SSL:

Merjenje: • koliko enot obsega
• Oceniti "obseg velikosti"

Fizikalne količine:

-

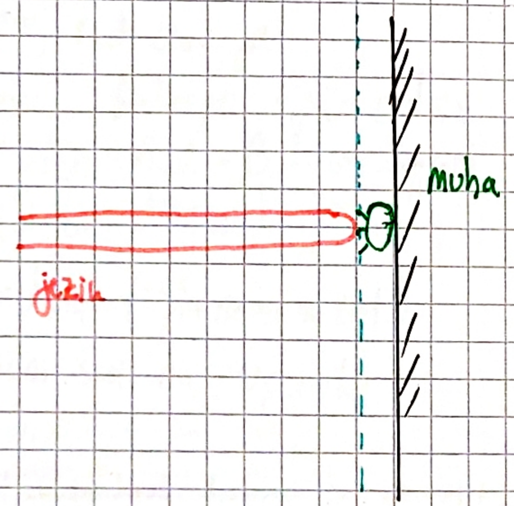
$$\vec{p} = m \cdot \vec{v}$$

- s predpisom povezano

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{\Delta\vec{x}}{\Delta t} \begin{matrix} \to \text{Umerler z metrom} \\ \to 1s \end{matrix}$$

Ocenjevanje Razdalje:
- Astro
- Nano
- Mikro
- Bio (plenilec → plen)

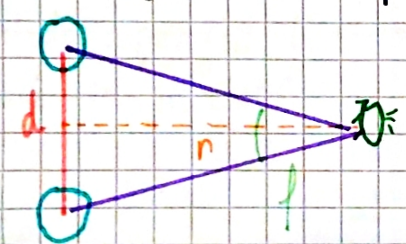
Dober kameleon. Njegov plen je muha.



Kaj mu to omogoča?

$\Delta x = 100 \mu m$

1.) Stereoskopsko gledanje $r = \left(\frac{d}{\varphi}\right)$

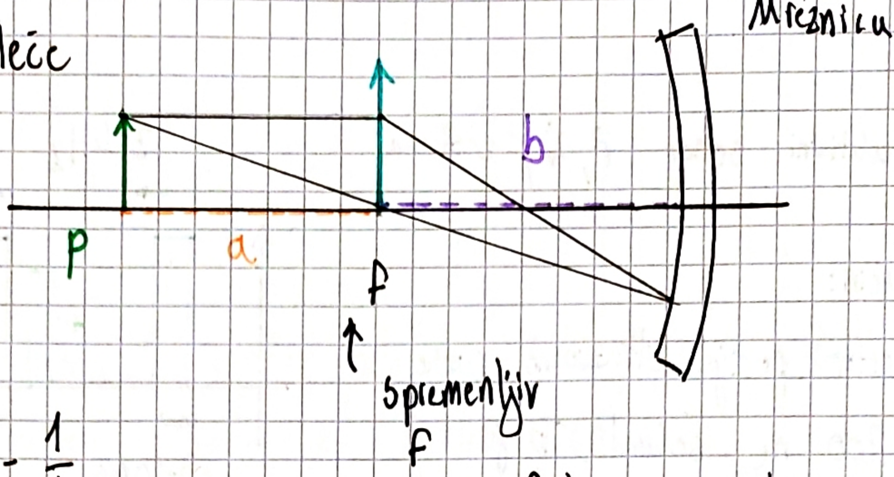


↑

Izmeri prek napetosti v mišici

Zaprli so mu z zaslonko eno oko in še vedno je zadel. Torej to ni mehanizem.

2.) Akomodacija leče

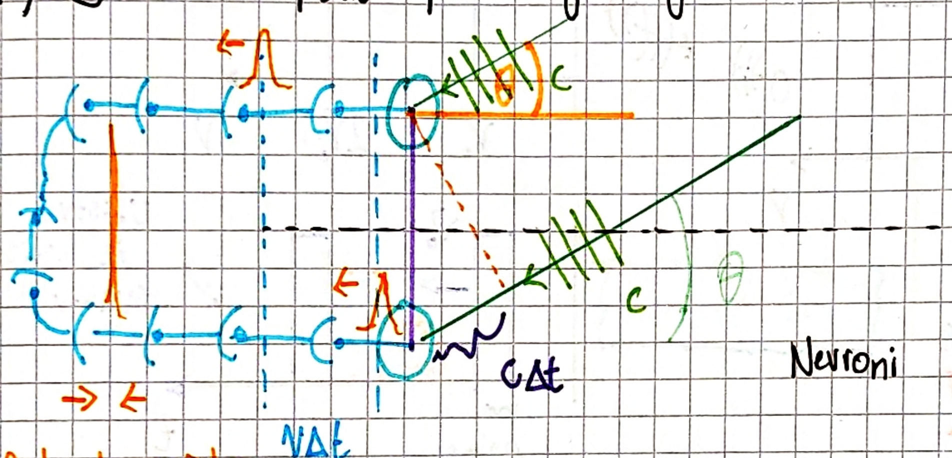


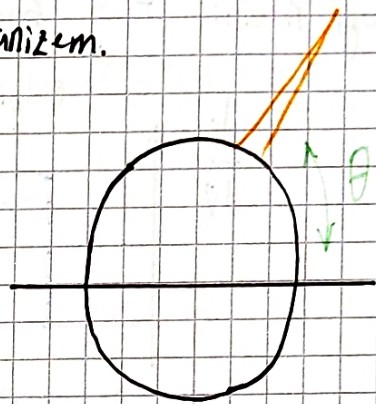
$$\frac{1}{a} + \frac{1}{b} = \frac{1}{f}$$

$$\Rightarrow \quad \frac{1}{a} = \underset{\uparrow\ \text{čuti}}{\frac{1}{f}} - \underset{\uparrow\ \text{fiksirano v očeh}}{\frac{1}{b}}$$

Dali so mu leče preko oči in je zgrešil. To je mehanizem.

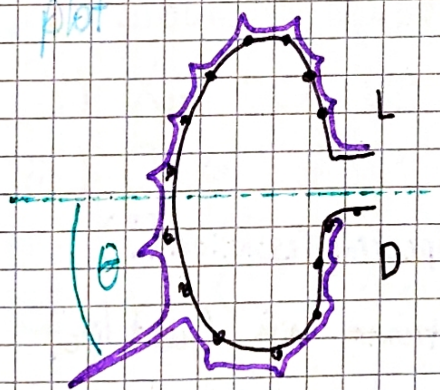
3.) Stereoskopsko poslušanje lege



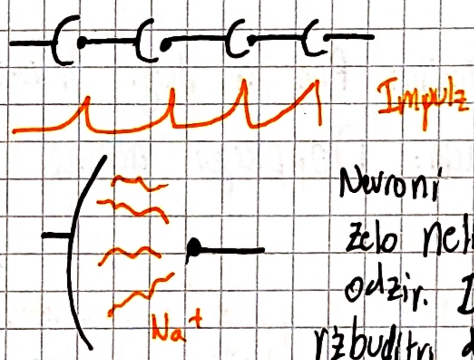


Pride do gašitve ne simetrično

Polar plot



Nevroni



Nevroni imajo zelo nelinearen odziv. Dve hkratni vzbuditvi dajo 10^3 močnejši kem. signal.

Kaj pa netopir?

