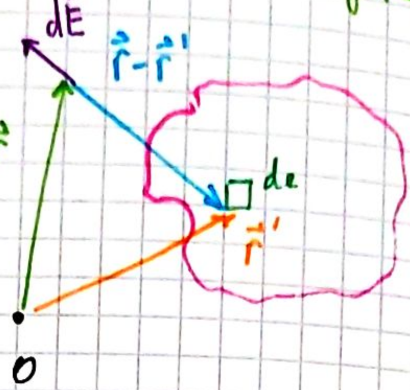
Električno polje porazdelitve nabojev

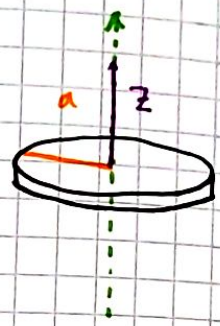
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\overbrace{\rho(\vec{r}\,')\,d^3\vec{r}\,'}^{de}}{|\vec{r}-\vec{r}\,'|^2}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

ρ = prostorninska gostota nabojev



1. [Električno polje nabite okrogle ploščice]

* σ... površinska gostota naboja



σ, a

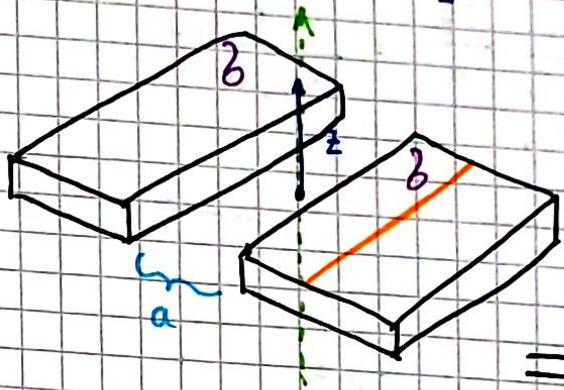
$E(z) = ?$

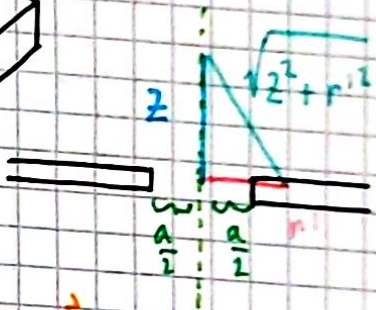
Limiti: $z \gg a$

$z \ll a$

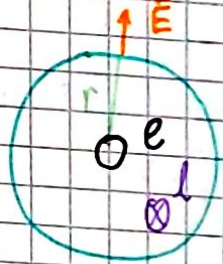
2. [Električno polje nabite velike ploščice z ravno režo]

Pozabil gostoto





Podproblem: Nabit ravni vodnik (dolg)



$$e = \varepsilon_0 \oint \vec{E} \cdot d\vec{S}$$

$$e = \varepsilon_0 E \cdot 2\pi r l$$

$$E = \frac{e}{2\pi \varepsilon_0 l r}$$

Od ene tičke

$$dE = dE' \cos\alpha = dE' \frac{z}{\sqrt{z^2+r'^2}} = \frac{\sigma l \, dr'}{2\pi \varepsilon_0 l \sqrt{z^2+r'^2}} \frac{z}{\sqrt{z^2+r'^2}}$$

$$dE = \frac{\sigma z \, dr'}{2\pi \varepsilon_0} \frac{1}{z^2+r'^2}$$

Ker sta dve plošči

$$E = 2\frac{\sigma z}{2\pi \varepsilon_0} \int_{a/2}^{\infty} \frac{dr'}{z^2+r'^2} = \frac{\sigma z}{\pi \varepsilon_0} \frac{1}{z} \left[\arctan \infty - \arctan\left(\frac{r'}{z}\right)\right] =$$

$$= \frac{\sigma}{\pi \varepsilon_0}\left[\frac{\pi}{2} - \arctan\left(\frac{a}{2z}\right)\right]$$

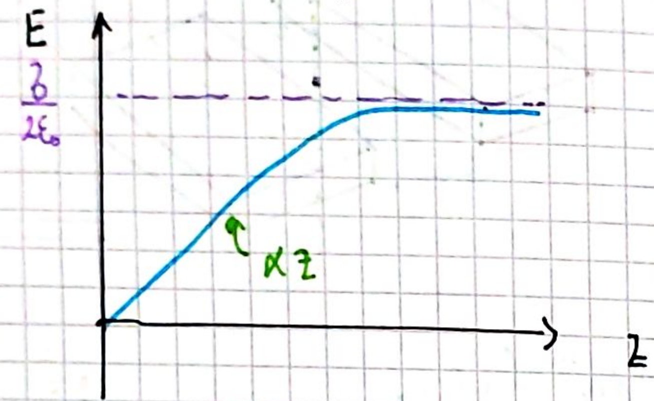
Limiti:

1. $z \gg a$: $E = \frac{\sigma}{\pi\varepsilon_0}\frac{\pi}{2} = \frac{\sigma}{2\varepsilon_0}$ neskončna ravna plošča (dolgi stran je tako zanemarljiva)

2. $z \ll a$: $\left(\arctan\frac{1}{x} = \frac{\pi}{2} - x + \dots\right)$

$$\arctan\frac{a}{2z} = \frac{\pi}{2} - \frac{2z}{a}$$

$$\Rightarrow \quad E = \frac{\sigma}{\pi\varepsilon_0}\frac{2z}{a} = \frac{2\sigma}{\pi\varepsilon_0 a}\, z$$ Linearno



Zadnjič pri predavanjih: $\rightarrow$ Prostorninska gostota naboja

Poissonova enačba

$$\left.\begin{aligned}\nabla\cdot\vec{E}(\vec{r}) &= \frac{\rho(\vec{r})}{\varepsilon_0}\\ \vec{E}(\vec{r}) &= -\vec{\nabla}U(\vec{r})\end{aligned}\right\} \quad \nabla^2 U(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

Fourierova Transformacija

$\rightarrow$ Valovni vektor

To je razvoj po ravnih valovih $e^{i\vec{k}\cdot\vec{r}}$

Ravni val (Bazne funkcije)

$$(*) \quad U(\vec{r}) = \underbrace{\int d^3\vec{k}}_{\text{„Vsota"}}\ \underbrace{U(\vec{k})}_{\text{Amplituda vala s } \vec{k} \text{ (koeficienti)}}\ e^{i\vec{k}\cdot\vec{r}}$$

/ · Skalarni produkt

Fourierova transformacija

$$\boxed{U(\vec{r}) = \int d^3\vec{k}\, U(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}}$$

Skalarni produkt z drugo bazno funkcijo ($\vec{k}'$)

$$\int d^3\vec{r}\, e^{-i\vec{k}'\cdot\vec{r}}$$

Naredimo to na obeh straneh (*):

$$\int d^3\vec{r}\, U(\vec{r})\, e^{-i\vec{k}'\vec{r}} = \iint d^3\vec{k}\, d^3\vec{r}\, U(\vec{k})\, e^{i(\vec{k}-\vec{k}')\vec{r}}$$

Integriramo najprej samo po $\vec{r}$

Ravni valovi po celem prostoru (kot sinus) dajo 0, razen ko $\vec{k}-\vec{k}'=0$, potem pa 1 integriramo po celem prostoru $\Rightarrow \infty$.

$$\int d^3\vec{r}\, U(\vec{r})\, e^{-i\vec{k}'\vec{r}} = \int d^3\vec{k}\, U(\vec{k})\, (2\pi)^3\, \delta(\vec{k}-\vec{k}')$$

$$= (2\pi)^3\, U(\vec{k}')$$

Amplituda poljubnega vala z $\vec{k}'$

$$\Rightarrow \boxed{U(\vec{k}) = \frac{1}{(2\pi)^3}\int d^3\vec{r}\, U(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}}$$

Inverzna FT

Kaj se zgodi z nablo?

$$\vec{\nabla} U(\vec{r}) = \int d^3\vec{k}\, U(\vec{k})\, \vec{\nabla} e^{i\vec{k}\cdot\vec{r}} =$$

$$\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = e^{i\vec{k}\cdot\vec{r}}\, \vec{\nabla}(i\vec{k}\cdot\vec{r}) = i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$$

$$= \int d^3k\, U(\vec{k})\, i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$$

Torej:

$$U(\vec{r}) \xrightarrow{FT} U(\vec{k})$$

$$\vec{\nabla} \xrightarrow{FT} i\vec{k}$$

$$\nabla^2 \xrightarrow{FT} -k^2$$

$$\delta(\vec{r}) \xrightarrow{FT} 1/(2\pi)^3$$

Kaj se zgodi z delto?

$$FT\,\delta(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3\vec{r}\, \delta(\vec{r})\, e^{-i\vec{k}\vec{r}} =$$

