}$ **ortonormirana baza** vektorskega prostora $\mathbb{R}^3$, zato je:

$$\vec{v} = \sum_{j=1}^{3} \langle \vec{v}, \vec{\eta_j} \rangle \vec{\eta_j} \qquad \forall \vec{v} \in \mathbb{R}^3$$

Torej za $\vec{v} = (\nabla_x u) \circ \vec{r}$ iz (◊) dobimo:

$$(\nabla_x u)(\vec{r}) = \sum_{j=1}^{3} \langle (\nabla_x u) \circ \vec{r}, \vec{\eta_j} \rangle \vec{\eta_j} = \sum_{j=1}^{3} \frac{1}{H_j} \cdot \frac{\partial U}{\partial \xi_j} \cdot \vec{\eta_j} = \left\langle \nabla_\xi U, \begin{bmatrix} \vec{\eta_1}/H_1 \\ \vec{\eta_2}/H_2 \\ \vec{\eta_3}/H_3 \end{bmatrix} \right\rangle$$

Izražava Laplaceovega operatorja

$$\Delta u = \frac{1}{H} \sum_{j=1}^{3} \frac{\partial}{\partial \xi_j} \left(\frac{H}{H_j^2} \cdot \frac{\partial U}{\partial \xi_j} \right) \circ \vec{R}$$

Kjer je $\vec{R} = \vec{r}^{-1}$ (lokalno): $x = \vec{r}(\xi) \Leftrightarrow \xi = \vec{R}(x)$

Za izražava obeh operatorjev poglej primere v zvezku. Se zdijo poučni ampak naporni za prepisat.

Zvezdasto območje

Območje $\Omega \subset \mathbb{R}^3$ je **zvezdasto**, če $\exists \omega_0 \in \Omega$ taksen, da za $\forall \omega \in \Omega$ je daljica:

$$[\omega_o, \omega] = \{(1-t)\omega_0 + t\omega; t \in [0,1]\}$$

Cela vsebovana v Ω.
Konveksna množica je zvezdasto območje, pri katerem je **vsak** element »dober« za ω_0.

Potencialnost polja

Naj bo:

- Ω **zvezdasto** območje v $\mathbb{R}^3$
- $\vec{F}$ **gladko** vektorsko polje na Ω, kjer je $rot\ \vec{F} = 0$

Tedaj je polje $\vec{F}$ **potencialno**. To pomeni, da $\exists u: \Omega \to \mathbb{R}$ tako, da je $\vec{F} = \nabla u$

Dokaz

Po definiciji zvezdastega območja $\exists \omega_0 \in \Omega$ tako, da je $[\omega_0, \omega] \subset \Omega\ \forall \omega \in \Omega$. Definiramo (ker vemo $\int \vec{F} ds = u(\vec{\beta}) - u(\vec{\alpha})$ če $\vec{F} = \nabla u$):

$$u(\omega) = \int_{[\omega_0, \omega]} \vec{F} d\vec{s}$$

Parametrizacijo daljice $[\omega_0 . \omega]$ zapisemo kot:

$$\vec{r}(t) = (1-t)\omega_0 + t\omega = \begin{bmatrix} (1-t)x_0 + tx \\ (1-t)y_0 + ty \\ (1-t)z_0 + tz_0 \end{bmatrix} = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix}$$

Sledi $\dot{\vec{r}}(t) = \omega - \omega_0$, zato je:

$$u(\omega) = \int_0^1 \langle F(\vec{r}(t)), \omega - \omega_0 \rangle_{\mathbb{R}^3} dt$$

Izračunamo $(\nabla u)(\omega) = (\partial_x u, \partial_y u, \partial_z u)(\omega)$. Dovolj bo, da dokažemo za $(\partial_x u)(\omega)$. Druga dva člena bi dokazali enako.

Velja:

$$(\partial_x u)(\omega) = \int_0^1 \frac{\partial}{\partial x}\langle F(\vec{r}(t)), \omega - \omega_0\rangle dt$$

Kar lahko po **Leibnizovem pravilu** zapišemo kot:

$$= \int_0^1 \left[\left\langle \frac{\partial}{\partial x}[F(\vec{r}(t))], \omega - \omega_0 \right\rangle + \left\langle F(\vec{r}(t)), \frac{\partial}{\partial x}(\omega - \omega_0)\right\rangle\right] dt$$

Ker je $\omega - \omega_0 = (x - x_0, y - y_0, z - z_0)$ je $\frac{\partial}{\partial x}(\omega - \omega_0) = (1,0,0)$
Pišemo $\vec{F} = (X, Y, Z)$; $X = X(a,b,c)$, $Y = Y(a,b,c)$, $Z = Z(a,b,c)$
Sledi:

$$\frac{\partial}{\partial x}[\vec{F}(\vec{r}(t))] = \frac{\partial}{\partial x}(X(\vec{r}), Y(\vec{r}), Z(\vec{r})) = \left(\partial_x(X(\vec{r})), \partial_x(Y(\vec{r})), \partial_x(Z(\vec{r}))\right)$$

Kar pa lahko odvajamo po verižnem pravilu in dobimo:

$$\partial_x(X(\vec{r})) = (\partial_a X)(\vec{r}) \cdot \partial_x(r_1(t)) + (\partial_b Y)(\vec{r}) \cdot \partial_x(r_2(t)) + (\partial_c Z)(\vec{r}) \cdot \partial_x(r_3(t))$$

$$\partial_x(r_1(t)) = t, \quad \partial_x(r_2(t)) = 0, \quad \partial_x(r_3(t)) = 0$$

Podobno naredimo se za ostali dve komponenti, dobimo:

$$\frac{\partial}{\partial x}[\vec{F}(\vec{r}(t))] = t\left((\partial_a X)(\vec{r}), (\partial_a Y)(\vec{r}), (\partial_a Z)(\vec{r})\right)$$

Uporabimo: $rot\ \vec{F} = (Z_b - Y_c, X_c - Z_a, Y_a - X_b) = (0, 0, 0)$

$$= t\left((\partial_a X)(\vec{r}), (\partial_b X)(\vec{r}), (\partial_c X)(\vec{r})\right) = t(\widetilde{\nabla} x)(\vec{r}); \qquad \widetilde{\nabla} = (\partial_a, \partial_b, \partial_c)$$

Tako smo dobili:

$$(\partial_x u)(\omega) = \int_0^1 \left[\langle t(\widetilde{\nabla} X)(\vec{r}), \omega - \omega_0\rangle + X(\vec{r})\right] dt = *$$

Uporabimo se dejstvo $t \cdot \frac{\partial \Lambda}{\partial t} + \Lambda = \frac{\partial}{\partial t}(t\Lambda)$ in dobimo:

$$* = \int_0^1 \frac{\partial}{\partial t}[tX(\vec{r}(t))] dt = [tX(\vec{r}(t))]\Big|_{t=0}^{t=1} = X(\vec{r}(1)) - 0 \cdot X(\vec{r}(0)) = X(\omega)$$

Torej je $\partial_x u = X$ in podobno lahko pokazemo $\partial_y u = Y$ in $\partial_z u = Z$, kar pa pomeni, da je $\vec{F} = \nabla u$.