Ima aktivni sonar $\vec{r}, \vec{v}$, sestava

Prvi način:

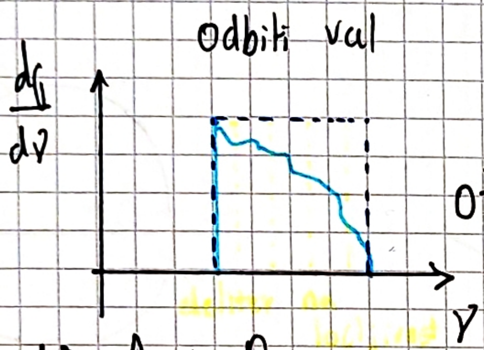
V prostor pošilja ultrazvočne sunke

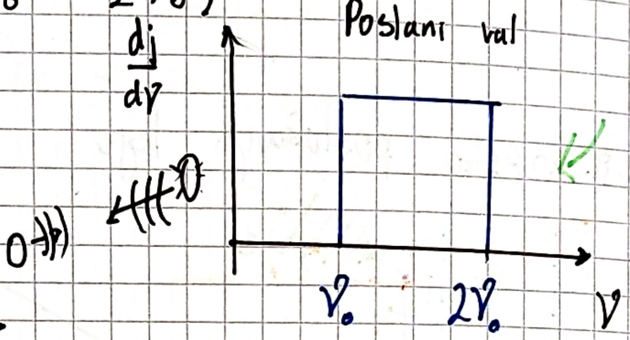
Okoli 200 na sekundo. To je

t.i. tipanje na daleč

Dobi lego

Drugi način: FM sunki ($\nu_0 \leftrightarrow 2\nu_0$)





Žvižga z celo oktavo

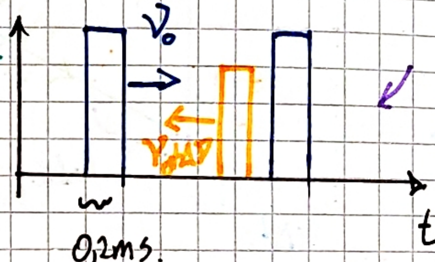
$1 = A + R$

adsorpcija — funkcija ν

moč

!

Dobi sestavo



Ubistvu iz krajših sunkov

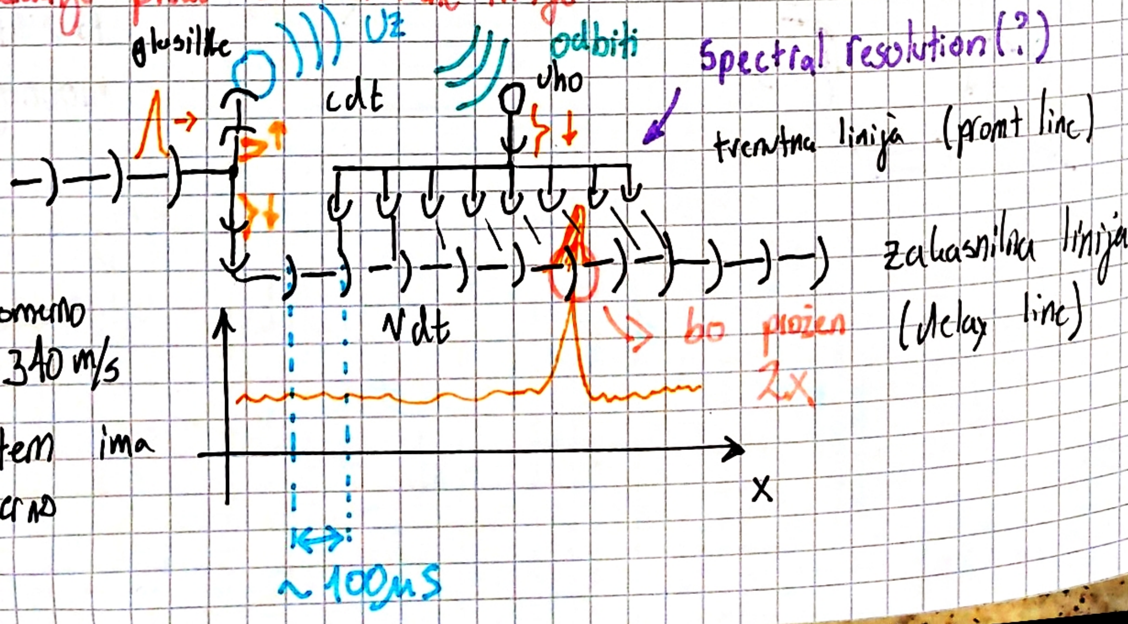
Tretji način: Prvi in drugi hkrati

Dobi

Četrti način: Dopplerjev premik $\nu = \nu_s(1 + 2\,v/c)$ Hitrost

Meritev razdalje preko "Zakasnitvene linije"

Spectral resolution (?)



Imamo enakomerno gibanje $c = 340\ m/s$

Modelski sistem ima tudi enakomerno gibanje.

Realni sistem — Modelski sistem

$S = ct$ ⟷ $S_M = v \cdot t$

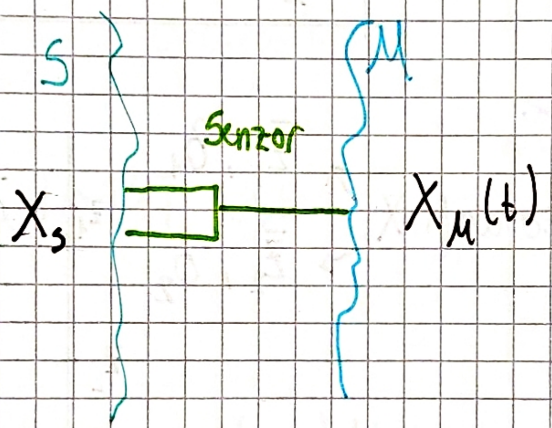
Senzor povezuje

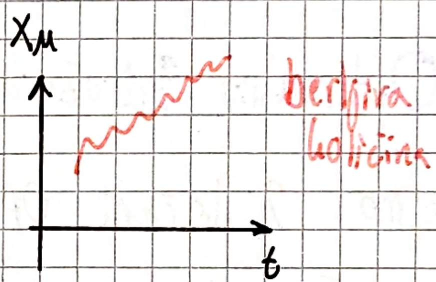
UHO

S_m je berljiva količina (ob vsakem primeru lahko pogledamo kolko je)
Tu senzor nima znatnega vpliva na realni sistem.

Optimalno filtriranje

Iščemo optimalen predpis za optimizacijo ~~modela~~ realnega sistema S na modelski sistem M.





Zahteve:

i) Šibka sklopitev S in M (čim manj vpliva)

ii) X_M mora biti berljiva količina (od t)

iii) Ocena stopnje usklajenosti

$$\lim_{t \to \infty} \langle (X_M - X_S)^2 \rangle = ?$$

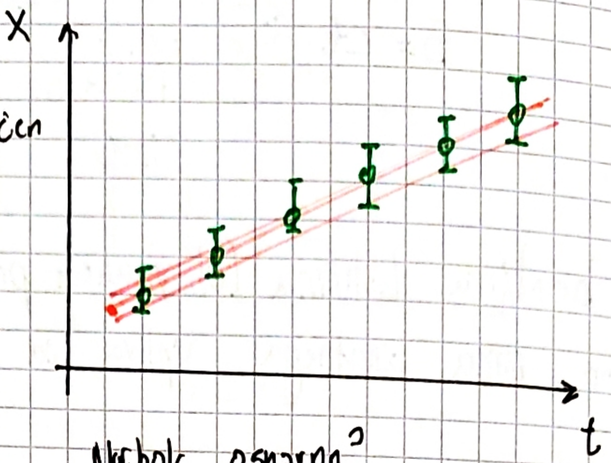
$\langle \dots \rangle$ ensambelsko povprečje

iv) Dinamika za X_S in X_M naj bo kar se da podobna

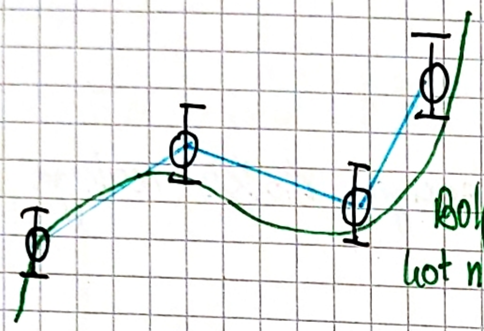
↳ Linearne diferencialne (diferenčne) enačbe