$4x(\vec{r}-0)$

Damo tu 0 ko integriramo delto

$$= 1/(2\pi)^3$$

3. [Poissonova enačba za točkasti naboj]

$$\rho(\vec r) = e\,\delta(\vec r) \qquad \text{Rešujemo: } \nabla^2 U(\vec r) = -\frac{e}{\varepsilon_0}\delta(\vec r)$$

$[As]$ ↳ $\left[\frac{1}{m^3}\right]$

Naredimo FT na celo enačbo:

Iz diferencialne enačbe smo dobili algebrajsko!

$$-k^2 U(\vec k) = -\frac{e}{\varepsilon_0}\frac{1}{(2\pi)^3}$$

Tako je:

$$U(\vec k) = \frac{e}{(2\pi)^3 \varepsilon_0 k^2}$$

$\vec k$ bo šlo po celem prostoru

Naredimo inverzno FT nazaj v $\vec r$ prostor:

$$U(\vec r) = \int \frac{e}{(2\pi)^3 \varepsilon_0 k^2} e^{i\vec k \vec r}\, d^3\vec k =$$

$d^3k = k^2\, dk\, d\varphi\, d(\cos\theta)$ v sferičnih

$$= \iiint \frac{e}{(2\pi)^3 \varepsilon_0 k^2} e^{ikr\cos\theta}\, k^2\, d\varphi\, dk\, d(\cos\theta) =$$

$$= \int_0^\infty \int_0^{2\pi} \int_{-1}^{1} \frac{e}{(2\pi)^3 \varepsilon_0} e^{ikr\cos\theta}\, dk\, d\varphi\, d(\cos\theta) =$$

$$= 2\pi \int_0^\infty \int_{-1}^{1} \frac{e}{(2\pi)^3 \varepsilon_0} e^{ikr\cos\theta}\, dk\, d(\cos\theta) =$$

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

$$= \frac{e}{(2\pi)^2 \varepsilon_0} \int_0^\infty \frac{1}{ikr} e^{ikr\cos\theta}\, dk \Big|_{-1}^{1} =$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \frac{-ie}{(2\pi)^2 \varepsilon_0 r}\left[\int_0^\infty \frac{dk}{k}\left(e^{ikr} - e^{-ikr}\right)\right] = -\frac{ie}{4\pi^2 \varepsilon_0 r}\int_0^\infty 2i\sin(kr)\frac{dk}{r} =$$

$$= \frac{2e}{4\pi^2 \varepsilon_0 r}\int_0^\infty \frac{\sin(kr)}{kr}\, d(kr) = \frac{e}{4\pi \varepsilon_0 r}$$ ✓

Rešili smo in našli pravzaprav ustrezno Greenovo funkcijo

Kako iz te rešitve sestavimo rešitev za splošen primer?

$$\nabla^2 U(\vec{r}) = -\frac{e}{\varepsilon_0}\,\delta(\vec{r}) \longrightarrow U(\vec{r}) = \frac{e}{4\pi\varepsilon_0 |\vec{r}|}$$

$$\nabla^2 U(\vec{r}) = -\frac{\rho(r)}{\varepsilon_0} \longrightarrow U(\vec{r}) = \int \frac{d^3\vec{r}\,'\rho(\vec{r}\,')}{4\pi\varepsilon_0|\vec{r}-\vec{r}\,'|}$$

de

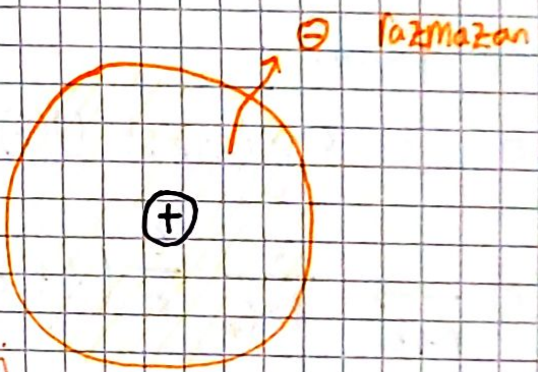
Ubistvu pa je to že znana vsota po točkastih nabojih

$$\rho(\vec{r}) = \int d^3\vec{r}\,'\rho(\vec{r}\,')\underbrace{\delta(\vec{r}-\vec{r}\,')}_{\text{Razvoj po deltah}}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{d^3\vec{r}\,'\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

Tudi že znan rezultat!

4. [Gostota naboja v vodikovem atomu]



Podan je potencial:

$$U(\vec{r}) = \frac{e}{4\pi\varepsilon_0}\,\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right);\ \alpha = \frac{2}{a_B}$$

Bohrov radij

$$\rho(\vec{r}) = ?$$

$$\rho(\vec{r}) = -\varepsilon_0\cdot\nabla^2 U(\vec{r})$$

V sferičnih koor. φ in θ const: $\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)$

$$U(\vec{r}) = \frac{e}{4\pi\varepsilon_0}\left(\frac{e^{-\alpha r}}{r} + \frac{\alpha e^{-\alpha r}}{2}\right)$$

$$r^2\frac{\partial U}{\partial r} = \frac{e r^2}{4\pi\varepsilon_0}\left(\frac{-\alpha e^{-\alpha r}\cdot r - e^{-\alpha r}}{r^2} - \frac{\alpha^2 e^{-\alpha r}}{2}\right) =$$

$$= \frac{e}{4\pi\varepsilon_0}\left(-\alpha e^{-\alpha r}\cdot r - e^{-\alpha r} - \frac{\alpha^2 r^2 e^{-\alpha r}}{2}\right) =$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = \frac{e}{4\pi\varepsilon_0}\left(\alpha^2 e^{-\alpha r} r - \alpha e^{-\alpha r} + \alpha e^{-\alpha r} + \frac{\alpha^3 r^2 e^{-\alpha r}}{2} - \frac{2\alpha^2 r e^{-\alpha r}}{2}\right) =$$

$$= \frac{e}{4\pi\varepsilon_0}\frac{\alpha^3 r^2 e^{-\alpha r}}{2}$$

Tako je:

$$\nabla^2 U = \frac{1}{r^2}\frac{e}{4\pi\varepsilon_0}\frac{\alpha^3 r^2 e^{-\alpha r}}{2} = \frac{e\alpha^3 e^{-\alpha r}}{8\pi\varepsilon_0}$$

$$\underline{\underline{\rho(\vec{r}) = - \frac{e\alpha^3 e^{-\alpha r}}{8\pi}}} = -\frac{e\alpha^3}{8\pi} e^{-\frac{2r}{a_0}} \propto |\Psi(r)|^2$$

negativen

$$\Rightarrow \Psi(r) \propto e^{-r/a_B}$$

Sklepamo, da gre za prispevek oblaka elektrona.

Res smo tako lani izpeljali za osnovno stanje (1s orbitala)

Kje je proton?

$r=0$ je singularnost $U(\vec{r})$. Poglejmo si $U(r)$, ko $r\to 0$:

$$U(r) \xrightarrow[r\to 0]{} \frac{e}{4\pi\varepsilon_0}\frac{1}{r}(1+0) = \frac{e}{4\pi\varepsilon_0 r}$$

Točkasti naboj

V izhodišču pa vidimo samo proton

$$\Rightarrow \underline{\underline{\rho(r) = e\delta(\vec{r})}}$$

Nauk!: Pazi pri odvajanju če so singularnosti.

Končni rezultat pa je vsota teh gostot:

$$\rho(r) = \underbrace{e\delta(r)}_{\text{proton}} - \underbrace{\frac{e\alpha^3}{8\pi} e^{-\alpha r}}_{\text{elektron}}$$

Poissonova enačba

$$\nabla^2 U(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0} \quad ; \rho(\vec{r}) = 0 \text{ skoraj povsod}$$

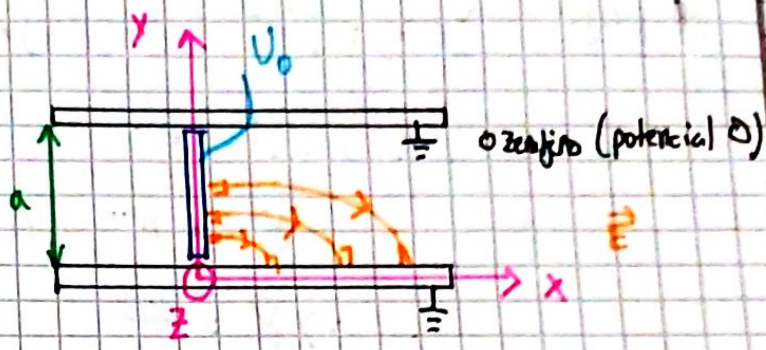
$$\Rightarrow \nabla^2 U(\vec{r}) = 0 \quad \text{Laplaceova enačba}$$

5. [Prečni trak v ploščatem kondenzatorju]

a, U_0

$U(x,y) = ?$

$\nabla^2 U(x,y) = 0$ znotraj



$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \; ;$$ Nastavek:

$U(x,y) = X(x)\,Y(y)$ (separacija spremenljivk)

$$X''Y + XY'' = 0 \; / \frac{1}{XY}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = \varkappa^2 \Rightarrow 2 \text{ enačbi} \begin{cases} X'' - \varkappa^2 X = 0 \\ Y'' + \varkappa^2 Y = 0 \end{cases}$$

za vsak x, za vsak y ⇒ konstanta

$$X'' - \varkappa^2 X = 0 \rightarrow X = Ce^{-\varkappa x} + De^{\varkappa x}$$

$$Y'' + \varkappa^2 Y = 0 \rightarrow Y = A\sin(\varkappa y) + B\cos(\varkappa y)$$

Robni pogoji:

1) $U(0,y) = U_0$

2) $U(x,0) = 0$

3) $U(x,a) = 0$

4) $U(x \to \infty, y) \neq \infty \Rightarrow \underline{\underline{D = 0}}$

iz RP2 ⇒ $U(x,0) = B \cdot Ce^{-\varkappa x} = 0 \Rightarrow B = 0$ (če bi bil $C = 0$ bi bilo X kar skos 0 in bi bilo vse 0).