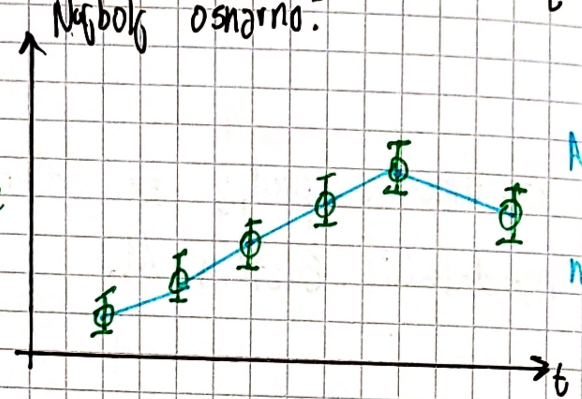
Zgled: [Premoenakomerno gibanje telesa]

1D, v = konst.



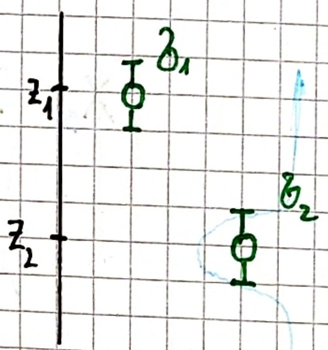
Že iz tega da $a < a_0$ vemo da takih nezvezno skokov ne dela ampak zakrivi okoli točk.





1. Optimalno združevanje

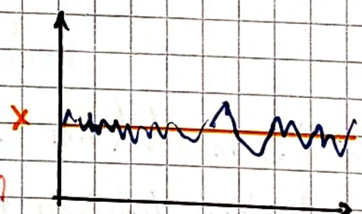
Imejmo 2 ločeni opazovanji X $\nearrow \bar{z}_1, \sigma_1$ $\searrow \bar{z}_2, \sigma_2$



Meritev $Z = X + r$

Izmerek (Z), prava vrednost (X)

naključna spremenljivka merilni šum (r)



Ocena = naključna sprem. porazdeljena po vrednosti X.

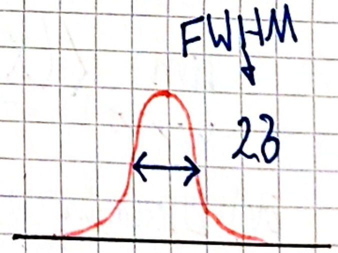
Zanima nas po kakšni porazdelitvi

$$\frac{dp}{dr} = N(0, \sigma) \,!$$

Po Gaussovi porazdelitvi $\Rightarrow$

$$\frac{dp}{dr} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-r^2/2\sigma^2}$$

disperzija (σ^2)



$$\langle r \rangle = \int_{-\infty}^{\infty} \frac{dp}{dr}\, r \, dr = 0$$

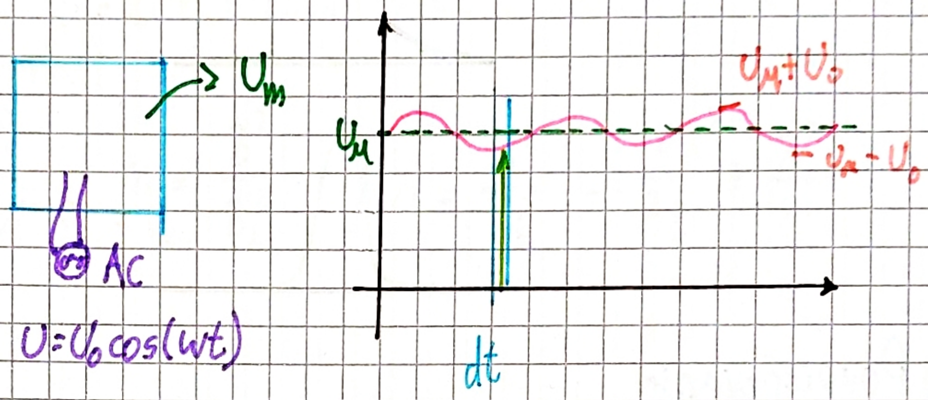
soda ($\frac{dp}{dr}$), liha (r)

$$\langle r^2 \rangle \neq 0 \Rightarrow \langle r^2 \rangle = \sigma^2$$

Vsota neodvisnih naključnih spremenljivk teži k normalni porazdelitvi.
Pomembno je da k šumu prispeva veliko malih prispevkov (skupaj Gauss) namesto enega dominantnega ne Gaussovskega

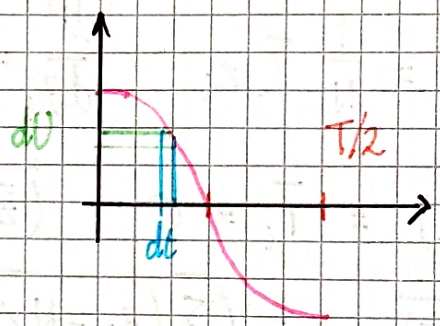
Brumov šum

Motnje zaradi napajanje z AC napetostjo



Verjetnost, da ob nekem času izmerimo meritev

$$\frac{dP}{dt} = \frac{1}{T/2} \text{ na } [0, T/2] \text{ konst.}$$



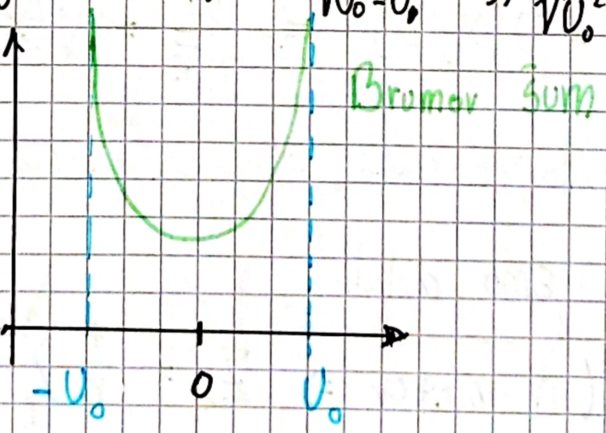
$$\frac{dP}{dU} \neq \text{konst}$$

$$dU = -U_0\omega\sin(\omega t)\,dt$$

$$\frac{dP}{dt}\left(\left|\frac{dt}{dU}\right|\right) = \frac{dP}{dt}\left|\left(\frac{1}{-U_0\sqrt{\sin^2\omega t}\,\omega}\right)\right| = \frac{dP}{dt}\,\frac{1}{U_0\sqrt{1-\cos^2\omega t}\,\omega} =$$

$$= \frac{dP}{dt}\,\frac{T}{2\pi\sqrt{U_0^2-U^2}} = \frac{1}{T/2}\,\frac{1}{2\pi}\,\frac{T}{\sqrt{U_0^2-U_0^2}} = \frac{1}{\pi}\,\frac{1}{\sqrt{U_0^2-U^2}}$$

Daleč od Gaussa,
CLT nas reši.
Imamo več prispevkov po brumu in ~~[illegible]~~.
Če jih je veliko gre proti Gauss.



a) Povprečevanje

$$\bar z = \frac{1}{N}\sum z_i \qquad \frac{dP}{dz_i} = N(x, \sigma)$$

$$\overline{(z_i - x)} = \bar r_i = 0 \qquad \overline{(\bar z - x)} = 0$$

$$\overline{(\bar z_i - x^2)} = \frac{1}{N^2}\overline{\left(\sum(z_i - Nx)\right)^2} = \frac{1}{N^2}\overline{\left(\sum(z_i - x)\right)^2} =$$

$$= \frac{1}{N^2}\left[\overline{\sum(z_i - x)^2} + \cancel{\overline{\sum_{i,j}(z_i - x)(z_j - x)}}\right] = \frac{1}{N^2}N\sigma^2 = \frac{\sigma^2}{N}$$

0 če so meritve neodvisne

$$\Rightarrow \frac{dP}{dz_i} = N(x, \sigma) \Rightarrow \frac{dP}{d\bar z} = N\left(x, \frac{\sigma}{\sqrt{N}}\right)$$

Porazdeljeno okoli istega x z ožjo Gaussovko

Recimo da smo delali

1. meritev (n meritev $\Rightarrow \bar z_1 = \frac{1}{n}\sum_1^n z_i \; ; \; \sigma_1 = \frac{\sigma}{\sqrt{n}}$)

2. meritev (m meritev $\Rightarrow \bar z_2 = \frac{1}{m}\sum_1^m z_i \; ; \; \sigma_2 = \frac{\sigma}{\sqrt{m}}$)

Kaj pa če je nekdo naredil n+m meritev

$$n+m \Rightarrow \bar{Z}_3 = \frac{1}{n+m}\sum_{1}^{n+m} z_i \qquad \sigma_3^2 = \frac{\sigma^2}{n+m}$$

$$= \frac{1}{n+m}\left[\sum_{1}^{n} z_i + \sum_{n+1}^{n+m} z_i\right] =$$

$$\bar{Z}_3 = \underbrace{\left(\frac{n}{n+m}\right)\bar{Z}_1 + \left(\frac{m}{n+m}\right)}_{\text{Dve različni uteži}}\bar{Z}_2$$

Izrazimo n, m z sigmami:

$$n = \frac{\sigma^2}{\sigma_1^2} \qquad m = \frac{\sigma^2}{\sigma_2^2} \qquad n+m = \frac{\sigma^2}{\sigma_3^2} = \underbrace{\frac{\sigma^2}{\sigma_1^2} + \frac{\sigma^2}{\sigma_2^2}}$$

ostranje

$$\frac{1}{\sigma_3^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

Torej:

$$\bar{Z}_3 = \frac{\sigma_3^2}{\sigma_1^2}\bar{Z}_1 + \frac{\sigma_3^2}{\sigma_2^2}\bar{Z}_2 =$$

$$\boxed{\bar{Z}_3 = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)\bar{Z}_1 + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\bar{Z}_2}$$

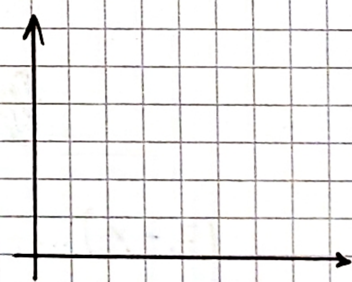
Zastopa

Izmerek, ki je bolj natančen je bolj (upoštevan) v optimalni združitveni oceni.

Merilni šum

- mnogo nepovezanih prispevkov sestavlja šum (CLI)

$$\frac{dP}{dr} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}e^{-r^2/2\sigma^2} \qquad z_i = X + r_i$$



$$\langle r^2 \rangle = \int \frac{dP}{dr} r^2 dr = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\int e^{-r^2/2\sigma^2} r^2 dr = \qquad u = r^2/2\sigma^2$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\int e^{-u^2} 2\sigma^2 u^2 \sqrt{2}\sigma = \frac{1}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-u^2}u^2 du}_{\frac{\sqrt{\pi}}{2}}(2\sigma^2) = \sigma^2$$

iz sodosti in lihosti

$$\langle r \rangle = \int_{-\infty}^{\infty}\frac{dP}{dr} r\, dr = 0$$

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \quad \Big/ \cdot \frac{d}{da}$$

$$\int_{-\infty}^{\infty} (-x^2)\,e^{-ax^2}\,dx = \sqrt{\pi}\left(-\frac{1}{2}\right) a^{-3/2}$$

$a \to 1$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Pričakovano odstopanje od povprečja

$\sigma = \sqrt{\langle r^2 \rangle}$

Konec ponavljanja

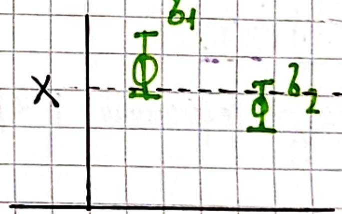
Z_3 lahko zapišemo rekurzivno:

$$Z_3 = \underbrace{Z_1}_{\text{stara meritev}} + \underbrace{\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}}_{\text{utež}} \overbrace{(Z_2 - Z_1)}^{\text{inovacija}}$$

b) Kvadratna forma

Do tega lahko pridemo tudi preko kvadratne forme $2J(x)$

$$2J(x) = \frac{(z_1 - x)^2}{\sigma_1^2} + \frac{(z_2 - x)^2}{\sigma_2^2}$$



$(z_1 - x)$ ... por. po $N(0, \sigma_1)$

$(z_2 - x)$ ... por. po $N(0, \sigma_2)$

$$\left.\begin{array}{l} \dfrac{(z_1 - x)}{\sigma_1} \text{ ... por. po } N(0,1) \\[2ex] \dfrac{(z_2 - x)}{\sigma_2} \text{ ... por. po } N(0,1) \end{array}\right\} \text{Normirani Gaussov šum}$$

Hočemo minimalni seštevek, torej zahtevamo $\frac{d}{dx}\chi^2(x) = 0$

$$\frac{-2(z_1 - x)}{\sigma_1^2} - \frac{2(z_2 - x)}{\sigma_2^2} = 0$$

$$x\left[\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right] = \frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}$$

Optimalen:

$$x = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1}\left[z_1 \frac{1}{\sigma_1^2} + z_2 \frac{1}{\sigma_2^2}\right] = z_3$$

---

c) Disperzija optimalno združene ocene

Ali je optimalno združevanje res optimalno?

$$(z_1, \sigma_1), \quad z_1 = x + r_1 \qquad \langle r_1 \rangle = 0, \quad \langle r_1^2 \rangle = \sigma_1^2$$

$$(z_2, \sigma_2), \quad z_2 = x + r_2 \qquad \langle r_2 \rangle = 0, \quad \langle r_2^2 \rangle = \sigma_2^2$$

Sestavimo z_3 kot linearno kombinacijo:

$$z_3 = \alpha z_1 + \beta z_2 = x + r$$

$z_3 = \hat{z}$

$$\Rightarrow z_3 = \alpha(x + r_1) + \beta(x + r_2) = x + r$$

$$= \underbrace{(\alpha + \beta)}_{1} x + \underbrace{\alpha r_1 + \beta r_2}_{r} = 1x + r$$

$$r = \alpha r_1 + (1 - \alpha) r_2$$

$$\langle r \rangle = 0$$

$$\langle r^2 \rangle = \alpha^2 \underbrace{\langle r_1^2 \rangle}_{=\sigma_1^2} + (1-\alpha)^2 \underbrace{\langle r_2^2 \rangle}_{=\sigma_2^2} + 2\alpha(1-\alpha)\cancel{\langle r_1 r_2 \rangle}$$