Tako dobimo: $U(x,y) = Ce^{-\varkappa x} A\sin(\varkappa y)$

$\hookrightarrow F \hookleftarrow$

RP3: $U(x,a) = F e^{-\varkappa x} \sin(\varkappa a) = 0$

$$\Rightarrow \sin(\varkappa a) = 0$$

$$\varkappa a = n\pi; \quad n \in \mathbb{N}$$

$$\Rightarrow \varkappa = \frac{n\pi}{a}$$

Torej bo rešitev superpozicija vseh teh

$$U(x,y) = \sum_{n=1}^{\infty} F_n e^{-\varkappa x} \sin(\varkappa_n y)$$

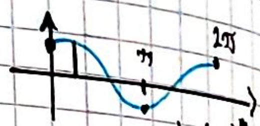
RP1:

Bazne funkcije

F_n dobimo z skalarnim produktom

$$U_0 = \sum_n F_n \sin(\varkappa_n y) \quad \Big/ \int_0^a dy \sin(\varkappa_m y)$$

Druga bazna funkcija



$\Rightarrow (-1)^m$

Leva stran:

$$U_0 \int_0^a \sin\left(\frac{m\pi}{a} y\right) dy = \frac{a U_0}{m\pi} \int_0^{m\pi} \sin(u) du = \frac{a U_0}{m\pi} \left(1 - \cos(m\pi)\right) =$$

$$u = \frac{m\pi}{a} y$$

$$= \frac{a U_0}{m\pi} \left(1 - (-1)^m\right)$$

$$\int_0^l \sin^2 x \, dx = \frac{1}{2} l$$

Desna stran:

$$\sum_n F_n \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{m\pi}{a} y\right) dy = F_m \int_0^a \sin^2\left(\frac{m\pi}{a} y\right) dy =$$

Bazne funkcije so ortogonalne $\propto \delta_{m,n}$

$$= \frac{F_m a}{2\pi m} \cdot m\pi = \frac{F_m a}{2}$$

Izenačimo strani:

$$\frac{\cancel{a} U_0}{m\pi} \left(1 - (-1)^m\right) = \frac{F_m \cancel{a}}{2}$$

$$\Rightarrow \quad F_m = \frac{2U_0}{m\pi}\left(1-(-1)^m\right)$$

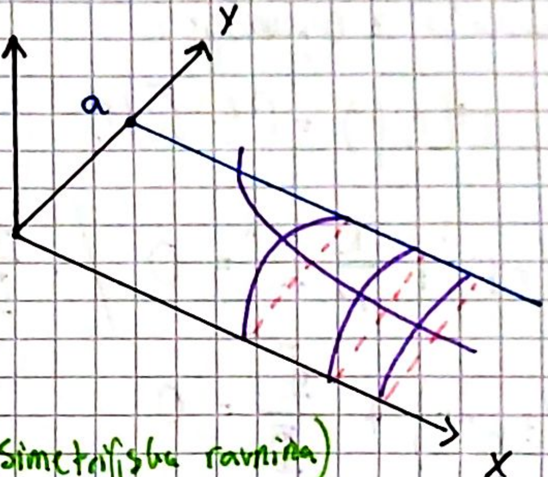
In rešitev je:

$$U(x,y) = \sum_{n=1}^{\infty} \frac{2U_0}{n\pi}\left(1-(-1)^n\right) e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$$

Posebna primera:

1) $x \gg a \rightarrow$ Samo $n=1$

$$U \simeq \frac{2U_0}{\pi} 2 e^{-\frac{\pi}{a}x} \sin\frac{\pi}{a} y$$



2) $y = \frac{a}{2}$ Sredina med ploščama (simetrijska ravnina)

Lepše bo, da izračunamo E

$$E\left(x, \frac{a}{2}\right) = ?$$

$$E_x = -\frac{\partial U}{\partial x}\bigg|_{y=\frac{a}{2}} = \sum_{n=1}^{\infty} \frac{2U_0}{n\pi}\left(1-(-1)^n\right)\left(-\frac{n\pi}{a}\right) e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{2}\right) =$$

$$= \frac{2U_0}{a} \sum^{\infty} \left(1-(-1)^n\right) \sin\left(\frac{n\pi}{a}\right) e^{-\frac{n\pi}{a}x} \qquad \left(e^{-\frac{\pi x}{a}}\right)^n = \alpha$$

Prepoznamo geometrijsko vrsto

Predfaktor

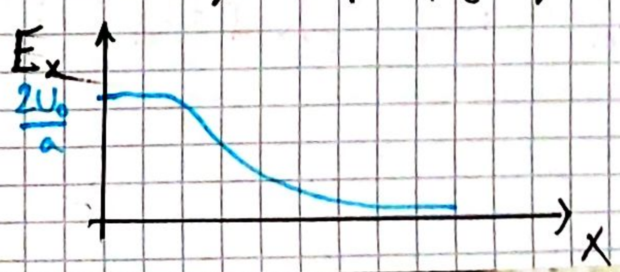
n	1	2	3	4	5	6
	2	0	-2	0	2	0

Izmenjuje se 2 in -2, sodih pa sploh ni

$$\overbrace{}^{\frac{1}{1+\alpha^2}}$$

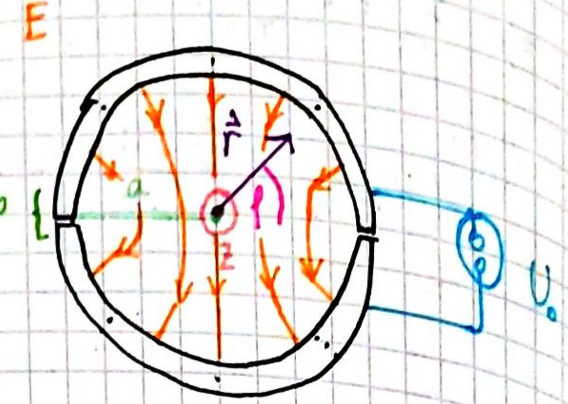
$$= \frac{2U_0}{a} 2\left[\alpha - \alpha^3 + \alpha^5 - \alpha^7 \pm\right] = \frac{4\alpha U_0}{a}\left[1-\alpha^2+\alpha^4-\alpha^6+\dots\right] =$$

$$= \frac{4\alpha U_0}{a(1+\alpha^2)} = \frac{4e^{-\frac{\pi x}{a}} U_0}{a\left(1+e^{-\frac{2\pi x}{a}}\right)} = \frac{4U_0}{a\left(e^{\frac{\pi x}{a}} + e^{-\frac{\pi x}{a}}\right)} = \frac{2U_0}{a \operatorname{ch}\left(\frac{\pi x}{a}\right)}$$



V E_y ni gradienta oz. E_y na sredini je 0

6. [Prepolovljena prevodna cev]



$$\nabla^2 U(r,\varphi) = 0$$

Pri nas 0

$$\nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}$$

Spet poskusimo z separacijo:

$$U(r,\varphi) = R(r)\Phi(\varphi)$$

$$\Phi\frac{1}{r}(rR')' + \frac{R}{r^2}\Phi'' = 0 \quad \Big/ \cdot \frac{1}{\Phi}\cdot\frac{r^2}{R}$$

$$\frac{r}{R}\left(R' + rR''\right) + \frac{\Phi''}{\Phi} = 0$$

Drugi odvodi torej vzamemo kvadrat

$$\frac{rR' + r^2R''}{R} = -\frac{\Phi''}{\Phi} = m^2$$

$$\Rightarrow \quad \Phi'' + m^2\Phi = 0 \rightarrow \Phi(\varphi) = A\sin(m\varphi) + B\cos(m\varphi)\ ; \ m = 1,2,3,\dots$$

$$r^2R'' + rR' - m^2R = 0 \quad \text{Pri } m=0 \ \ \Phi'' = 0 \Rightarrow \Phi(\varphi) = a\varphi + b\ ; \ m=0$$

$$\hookrightarrow R(r) = Cr^m + Dr^{-m}, \quad m = 1,2,3,\dots$$

$m=0$

$$r^2R'' + rR' = 0$$

Konstanta

$$(\ln R')' = \frac{R''}{R'} = -\frac{1}{r} \quad \Big/ \int dr \Rightarrow \ln R' = \ln c - \ln r = \ln\frac{c}{r} \quad \Big/ \exp$$