0 če nista korelirani

$$= \alpha^2 \sigma_1^2 + (1-\alpha)^2 \sigma_2^2$$

Minimiziramo to po α $\frac{d}{d\alpha}\langle r^2\rangle = 0$

$$2\alpha\sigma_1^2 + 2(1-\alpha)(-1)\sigma_2^2 = 0$$

$$\alpha(\sigma_1^2+\sigma_2^2) = \sigma_2^2$$

$$\alpha = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \qquad \beta = 1-\alpha = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$$

② Korelacija med izmerki (Ocenami)

Imamo 2 seta meritev x, y

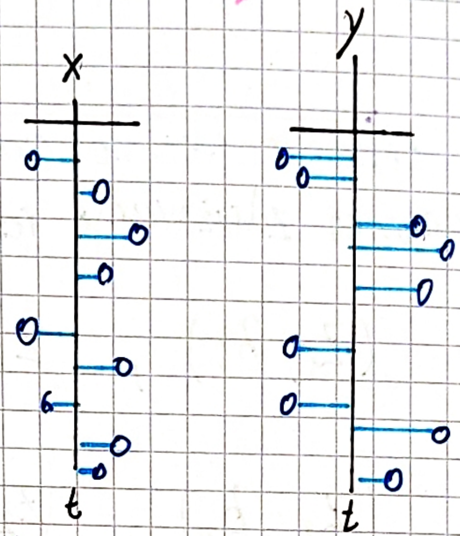
$$\overline{r}_x = \overline{r}_y = 0$$

$$\sigma_x^2 = \overline{r_x^2} \neq 0$$

$$\sigma_y^2 = \overline{r_y^2} \neq 0$$

$$\overline{r_x r_y} \neq 0$$

Obstaja korelacija (povezava med šumoma x in y meritev)



N meritev $\{x_i\}_N$ N meritev $\{y_i\}_N$

indeks je prav isti

Definiramo kovarianco

$$\sigma_{xy} = \overline{(x-\bar{x})(y-\bar{y})} = \rho\,\sigma_x\sigma_y$$

korelacijski koeficient $|\rho| \leq 1$

Negativen ρ pomeni antikorelacijo

To lahko izvedemo tudi drugače (vstavimo povprečje)

$$\sigma_{xy} = \frac{1}{N}\sum_i (x-\bar{x})(y-\bar{y}) =$$

$$= \frac{1}{N}\sum_i (xy - \bar{x}y - x\bar{y} + \bar{y}\bar{x}) =$$

$$= \frac{1}{N}\sum xy - \frac{1}{N}\bar{x}\sum y - \frac{1}{N}\bar{y}\sum x + \frac{1}{N}\sum 1\,\bar{x}\bar{y} =$$

$$= \overline{xy} - \bar{x}\bar{y} - \bar{y} - \bar{x} + \bar{x}\bar{y}$$

$$\sigma_{xy} = \overline{xy} - \bar{x}\cdot\bar{y}$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \quad \Big\} \text{ Korelacijski koeficient}$$

③ Združevanje koreliranih meritev/ocen

W... je merilni šum (v vlogi tistega kar je bilo prej r)

$$\left.\begin{matrix}(z_1, \sigma_1) \\ (z_2\ \sigma_2)\end{matrix}\right\}\rho$$

$$z_1 = X + W_1 \qquad \langle W_1^2\rangle = \sigma_1^2$$

$$z_2 = X + W_2 \qquad \langle W_2^2\rangle = \sigma_2^2$$

$$\langle W_1 W_2\rangle \neq 0$$

$$\langle W_1 W_2\rangle = \langle (z_1 - x)(z_2 - x)\rangle = \rho\,\sigma_1\sigma_2 \quad \text{kovarianca}$$

Šum ene meritve zapišemo kot lin. kombinacijo šuma druge meritve in neodvisnega dela (šum ubistvu razstavimo)

$$W_1 = \alpha W_2 + W \qquad \text{neodvisni šum} \qquad \longleftrightarrow \qquad (z_1 - x) = \alpha(z_2 - x) + W$$

$$\langle W^2\rangle = \sigma_w^2$$

$$\langle W\rangle = 0$$

$$\langle W W_2\rangle = 0 \quad (\text{nekorelirano})$$

Izrazimo s tem kovarianco:

$$\rho\,\sigma_1\sigma_2 = \langle W_1 W_2\rangle = \langle(\alpha W_2 + W)W_2\rangle =$$

$$= \alpha\langle W_2^2\rangle + \langle W W_2\rangle$$

$$\rho\sigma_1\sigma_2 = \alpha\sigma_2^2$$

$$\alpha = \rho\frac{\sigma_1}{\sigma_2}$$

Še iz disperzije prvega šuma:

$$\sigma_1^2 = \langle w_1^2\rangle = \langle(\alpha w_2 + w)^2\rangle = \alpha^2\langle w_2^2\rangle + \langle w^2\rangle + \alpha\langle w_2 w\rangle$$

$$\sigma_1^2 = \alpha^2\sigma_2^2 + \sigma_w^2$$

$$= \rho^2\frac{\sigma_1^2}{\sigma_2^2}\sigma_2^2 + \sigma_w^2$$

$$\sigma_1^2(1-\rho^2) = \sigma_w^2$$

Sestavimo sedaj kvadratno formo $2J(x)$:

ime (ne moreš pokrajšat 2)

$$2J(x) = \left(\frac{w_2}{\sigma_2}\right)^2 + \left(\frac{w}{\sigma_w}\right)^2 =$$

$$= \frac{w_2^2}{\sigma_2^2} + \frac{(w_1-\alpha w_2)^2}{\sigma_w^2} = \frac{w_2^2}{\sigma_2^2} + \frac{w_1^2 - 2\alpha w_1 w_2 + \alpha^2 w_2^2}{\sigma_w^2} =$$

Vstavimo:

$$= \frac{w_2^2}{\sigma_2^2} + \frac{w_2^2\rho^2\sigma_1^2/\sigma_2^2}{\sigma_1^2(1-\rho^2)} + \frac{w_1^2}{\sigma_1^2(1-\rho^2)} - \frac{2\rho\frac{\sigma_1}{\sigma_2}w_1 w_2}{\sigma_1^2(1-\rho^2)} =$$

$$= \frac{w^2}{\sigma_2^2}\underbrace{\left(1+\frac{\rho^2}{1-\rho^2}\right)}_{1/(1-\rho^2)} + \frac{w_1^2}{\sigma_1^2(1-\rho^2)} - \frac{2\rho w_1 w_2}{\sigma_1\sigma_2(1-\rho^2)} =$$

$$2J(x) = \left(\overset{(z_2-x)^2}{\frac{w_2^2}{\sigma_2^2}} + \overset{\to(z_1-x)^2}{\frac{w_1^2}{\sigma_1^2}} - \overset{(z_1-x)(z_2-x)}{\frac{2\rho w_1 w_2}{\sigma_1\sigma_2}}\right)\frac{1}{1-\rho^2}$$

Da je združevanje optimalno $\frac{d}{dx}(2J(x)) = 0$

$$+\frac{\cancel{2}(z_2-x)}{\sigma_2^2} + \frac{\cancel{2}(z_1-x)}{\sigma_1^2} + \frac{\cancel{2}\rho}{\sigma_1\sigma_2}\left(2x-(z_1+z_2)\right) = 0$$

$$\left[\frac{z_2}{\sigma_2^2} + \frac{z_1}{\sigma_1^2} - \frac{\rho(z_1+z_2)}{\sigma_1\sigma_2}\right] = x\left[\frac{1}{\sigma_2^2} + \frac{1}{\sigma_1^2} - \frac{2\rho}{\sigma_1\sigma_2}\right]$$