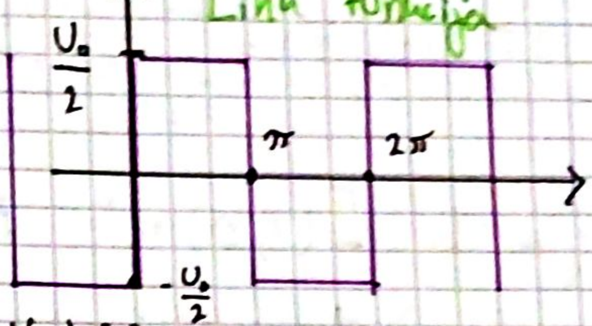
$$R' = \frac{c}{r} \quad \Big/ \int dr \quad \Rightarrow \quad R(r) = c\ln r + d\ ; \ m=0$$

Rešitev:

$$U(r,\varphi) = \sum_{m=1}^{\infty}\left(A_m\sin m\varphi + B_m\cos m\varphi\right)\left(C_m r^m + D_m r^{-m}\right) + \underbrace{(a\varphi + b)(c\ln r + d)}_{m=0}$$

Robni pogoji

Liha funkcija



RP1: $U(a,\varphi) = \begin{cases} \frac{U_0}{2}, & 0 \le \varphi < \pi \\ -\frac{U_0}{2} & \pi \le \varphi < 2\pi \end{cases}$

RP2: $U(r \to 0, \varphi) \neq \infty \Rightarrow D_m = 0 \quad \forall m \setminus \{0\}$

Zaradi lihosti $A_m = 0$ (nimamo sodih prispevkov)

Torej je naš nastavek ($F_m = B_m \cdot C_m$)

$$U(r,\varphi) = \sum_{m=1}^{\infty} F_m \sin(m\varphi)\, r^m$$

Skalarni produkt z dano bazno funkcijo (določimo koef.)

Iz RP1:
$$U(a,\varphi) = \sum_{m=1}^{\infty} F_m \sin(m\varphi) a^m \quad / \cdot \int_0^{2\pi} \sin(n\varphi)\, d\varphi$$

Leva:
$$\frac{1}{n}\int_0^{\pi} \sin(n\varphi)\frac{U_0}{2}\, d(n\varphi) - \frac{1}{n}\int_{\pi}^{2\pi} \frac{U_0}{2} \sin(n\varphi)\, d(n\varphi) =$$
$$= -\frac{U_0}{2n}\Big(\overbrace{\cos(n\pi)}^{(-1)^n} - 1\Big) + \frac{U_0}{2n}\Big(\underbrace{\cos(2n\pi)}_{1} - \overbrace{\cos(n\pi)}^{(-1)^n}\Big) =$$
$$= \frac{U_0}{n}\left(1 - (-1)^n\right)$$

Desna:
$$\int_0^{2\pi} \sum_{m=1}^{\infty} a^m F_m \sin m\varphi \sin n\varphi \, d\varphi = \sum_{m=1}^{\infty} F_m a^m \int_0^{2\pi} \sin(m\varphi)\sin(n\varphi)\, d\varphi = \sum_{m=1}^{\infty} F_m a^m \int_0^{2\pi} \sin^2(n\varphi)\,\delta_{m,n}\, d\varphi$$
$$= \sum_{m=1}^{\infty} F_m a^m \pi \,\delta_{m,n} = a^n F_n \pi$$

Izenačimo:
$$F_n = \frac{U_0}{n\pi a^n}\left(1 - (-1)^n\right)$$

$$\Rightarrow \quad U(r,\varphi) = \sum_{m=1}^{\infty} \frac{U_0}{m\pi}\left(\frac{r}{a}\right)^m \left(1-(-1)^m\right)\sin(m\varphi)$$

Poglejmo si $\vec{E}(r,0) = ?$ Zanima nas komponenta E_φ

$$\vec{E} = -\nabla U$$

$$E_\varphi = -\frac{1}{r}\cdot\frac{\partial U}{\partial \varphi}\Big|_{\varphi=0}$$

$$\Rightarrow E_\varphi = -\frac{1}{r}\frac{\partial}{\partial\varphi}\left(\sum_{m=1}^{\infty}\frac{U_0}{m\pi}\left(\frac{r}{a}\right)^m\left(1-(-1)^m\right)\sin(m\varphi)\right)\Big|_{\varphi=0} =$$

$$= -\frac{1}{r}\sum_{m=1}^{\infty}\frac{U_0}{\cancel{m}\pi}\left(\frac{r}{a}\right)^m\left(1-(-1)^m\right)\underbrace{\cos(m\varphi)}_{1}\cancel{m}\Big|_{\varphi=0} =$$

Sodi m so 0

$$= -\frac{1}{r}\sum_{m=1}^{\infty}\frac{U_0}{\pi}\left(\frac{r}{a}\right)^m\left(1-(-1)^m\right) = \qquad \frac{1}{1-x} = 1+x+x^2+\dots$$

$$= -\frac{2}{r}\frac{U_0}{\pi}\left[\frac{r}{a}+\left(\frac{r}{a}\right)^3+\left(\frac{r}{a}\right)^5+\dots\right] = -\frac{2}{a}\frac{U_0}{\pi}\left[1+\left(\frac{r}{a}\right)^2+\dots\right] =$$

$$= -\frac{2U_0}{a\pi}\frac{1}{1-(r/a)^2}$$

; $r \to a$ divergira

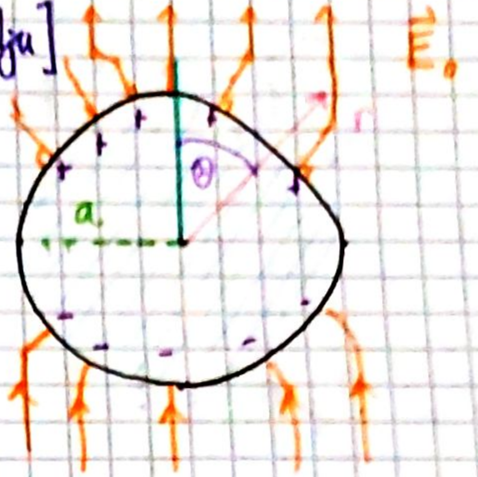
(majhena razdalja, končna napetost $\Rightarrow \to \infty$ polje)

Res kaže polje dol na sredini

Poglejmo še navpično ravnino:

DN $\quad E_r\left(r,\frac{\pi}{2}\right) = \dots = -\frac{2U_0}{a\pi}\frac{1}{1+\left(\frac{r}{a}\right)^2}$; Torej ne divergira

7. [Prevodna krogla v homogenem električnem polju]



E_0, a Iščemo osno simetrične rešitve

$$\nabla^2 U(r,\theta) = 0$$

$$U(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

Legendrovi polinomi:

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Med sabo so ortogonalni!

Robni pogoji:

Nastavimo na 0 oz. neko konst.

RP1: $U(a,\theta) = 0$

RP2: $U(r\to\infty,\theta) = ?$

$\vec{E}_0$

$U = -E_0 z$

$= -E_0 r\cos\theta$

Razvoj po Legendrovih

RP2: $\rightarrow$ $U(r\to\infty,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta \Big/ \int_{-1}^{1} P_{l'}(\cos\theta)\, d(\cos\theta)$

To sicer ni potrebno ker

Razvijamo P_1 po Legendrovih $P_1(\cos\theta)$ ki je ravno P_1 ... $\Leftarrow$

$\Rightarrow U(r\to\infty,\theta) \Rightarrow A_1 r P_1(\cos\theta) = -E_0 r P_1(\cos\theta)$

$$\Rightarrow \quad A_1 = -E_0$$

$$A_{l\neq 1} = 0$$

~~RP1~~

Vmesna rešitev: $U(r,\theta) = \sum_{l=0}^{\infty} \left(-E_0 r \underbrace{\cos\theta}_{P_1(\cos\theta)} + B_l r^{-(l+1)} P_l(\cos\theta)\right)$

RP1 $\rightarrow$

$$U(a,\theta) = -E_0 r\cos\theta + \sum_{l=0}^{\infty} B_l a^{-(l+1)} P_l(\cos\theta) = 0$$

$$\Rightarrow \quad \sum_{l=0}^{\infty} B_l\, a^{-(l+1)} P_l(\cos\theta) = E_0\, a \cos\theta$$

↑ spet P_1 razvijamo po P_l
ostane le $l=1$

$$\Rightarrow \quad B_1 a^{-2} = E_0 a \quad \Rightarrow \quad B_1 = E_0 a^3$$

$$B_{l\neq 1} = 0$$

Tako dobimo rešitev:

Homogeno polje

Naboji na krogli (zmotijo polje) enake oblike kot točkasti dipol

$$U(r,\theta) = \underbrace{-E_0 r\cos\theta}_{\text{Homogeno polje}} + \overbrace{E_0 a^3 r^{-2}\cos\theta}^{\text{Naboji na krogli}} =$$

$$= E_0 \cos\theta \left(\frac{a^3}{r^2} - r\right)$$

$\frac{\cos\theta}{r^2}$ Točkasti dipol

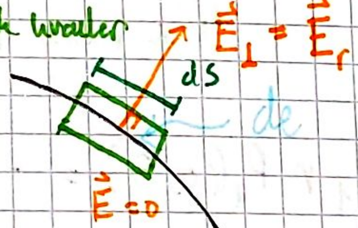
$$U_{dip}(\vec r) = \frac{\vec p_e \cdot \vec r}{4\pi\varepsilon_0 r^3} = \frac{p_e \cos\theta}{4\pi\varepsilon_0 r^2} \propto \frac{\cos\theta}{r^2}$$

$$\frac{p_e}{4\pi\varepsilon_0} = E_0 a^3 \quad \Rightarrow \quad p_e = 4\pi\varepsilon_0 E_0 a^3$$

Dipolni moment

$$\vec p_e = \int \vec r\,' \, de = \int \vec r\,' \rho(\vec r\,') \, d^3\vec r\,'$$

Poglejmo rob krogle:



Gaussov izrek

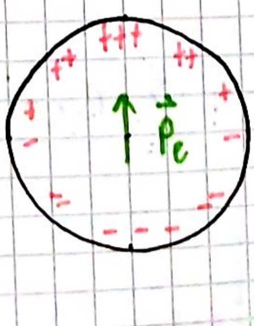
$$de = \varepsilon_0 E_\perp dS$$

$$\sigma = \frac{de}{dS} = \varepsilon_0 E_\perp$$

$$E_r = -\frac{\partial U}{\partial r}\bigg|_{r=a} = -\left(-E_0\cos\theta - 2\frac{E_0 a^3}{r^3}\cos\theta\right)\bigg|_{r=a} =$$

$$= 3E_0\cos\theta$$

$$\Rightarrow \quad \sigma = 3\varepsilon_0 E_0 \cos\theta$$



$\vec{p}_e = \int \vec{r}\,' de \qquad p_e = \int z' de \quad ; \quad de = \sigma a^2 d\varphi\, d(\cos\theta)$

$z' = a\cos\theta$

$$p_e = \int_0^{2\pi} d\varphi \int_{-1}^{1} \sigma a^2 d(\cos(\theta)) \cdot a\cos\theta =$$

$$= 2\pi a^3 \int \sigma(\theta)\cos\theta\, d(\cos(\theta)) = 2\pi a^3 \varepsilon_0 \cdot 3E_0 \int_{-1}^{1} \cos^2(\theta)\, d(\cos(\theta)) =$$

$$= 2\pi a^3 \varepsilon_0 E_0 \cos^3(\theta)\Big|_{-1}^{1} = \underline{\underline{4\pi a^3 \varepsilon_0 E_0}}$$

Dipolni moment, enako kot prej z ostrim pogledom.

8. [Točkasti dipol v središču krogelne votline]

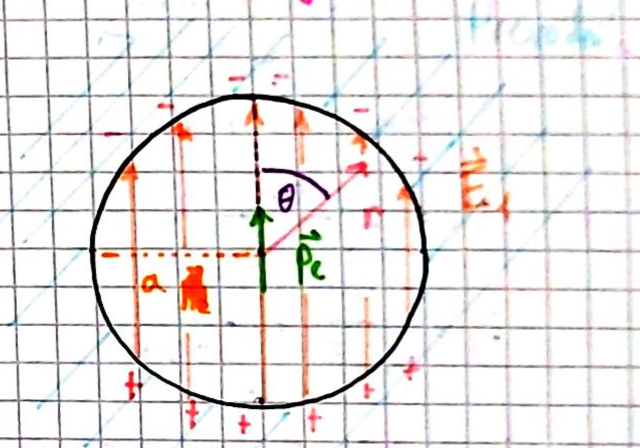
$a,\ p_e$

$U(r,\theta) = ?$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$



Splošna rešitev:

$$U(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta)$$

Robni pogoji:

RP1: $U(a,\theta) = 0$

RP2: $\lim_{r\to 0} U(0,\theta) = U_{Dipol} = \frac{\vec{p}_e\cdot\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{p_e\cos\theta}{4\pi\varepsilon_0 r^2}$ Potencial dipola

$P_1(\cos\theta)$

$$RP2 \Rightarrow \frac{p_e\cos\theta}{4\pi\varepsilon_0 r^2} = \sum_{l=0}^{\infty} \left[\cancel{A_l r^l}^{\,0} + B_l r^{-(l+1)}\right] P_l(\cos\theta) \quad ; \quad \text{samo } l = 1$$

$$\frac{p_e}{4\pi\varepsilon_0 r^2} \cancel{P_1(\cos\theta)} = B_1 r^{-2} \cancel{P_1(\cos\theta)} \Rightarrow \underline{\underline{B_1 = \frac{p_e}{4\pi\varepsilon_0}}}$$

$$\underline{\underline{B_{l\neq 1} = 0}}$$

Vmesna rešitev:

$$U(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) + \frac{p_e}{4\pi\varepsilon_0} P_1(\cos\theta)$$

RP1:

$$\Rightarrow \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = -\frac{p_e}{4\pi\varepsilon_0 a^2} P_1(\cos\theta) \Rightarrow \boxed{A_1 = \frac{-p_e}{4\pi\varepsilon_0 a^3}}$$

$$\boxed{A_{l\neq 0,1} = 0}$$

Spet zaradi P_1 in P_l $l = 1$ samo

Tako je rešitev:

$$U(r,\theta) = \underbrace{-\frac{p_e}{4\pi\varepsilon_0 a^3}\overbrace{r\cos(\theta)}^{z}}_{\text{Potencial nabojev na robu votline}} + \underbrace{\frac{p_e}{4\pi\varepsilon_0 r^2}\cos(\theta)}_{\text{Potencial dipola}}$$

$\Downarrow$

Homogeno polje

$$U_1 = -\frac{p_e}{4\pi\varepsilon_0 a^3} z$$

$$E_1 = -\frac{\partial U}{\partial z} = \frac{p_e}{4\pi\varepsilon_0 a^3}$$

b) $\sigma(\theta) = -\varepsilon_0 E_\perp = -\varepsilon_0 E_r$

$\hookrightarrow$ Zunaj je polje 0 torej kaže na noter, ravno obratno kot pri prejšnji nalogi

$$\sigma(\theta) = -\varepsilon_0 \left(\frac{\partial U}{\partial r}\right)\Bigg|_{r=a} = \cancel{\varepsilon_0}\left(-\frac{p_e}{4\pi\cancel{\varepsilon_0} a^3}\cos\theta - 2\frac{p_e}{4\pi\cancel{\varepsilon_0} a^3}\cos\theta\right) =$$

$$= -3p_e \frac{\cos\theta}{4\pi a^3}$$ Spet odvisnost $\cos\theta$

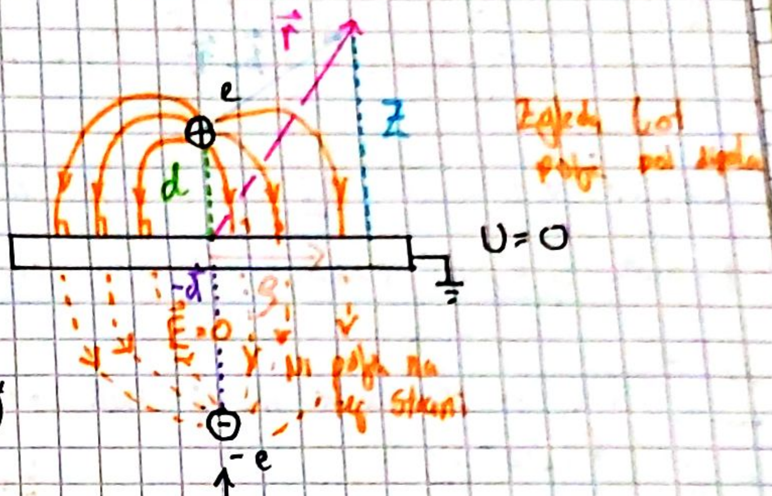
9. [Točkasti naboj nad prevodno ploščo]

e, d

$U(\vec{r}) = ?$

$\sigma_{ind}(\rho) = ?$

$e_{ind} = ?$



Virtualni naboj (če odmaknemo ploščo)