Dobimo, da je optimalen $\hat{z} = x$

$$\hat{z} = \underbrace{\left[\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{2\rho}{\sigma_1\sigma_2}\right]^{-1}}_{\hat{\sigma}^2 = (1-\rho^2)\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{2\rho}{\sigma_1\sigma_2}\right)^{-1}}\left(\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2} - \frac{\rho(z_1+z_2)}{\sigma_1\sigma_2}\right)$$

ta del smo pri odvodu pohrofšali

Mejni primeri:

1.) $\rho = 0$ $\Rightarrow$ dobimo prejšnjo formulo

2.) $\rho = 1$ (popolna korelacija $z_1 = z_2$)

$$x = \left(\frac{\sigma_2^2 + \sigma_1^2 - 2\sigma_1\sigma_2}{\cancel{\sigma_1^2\sigma_2^2}}\right)^{-1}\left[\frac{z_1\sigma_2^2 + z_2\sigma_1^2 - (z_1+z_2)\sigma_1\sigma_2}{\cancel{\sigma_1^2\sigma_2^2}}\right]$$

$$= (\sigma_2-\sigma_1)^{-2}\left[z_1\sigma_2(\sigma_2-\sigma_1) - z_2\sigma_1(\sigma_2-\sigma_1)\right]$$

$$= (\sigma_2-\sigma_1)^{-2}(\sigma_2-\sigma_1)^2 z_2 = z_2$$

$$\hat{\sigma}^2 = \sigma_2^2 = \sigma_1^2$$

3) Enaka disperzija $\sigma_1 = \sigma_2$

ρ kurkoli

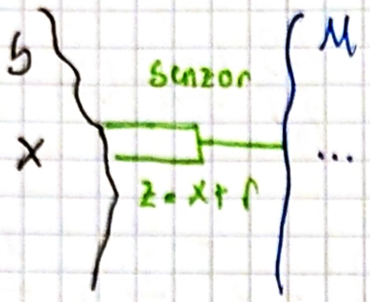
$\sigma_1 = \sigma_2 = \sigma$

$$x = (2-2\rho)^{-1}\left[(z_1+z_2) - \rho(z_1+z_2)\right]$$

$$= \frac{1}{2(1-\rho)}(z_1+z_2)(1-\rho) = \frac{z_1+z_2}{2}$$

Običajno povprečevanje

Sledenje (merjenje) konstantni skalarni količini



Kako dotok novih informacij preko meritev

$$z_i = x + r_i$$

Sinhronizira modelski sistem z realnim.

$\hat{x}$ ... Ocena k x

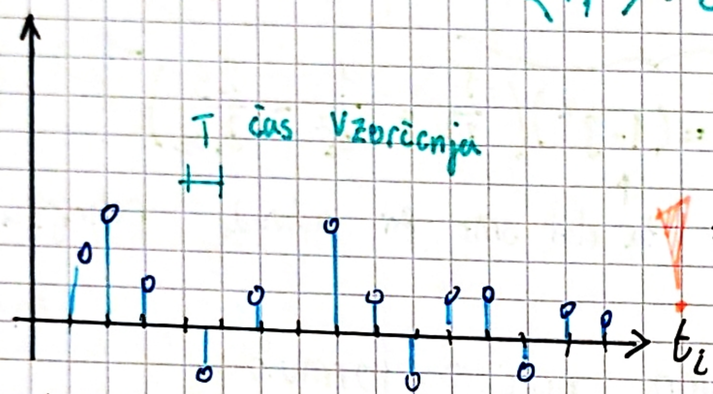
Lastnosti merilnega šuma: $z_i = x + r_i$

r_i ... merilni šum

Naključna spremenljivka por. po $N(0,\sigma)$

$\langle r_i \rangle = 0$

$\langle r_i^2 \rangle = \sigma^2$

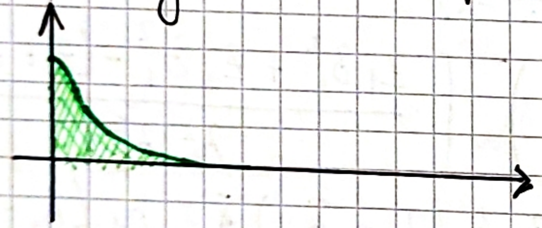


Merilni šum je nekoreliran.

$\langle r_i r_j \rangle = \delta_{ij} \sigma^2$ Avtokorelacijska funkcija

To je idealizacija (če bi čas vmes → 0 ima neka fluktuacija nek vzrok)

Fizikalno gledano ni δ ampak

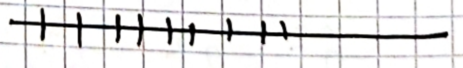


Šum v vsakem trenutku je popolnoma nepovezan s šumom v prejšnjih trenutkih.

$$\langle (\hat{x} - x) \rangle = \sigma^2$$

↳ Ocena sinhronizacijo

Shema za sledenje



$$(n \cdot T): \quad \hat{x}_n = \frac{1}{n} \sum_{i=1}^{N} z_i$$ (že imamo n meritev)

Korak

$$\hat{\sigma}^2_{0n} = \frac{\sigma^2}{n}$$

$$((n+1) \cdot T): \quad \hat{x}_{n+1} = \frac{1}{n+1} \sum_{i}^{n+1} z_i = \frac{1}{n+1} \left(\sum_{i=1}^{N} z_i + z_{n+1} \right)$$

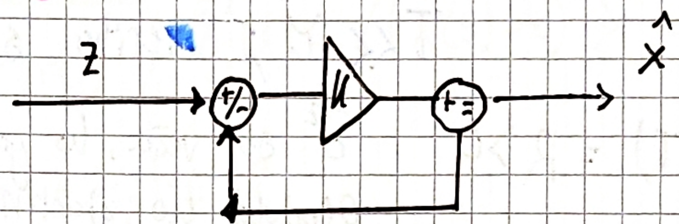
$$= \frac{1}{n+1}\hat{x}_n + \frac{1}{n+1} z_{n+1} = \hat{x}_n + \frac{1}{n+1}(z_{n+1} - \hat{x}_n)$$

Povratna zanka.

$$\hat{x}_{n+1} = \hat{x}_n + \frac{1}{n+1}(z_{n+1} - \hat{x}_n)$$

$$\hat{x}_{n+1} = \hat{x}_n + \underbrace{\frac{\hat{\sigma}^2_{n+1}}{\sigma^2}}_{\text{Utež}}\underbrace{(z_{n+1} - \hat{x}_n)}_{\text{inovacija}}$$

$$\sigma^{-2}_{n+1} = \sigma_n^{-2} + \sigma^{-2}$$



Shema sledenja konstanti

$$K(t_i) = \frac{\hat{\sigma}^2_{n+1}}{\sigma^2}$$

Ocena konvergence $\hat{x} \to x$?