Ali plošča ali nasprotni (virtualni) naboj spodaj → Zgoraj bo enako polje

$$|\vec{r} - \vec{d}|^2 = \rho^2 + (z-d)^2$$

$$U(\vec{r}) = \frac{e}{4\pi\varepsilon_0 |\vec{r}-\vec{d}|} - \frac{e}{4\pi\varepsilon_0 |\vec{r}+\vec{d}|} \Rightarrow U(\rho, z) = \frac{e}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{\rho^2+(z-d)^2}} - \frac{1}{\sqrt{\rho^2+(z+d)^2}}\right]$$

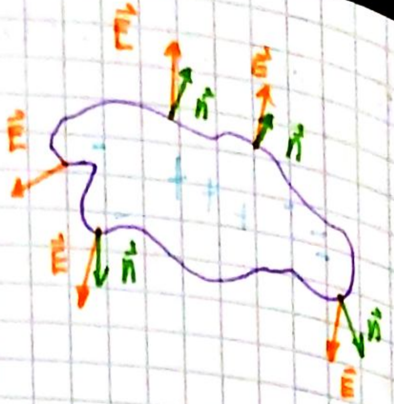
$$\sigma_{ind}(\rho) = \varepsilon_0 E_\perp = \varepsilon_0 E_z$$

$$E_z = -\frac{\partial U}{\partial z}\bigg|_{z=0} = \frac{e}{4\pi\varepsilon_0}\left[\frac{1\cdot 2(z-d)}{2(\rho^2+(z-d)^2)^{\frac{3}{2}}} - \frac{2(z+d)}{2(\rho^2+(z+d)^2)^{\frac{3}{2}}}\right]\Bigg|_{z=0} =$$

$$= \frac{e}{4\pi\varepsilon_0}\left[\frac{-d}{(\rho^2+d^2)^{\frac{3}{2}}}\right.$$

Celotno polje!

Sila: $\vec{F}_e = \varepsilon_0 \oint \left[\vec{E}(\vec{E}\cdot\vec{n}) - \frac{1}{2}E^2\vec{n}\right] dS$



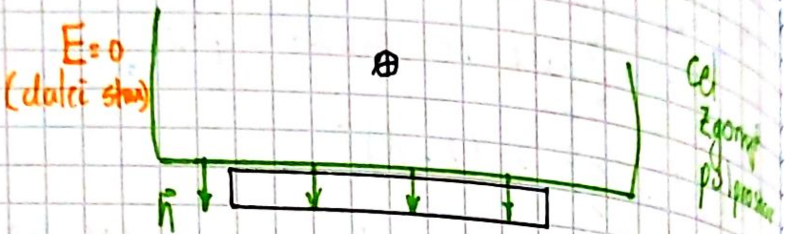
Dobimo silo na vse kar je znotraj ploskve. Izbira ploskve je ključna zadeva!

10. [Sila na točkast naboj nad prevodno ploščo]

e, d

$F_e = ?$

Prispeva samo spodnja ploskev:



$\vec{n} \parallel \vec{E}$

$\vec{E}\cdot\vec{n} = E$

$\vec{E}(\vec{E}\cdot\vec{n}) = E^2\vec{n}$

$\vec{E}(\vec{E}\cdot\vec{n}) - \frac{1}{2}E^2\vec{n} = \frac{1}{2}E^2\vec{n}$

v tem primeru konst.

$$\vec{F}_e = \varepsilon_0 \int_{\text{spodnja ploskev}} \frac{1}{2}E_z^2\,\vec{n}\,dS = \frac{\varepsilon_0}{2}\vec{n}\int_0^\infty \underbrace{\frac{e^2 d^2}{4\pi^2\varepsilon_0^2}\frac{1}{(\rho^2+d^2)^3}}_{E_z^2} 2\pi\rho\, d\rho =$$

$$= \frac{e^2 d^2}{4\pi\varepsilon_0}\int_0^\infty \frac{\rho}{(\rho^2+d^2)^3} = \frac{e^2 d^2}{4\pi\varepsilon_0\cdot 2}\int_{d^2}^\infty \frac{du}{u^3} = \frac{e^2 d^2}{16\pi\varepsilon_0 d^4} =$$

$\rho^2 + d^2 = u$

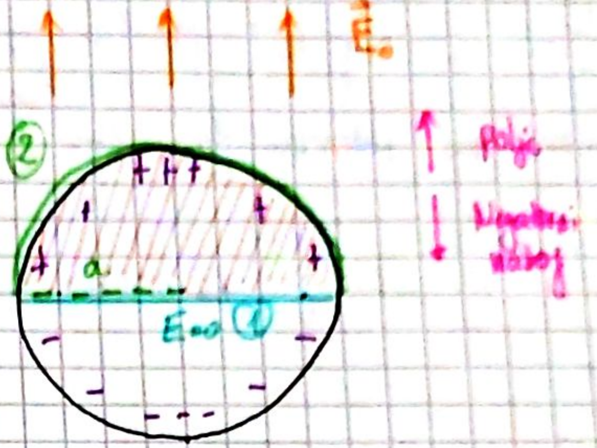
$$= \frac{e^2}{16\pi\varepsilon_0 d^2} = \underbrace{\frac{e^2}{4\pi\varepsilon_0 (2d)^2}}_{\text{Sila na ta dva točkasta naboja!}}$$

11. [Sila na polovico prevodne krogle]

a, E_0

$F_e = ?$

Zadnjič smo vzeli polprostor ker je prispevek polju v ∞ padel na 0.
Tukaj ga pa mamo tak, da objamemo kroglo na tesno



① $E_1 = 0 \Rightarrow \vec{F}_{e_1} = 0$

② $$\vec{F}_{e_2} = \varepsilon_0 \int_{②} \left[\vec{E}(\vec{E}\cdot\vec{n}) - \frac{1}{2}E^2\vec{n}\right] dS = \frac{\varepsilon_0}{2}\int E^2 \vec{n}\, dS$$

$$\vec{E} \parallel \vec{n} \rightarrow \vec{E}\cdot\vec{n} = E$$
$$\vec{E}(\vec{E}\cdot\vec{n}) = \vec{E}E = E^2\vec{n}$$

od prej: $U(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$

$$E = -\frac{\partial U}{\partial r}\Big|_{r=a} = \left[E_0\cos\theta + 2\frac{E_0 a^3}{r^3}\cos\theta\right]\Big|_{r=a} = 3E_0\cos\theta$$

Važno

$$\vec{n} = \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} \quad \text{in} \quad dS = a^2\, d(\cos\theta)\, d\varphi$$

Po celi periodi $\Rightarrow$ Prvi dve komponenti sta 0

$$\vec{F}_{e_2} = \frac{\varepsilon_0}{2}\int E^2\vec{n}\, dS = \frac{\varepsilon_0}{2}\int_0^{2\pi}\int_0^1 9E_0^2\cos^2\theta \begin{bmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{bmatrix} a^2\, d(\cos\theta)\, d\varphi =$$

$$= \frac{9E_0^2 a^2\varepsilon_0}{2}\begin{bmatrix} 0 \\ 0 \\ 2\pi\int_0^1 \cos^3\theta\, d(\cos\theta) \end{bmatrix} = \frac{9E_0^2 a^2\varepsilon_0}{2}\begin{bmatrix} 0 \\ 0 \\ \frac{\pi}{2} \end{bmatrix} = \frac{9\pi}{4}\varepsilon_0 a^2 E_0^2\, \hat{e}_z$$

Res kaže v $\hat{e}_z$ in celo navzgor.

12. [Točkasti naboj med prevodnima ploščama] (↑ neskončnima)

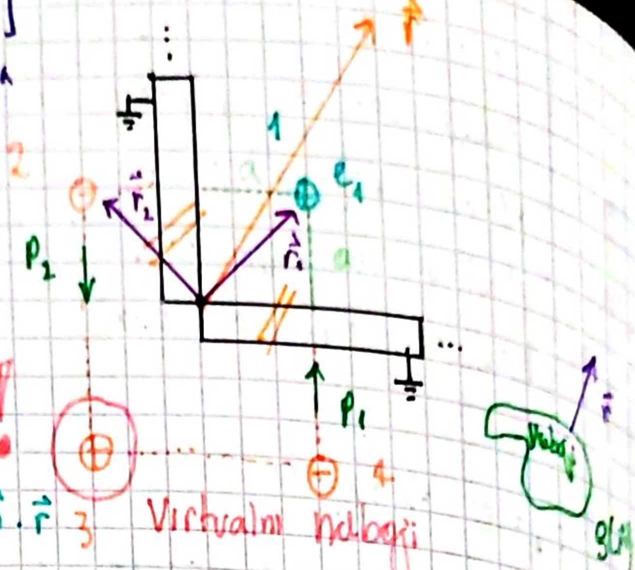
Zanima nas potencial daleč stran

pri $r \gg a$

$\Rightarrow$ Multipolni razvoj potenciala

e_1, a

$U(\vec{r}) = ?$



Multipolni razvoj

$$U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\frac{e}{r}}_{\text{Monopolni člen}} + \underbrace{\frac{1}{r^3}\sum_{i=1}^{3} p_i r_i}_{\text{Dipolni člen}} + \underbrace{\frac{1}{r^5}\sum_{i,j=1}^{3} Q_{ij} r_i r_j}_{\text{Kvadrupolni člen}} + \ldots\right]$$

$Q_{ij} r_i r_j = \vec{r} \cdot (\underline{Q}\,\vec{r})$

Monopolni moment (cel naboj):

$$e = \int \rho(\vec{r}')\, d^3\vec{r}'$$

Dipolni moment:

$$p_i = \int r_i' \rho(\vec{r}')\, d^3\vec{r}'$$

Kvadrupolni moment:

$$Q_{ij} = \int \left[3 r_i' r_j' - \delta_{ij} r'^2\right] \rho(\vec{r}')\, d^3\vec{r}'$$

Simetričen tenzor! Brezsleden!