Vzemimo čaz vzorčenja $T \to 0$. V limiti $\lim T \to 0$ ratajo naše diskretne spremenljivke zvezne.

$$\hat{x}_n \to \hat{x}(t) \qquad z_n \to z(t)$$

$$\hat{\sigma}^2_n \to \hat{\sigma}^2_x(t)$$

Poglejmo:

$$\lim_{T\to 0} \frac{\hat{x}_{n+1} - \hat{x}_n}{T} = \dot{\hat{x}}(t) = \frac{\hat{\sigma}^2_{n+1}}{\sigma^2 T}(z_n - \hat{x}_n)$$

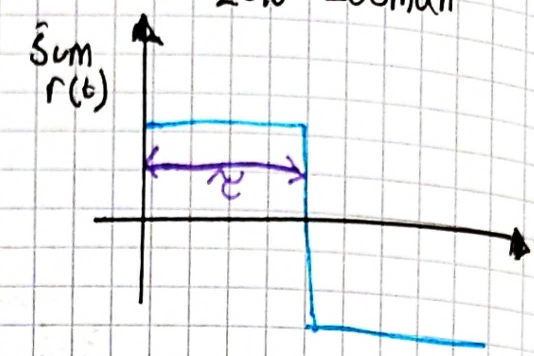
Vstavimo "limite"

$$\Rightarrow \quad \dot{\hat{x}}(t) = \frac{\hat{\sigma}^2_x}{(\sigma^2 T)}(z(t) - \hat{x}(t))$$

$\lim_{T\to 0}(\sigma^2 T) = R(t) > 0$ Zahtevamo, da je to tako!

$$\Rightarrow \quad \dot{\hat{x}}(t) = \frac{\hat{\sigma}_x^2}{R(t)}(z - x)$$

Zelo zooman



τ... čas minimalnih flutuacij

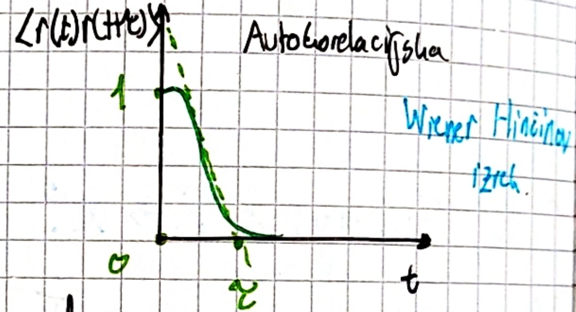
Kako je z nekoreliranostjo?

$T \gg \tau$; meritve so nekorelirane

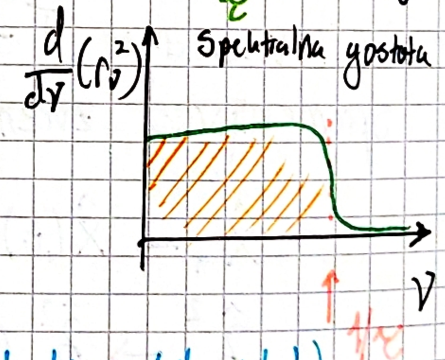
$T \ll \tau$; meritve so korelirane

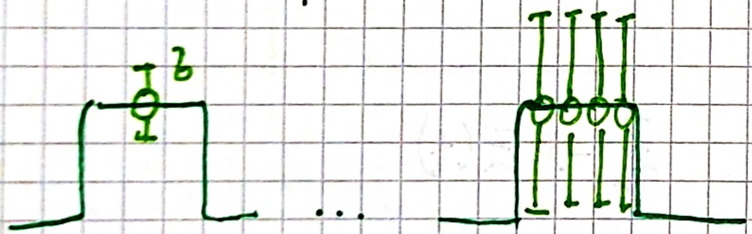
$\underbrace{\lim_{T\to 0}(\sigma^2 T) = R > 0}$ σ^2 se veča, ko gre $T \to 0$, tako, da je produkt konstanten.

Popravljanje pri zelo majhnih časih vzorčenja



Torej če znotraj ene flutuacije vzorčimo večkrat se σ poveča.





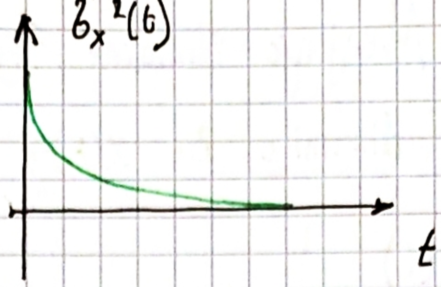
Konvergenca disperzije združene meritve (v kontinuumskih slikah)

Poglejmo si: $\frac{1}{T}\hat{\sigma}^2_{n+1} - \hat{\sigma}^2_n = \left(\frac{\hat{\sigma}^2_n \sigma^2}{\hat{\sigma}^2_n - \sigma^2} - \hat{\sigma}^2_n\right)\frac{1}{T} =$

$$= \frac{-(\hat{\sigma}^2_n)^2}{(\hat{\sigma}^2_n \cancel{T} + \sigma^2 T)} = \dot{\hat{\sigma}}^2_x$$

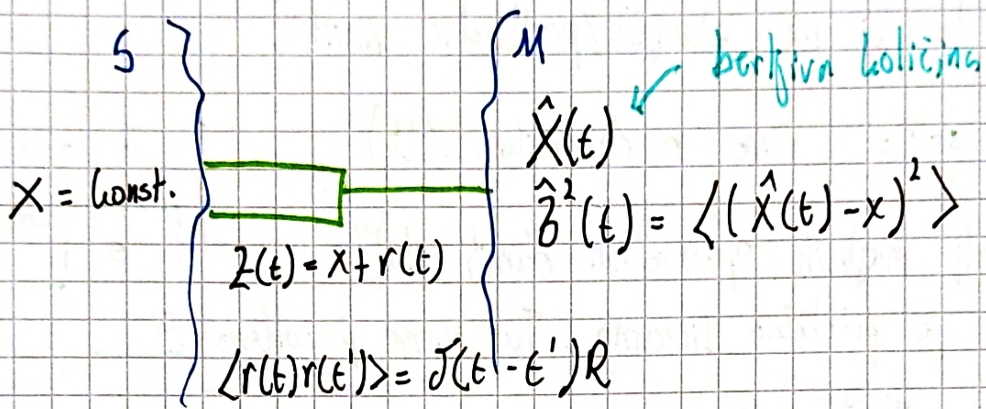
in tu sedaj limitiramo $T \to 0$.

$$\dot{\hat{\sigma}}_x^2 = -\frac{(\hat{\sigma}_x^2)^2}{R}$$



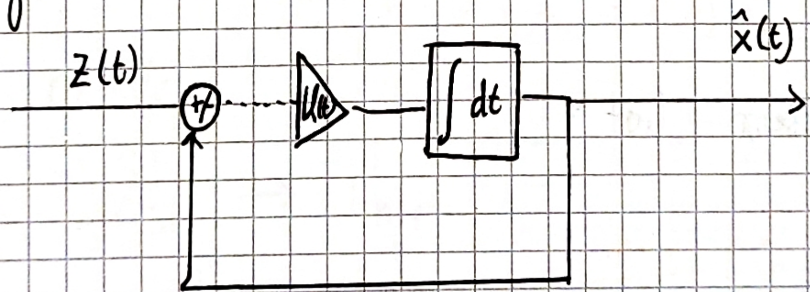
$$\hat{\sigma}_x^2 \to 0 = \langle(\hat{x} - x)^2\rangle$$

Ocena konvergira k pravi vrednosti, ko dotekamo informacije. Na koncu dobimo točno sinhronizacijo med realnim in modelskim sistemom.



$$\dot{\hat{X}} = \frac{\hat{\sigma}_x^2}{R}(z - \hat{X}) \qquad \dot{\hat{\sigma}}_x^2 = -\frac{(\hat{\sigma}_x^2)^2}{R}$$

Shema sledenja konstanti:

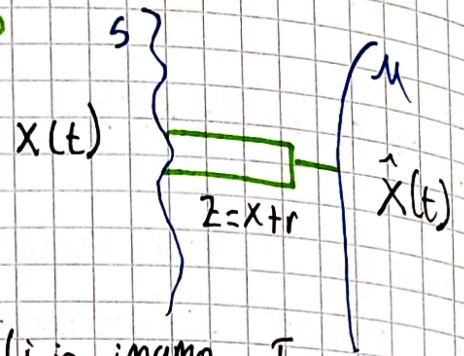


Merjenje skalarne spremenljivke

1.) Če ne poznamo dinamike za $x(t)$ v S

$$\hat{x} = Z(t)$$

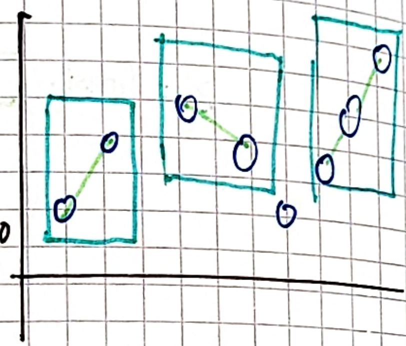
$$\hat{\sigma}^2 = R(t)$$



V vsakem trenutku je meritev edina ocena, ki jo imamo. To je kot, da bi na starejše/predhodne meritve.