V našem primeru (ko upoštevamo še virtualne naboje) je:

$$e = e_1 - e_1 + e_1 - e_1 = 0$$

$$p_i = p_1 - p_2 = 0$$

Diskretna verzija:

$$Q_{ij} = \sum_n \left[3 (r_n')_i (r_n')_j - \delta_{ij} r_n'^2\right] e_n$$

↑ po nabojih

$$Q_{xx} = Q_{11} = (3a^2 - 2a^2)e_1 - (3(-a)(-a) - (\sqrt{2}a)^2)e_1 + (3(-a)^2 - 2a^2)e_1 - (3a^2 - 2a^2)e_1 = \underline{\underline{0}}$$

$Q_{yy} = \underline{\underline{0}}$ → Isto dlog kot xx

$Q_{zz} = \underline{\underline{0}}$ → Smo v ravnini z koordinate so 0 $\Rightarrow \underline{Q} = \begin{bmatrix} 0 & 12a^2e_1 & 0 \\ 12a^2e_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\left.\begin{array}{l} Q_{xz} = 0 \\ Q_{yz} = 0 \end{array}\right\}$ 0 ker so z koordinate 0

$$Q_{xy} = 3a^2e_1 + 3a^2e_1 + 3a^2e_1 + 3a^2e_1 = 12a^2e_1$$

To vstavimo v razvoj:

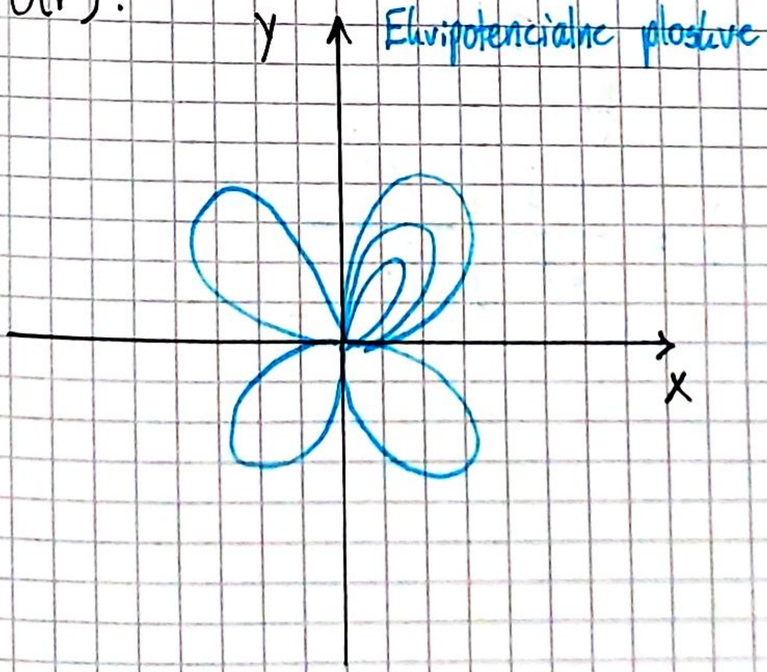
Enako oblika kot d_{xy} orbitala

$$U(\vec{r}) = \frac{12a^2e_1}{4\pi\varepsilon_0 r^5}(2xy) = \frac{6a^2e_1}{\pi\varepsilon_0}\frac{xy}{r^5} \quad ; \quad r = \sqrt{x^2+y^2+z^2}$$

Preverimo, da je $tr(\underline{Q}) = 0$

$$tr\underline{Q} = Q_{11} + Q_{22} + Q_{33} = \int \rho(\vec{r}')d^3\vec{r}'\,[\underbrace{3x'^2 - r'^2 + 3y'^2 - r'^2 + 3z'^2}_{3r'^2} - r'^2] = \underline{\underline{0}}$$

Kako zgleda $U(\vec{r})$?



Magnetostatika

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \nabla\times\vec{A}$: Vektorski potencial

$\vec{\nabla}\times\vec{B} = \mu_0 \vec{j}$ gostota električnega toka

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \mu_0\vec{j} \rightarrow \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{A}}_{\text{izberemo da } 0}) - \nabla^2\vec{A} = \mu_0\vec{j}$$

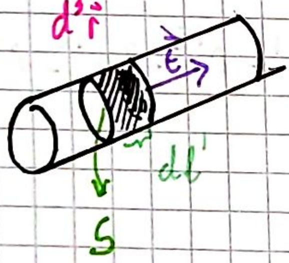
$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{j}}$$ 3D Poissonova enačba (Kirchoffova enačba)

Spomnimo se in uganemo rešitev:

$$\nabla^2 U = -\frac{\rho}{\varepsilon_0} \rightarrow U(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,')\,d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

$$\nabla^2\vec{A} = -\mu_0\vec{j} \rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{r}\,')\,d^3\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

Vodniki:

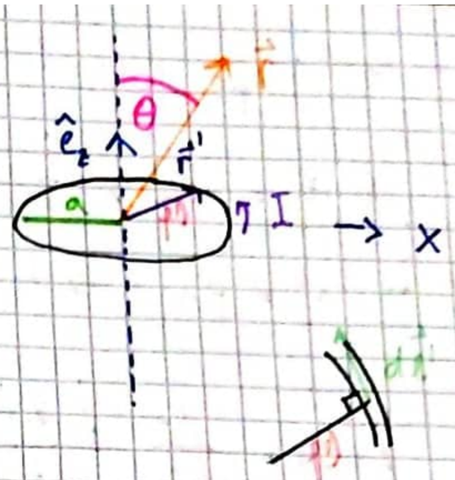


$$\vec{j}(\vec{r}\,')\,d^3\vec{r}\,' = \underbrace{\vec{j}}_{I\vec{e}}\cdot S\,dl' = I\,d\vec{l}\,'$$

$$\Downarrow$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|}$$

13. [Magnetno polje krožne tokovne zanke]



a, I

$\vec A(\vec r) = ?$ daleč stran $r \gg a$

$$\vec r' = a\begin{bmatrix}\cos\varphi'\\ \sin\varphi'\\ 0\end{bmatrix}$$

$d\vec l'$ krožni lok

$$d\vec l' = \begin{bmatrix}-\sin\varphi'\\ \cos\varphi'\\ 0\end{bmatrix} a\,d\varphi'$$

$$\vec A = \frac{\mu_0 I}{4\pi}\int \frac{d\vec l}{|\vec r - \vec r'|}$$

$$\vec r = \begin{bmatrix}\sin\theta\\ 0\\ \cos\theta\end{bmatrix} r$$

Torej: $$|\vec r - \vec r'| = \left\|\begin{pmatrix} r\sin\theta - a\cos\varphi'\\ -a\sin\varphi'\\ r\cos\theta\end{pmatrix}\right\| = \sqrt{r^2\sin^2\theta - 2ra\sin\theta\cos\varphi' + a^2\cos^2\varphi' + a^2\sin^2\varphi' + r^2\cos^2\theta}$$

$$= \sqrt{a^2 + r^2 - 2ar\sin\theta\cos\varphi'}$$

Integral tega bi bil kompliciran (eliptične funkcije). Poenostavimo za $r \gg a$

Velja: $(1+\varepsilon)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\varepsilon$

Razvijemo

$$\frac{1}{|\vec r - \vec r'|} = \left(r\sqrt{\underbrace{\cancel{\frac{a^2}{r^2}}}_{0} + 1 - 2\frac{a}{r}\sin\theta\cos\varphi'}\right)^{-1}$$

$$\Rightarrow \frac{1}{|\vec r - \vec r'|} \approx \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\cos\varphi'\right)$$

Tako je: $\int_0^{2\pi}\sin\varphi\cos\varphi = \int_0^{2\pi}\frac{1}{2}\sin 2\varphi = 0$

$$\vec A = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\begin{bmatrix}-\sin\varphi'\\ \cos\varphi'\\ 0\end{bmatrix}\frac{1 + \frac{a}{r}\sin\theta\cos\varphi'}{r}\,d\varphi' =$$

$$= \hat e_y \frac{\mu_0 I a^2}{4\pi r^2}\sin\theta\int_0^{2\pi}\cos^2\varphi'\,d\varphi' = \hat e_y\frac{\mu_0 I a^2\sin\theta}{4r^2}$$

Prepišemo to v lepšo obliko, da ga prepoznamo:

$$\hat{e}_y \sin\theta = \hat{e}_z \times \frac{\hat{r}}{r}$$

$$I \cdot \pi \cdot a^2 = p_m \quad \rightarrow \text{Kaže v smeri } \hat{e}_z$$

Za zanko:

$$\vec{p}_m = I\vec{S}$$

Dobimo:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{p}_m \times \vec{r}}{r^3}$$

Magnetni potencial točkastega dipola (kar zanka je daleč stran)

Enačba za magnetni dipolni moment v splošnem:

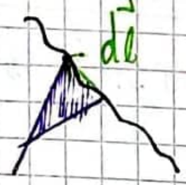
$$\cancel{\vec{p}_m = \frac{1}{2}\int \vec{r}\,' \times d\vec{l}\,' \, \vec{j}(\vec{r})}$$

$$\vec{p}_m = \frac{1}{2}\int \vec{r}\,' \times \vec{j}(\vec{r}\,')\, d^3\vec{r}\,'$$

Splošno, za zvezno porazdeljen tokov

Za zanke:

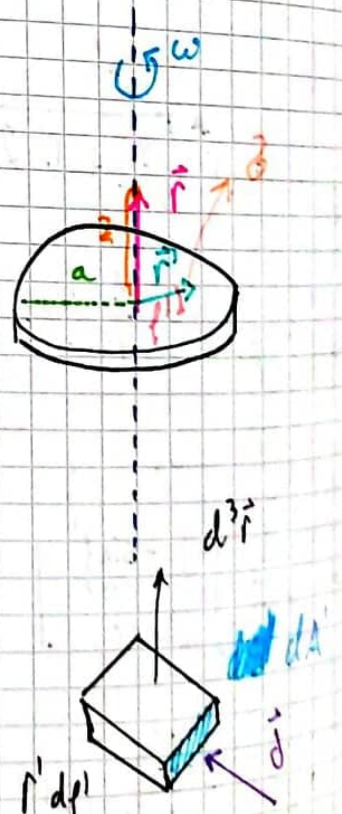
$$\vec{p}_m = \frac{I}{2}\int \vec{r}\,' \times d\vec{l}\,'$$



14. [Magnetno polje nabite vrteče se okrogle plošče]

a, σ, ω

$B(z) = ?$



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3\vec{r}\,'$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, d^3\vec{r}\,'$$

Nastavimo vektorje:

$$\vec{r} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \qquad \vec{r}' = r' \begin{bmatrix} \cos\varphi' \\ \sin\varphi' \\ 0 \end{bmatrix}$$

$$\omega = \frac{d\varphi}{dt} \qquad \vec{j} = j \begin{bmatrix} -\sin\varphi' \\ \cos\varphi' \\ 0 \end{bmatrix}$$

$$\vec{j} \times (\vec{r} - \vec{r}') = \begin{vmatrix} i & j & z \\ -\sin\varphi' & \cos\varphi' & 0 \\ r'\cos\varphi' & -r'\sin\varphi' & z \end{vmatrix} j = \begin{bmatrix} z\cos\varphi' \\ z\sin\varphi' \\ r'\sin^2\varphi + r'\cos^2\varphi \end{bmatrix} = \begin{bmatrix} z\cos\varphi' \\ z\sin\varphi' \\ r' \end{bmatrix}$$

$$|\vec{r} - \vec{r}'|^3 = \left(r'^2\cos^2\varphi' + r'^2\sin^2\varphi' + z^2\right)^{\frac{3}{2}} = \left(r'^2 + z^2\right)^{\frac{3}{2}}$$

To lahko sedaj vstavimo:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int j \begin{bmatrix} z\cos\varphi' \\ z\sin\varphi' \\ r' \end{bmatrix} \frac{1}{(r'^2 + z^2)^{\frac{3}{2}}} r' d\varphi' dS'$$

Poglejmo kaj se dogaja s tokom:

$$dI(r') = \frac{de}{t_0} = \frac{\sigma dS'}{t_0} = \qquad \sigma = \frac{de}{dS'}$$

Obhodni čas

ω dr'

$$= \frac{\sigma\, 2\pi r' \cancel{d} }{t_0} = \sigma \omega r' \cancel{\ } dr'$$

Torej:

Periodični po celem intervalu

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_0^{2\pi}\int_0^{a} \begin{bmatrix} z\cos\varphi' \\ z\sin\varphi' \\ r' \end{bmatrix} \frac{\sigma\omega r'^2 dr'}{(r'^2 + z^2)^{\frac{3}{2}}} d\varphi' =$$

$$= \hat{e}_z \frac{\mu_0}{4\pi} \omega\sigma 2\pi \underbrace{\int_0^a \frac{r'^3}{(r'^2 + z^2)^{\frac{3}{2}}} dr'}_{\mathcal{I}}$$

Posebej izračunamo:

$$I = \int_0^a \frac{r'^3}{(r'^2+z^2)^{3/2}}\, dr' = \frac{1}{2} \int_{z^2}^{a^2+z^2} \frac{u - z^2}{u^{3/2}}\, du =$$

$$z^2 + r'^2 = u \qquad 2r'dr' = du$$

$$= \frac{1}{2} \int_{z^2}^{a^2+z^2} \left(\frac{1}{u^{1/2}} - \frac{z^2}{u^{3/2}} \right) du = \frac{1}{2} \left[2\sqrt{u} + 2\frac{z^2}{\sqrt{u}} \right] \Bigg|_{z^2}^{a^2+z^2} =$$

$$= \sqrt{a^2+z^2} - z + \frac{z^2}{\sqrt{a^2+z}} - z = \frac{a^2+2z^2}{\sqrt{a^2+z^2}} - 2z$$

In tako je:

$$\vec{B}(\vec{r}) = \frac{1}{2} \mu_0 \omega \sigma \overbrace{\left(\frac{a^2+2z^2}{\sqrt{a^2+z^2}} - 2z \right)}^{\theta} \hat{e}_z$$

Poglejmo si limito $z \gg a$

Velja $(1+\varepsilon)^p = 1 + p\varepsilon + \ldots$

$$(1+\varepsilon)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2$$

$$\theta = \left(\frac{z^2 \left(\frac{a^2}{z^2} + 2\right)}{z\sqrt{\frac{a^2}{z^2}+1}} - 2z \right) = z \left(\frac{\left(\frac{a^2}{z^2}\right) + 2}{\sqrt{\frac{a^2}{z^2}+1}} - 2 \right) =$$

$$= z \left(\left(\left(\frac{a^2}{z^2}\right) + 2 \right) \left(1 - \frac{1}{2}\frac{a^2}{z^2} + \frac{3}{8}\left(\frac{a^2}{z^2}\right)^2 \right) - 2 \right) =$$

višjega reda ≈ 0

$$= z \left(\cancel{\frac{a^2}{z^2}} + \cancel{2} - \frac{1}{2}\frac{a^4}{z^4} - \cancel{1\frac{a^2}{z^2}} + \frac{3}{8}\frac{a^6}{z^6} + \frac{3}{4}\frac{a^4}{z^4} - \cancel{2} \right) =$$

$$= \frac{z}{4} \frac{a^4}{z^4} + \mathcal{O}(z^{-6}) = \frac{a^4}{4z^3}$$

$$\Rightarrow \quad B(\vec{r}) = \frac{1}{2} \mu_0 \omega \sigma \frac{a^4}{4z^3} \hat{e}_z$$

Splošno dipol:

$$B_{dip} = \frac{\mu_0}{4\pi} \frac{3\vec{r}(\vec{r}\cdot\vec{p}_m) - r^2 \vec{p}_m}{r^5}$$

$\vec{p}_m \parallel \vec{r}$

$$B_{dip} = \frac{\mu_0}{4\pi} \frac{2r^2 p_m}{r^5} = \frac{\mu_0 p_m}{2\pi r^3}$$

Rabimo le še p_m, da preverimo, če smo res dobili dipol

cela! $\pi r'^2$

$$dp_m = dI\,S$$

$$\Rightarrow p_m = \int_0^a \pi r'^2 \zeta \omega r' dr' = \frac{\pi \zeta \omega}{4} a^4$$

Vidimo, da res pride enako:

$$B_{dip} = \frac{\mu_0}{2\pi} \frac{a^4 \pi \zeta \omega}{4 \cancel{\pi} r^3} = \frac{\mu_0}{8} a^4 \zeta \omega \frac{1}{r^3}$$

Magnetne sile:

$$\vec{F}_m = \frac{1}{\mu_0} \oint \left[\vec{B}(\vec{B} \cdot \vec{n}) - \frac{1}{2} B^2 \vec{n} \right] dS$$

15. [Magnetna sila v koaksialnem kablu]

a, I Napetost v kablu (plašču kablu)

$\frac{F}{l} = ?$