2.) Nekaj gotovo vemo o dinamiki $x(t)$

Na dovolj majhnih opazovalnih oknih lahko rečemo, da je približno linearno. Tu manj pozabljamo na prejšnje meritve.



3.) Kako opišemo dinamiko v S?

Opišemo jo z linearno diferencialno enačbo 1. reda

$$\dot{x}(t) = Ax(t) + c(t)$$

Diskretno to zapišemo kot:

$$\dot{x} = \frac{x_{n+1} - x_n}{T}$$

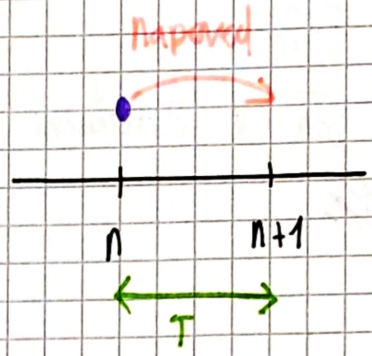
$$\Rightarrow x_{n+1} = (1 + A(t_n)T)x_n + C(t_n)T$$

$$x_{n+1} = \phi_n x_n + c_n$$ → Linearna diferenčna enačba (1. reda seveda)

$$\phi_n = 1 + A(t_n)T$$

$$c_n = C(t_n)T$$

Postopek optimalne sinhronizacije



Imejmo v trenutku (nekem) $\hat{x}_n, \hat{\sigma}_n^2$

Napovedna ocena: $\bar{x}_{n+1} = \phi_n \hat{x}_n + c_n$

Napoved disperzije za $(n+1) \cdot T$: $\bar{\sigma}^2_{n+1} = \phi_n^2 \hat{\sigma}_n^2$

$\downarrow$ Def

$$\langle(\bar{x}_{n+1} - x_{n+1})^2\rangle = \langle(\phi_n \hat{x}_n + c_n - \phi_n x_n - c_n)^2\rangle =$$
$$= \phi_n^2 \langle(\hat{x}_n - x_n)^2\rangle = \phi_n^2 \hat{\sigma}_n^2$$

Sedaj pa dobimo nov izmerek v času $n+1$ z_{n+1}, σ. Naredimo izostreno oceno po znanem postopku:

$$\left.\begin{aligned} \hat{x}_{n+1} &= \bar{x}_{n+1} + \frac{\hat{\sigma}^2_{n+1}}{\sigma^2}(z_{n+1} - \bar{x}_{n+1}) \\ \hat{\sigma}^{-2}_{n+1} &= \bar{\sigma}^{-2}_{n+1} + \sigma^{-2} \end{aligned}\right\} \text{Postopek optimalnega sledenja}$$

Uvedemo nove oznake:

$\hat{\sigma}^2_{n+1} \to P_{n+1}$ Kovarianca izostrene ocene (kovariančna matrika ocene)

$\bar{\sigma}^2_{n+1} \to M_{n+1}$ Kovarianca napovedi (kovariančna matrika napovedi)

$K_{n+1} = \frac{\hat{\sigma}^2_{n+1}}{\sigma^2} = \frac{P_{n+1}}{\sigma^2}$ Ojačevalni faktor inovacije

$$M_{n+1} = \phi_n^2 P_n \qquad P_{n+1} = \frac{M_{n+1}\sigma^2}{M_{n+1} + \sigma^2} = M_{n+1} - \frac{M_{n+1}^2}{M_{n+1} - \sigma^2}$$

$$\bar{x}_{n+1} = \phi_n \hat{x}_n + c_n$$

Dinamični šum

Φ_n in C_n sta v diferenčni sliki nepopolna. Dodati moramo še nekaj. ~~To~~

$$X_{n+1} = \Phi_n X_n + C_n + \underbrace{\Gamma_n}_{\text{multiplikativni faktor}} W_n \rightarrow \text{Dinamični šum}$$

W_n obravnavajmo to kot šum. Tisto kar ne poznamo o dinamiki skrijemo v dinamični šum (isto Gaussovsko porazdeljen nekoreliran (bel) šum).

$$\langle W_n W_{n'} \rangle = \delta_{nn'} Q_n$$

$Q = 0 \Rightarrow$ natančno znana dinamika

Ta ne vpliva na našo oceno $\hat{X}_{n+1}$. Vpliva pa na kovariance:

$$M_{n+1} = \langle (\bar{X}_{n+1} - X_{n+1})^2 \rangle = \langle (\Phi_n \hat{X}_n + \cancel{C_n} - \Phi_n X_n - \cancel{C_n} - \Gamma_n W_n)^2 \rangle$$

$$= \langle (\Phi_n (\hat{X}_n - X_n) - \Gamma_n W_n)^2 \rangle =$$

$$= \Phi_n^2 \langle (\hat{X}_n - X_n)^2 \rangle + \Gamma_n^2 \langle W_n^2 \rangle - 2\Gamma_n \Phi_n \underbrace{\langle (\hat{X}_n - X_n) W_n \rangle}_{0} =$$

popolnoma nepovezani

$$= \Phi_n^2 P_n + \Gamma_n^2 Q_n$$

$$\Rightarrow M_{n+1} = \Phi_n^2 P_n + \Gamma_n^2 Q_n$$

Zaradi nepoznavanja dinamike se nam kovarianca lahko le poveča.

Prehod v kontinuumsko sliko:

V sistemu M smo rekli: $\hat{X}_{n+1} = \bar{X}_{n+1} + K_{n+1}(Z_{n+1} - \bar{X}_{n+1})$

$$\bar{X}_{n+1} \downarrow \Phi_n \hat{X}_n + C_n$$

Poglejmo si $\lim_{T\to 0} \frac{\hat{x}_{n+1} - \hat{x}_n}{T} = \underbrace{\frac{(\phi_n - 1)\hat{x}_n}{T}}_{A(t)} + \underbrace{\frac{c_n}{T}}_{C(t)} + \frac{P_{n+1}}{T\sigma^2}(z_{n+1} - \hat{x}_{n+1}) =$

$\phi_n = 1 + A(nT)T$

$c_n = C(nT)T$

$$\Rightarrow \quad \dot{\hat{x}}(t) = A(t)\hat{x}(t) + C(t) + \frac{P(t)}{R}(z(t) - \hat{x}(t))$$

Poglejmo prehod še za kovarianco:

$$\underbrace{P_{n+1}}_{\text{izostrena}} = \underbrace{M_{n+1}}_{\text{napoved}} - \frac{M_{n+1}^2}{M_{n+1} + \sigma^2}$$

$$P_{n+1} = \underbrace{\phi_n^2}_{\to 1} P_n + \Gamma_n^2 Q_n - \frac{(\phi_n^2 P_n + \Gamma_n^2 Q_n)^2}{(M_{n+1} + \sigma^2)}$$

$(\phi_n + 1)(\phi_n - 1)$

$\frac{M_n^+}{T} \quad Q_n^2 T^2 \cdot T^{-2} \to 0$

Spet pogledamo limito

$$\lim \frac{P_{n+1} - P_n}{T} = \frac{(\phi_n^2 - 1)P_n}{T} + \frac{\Gamma^2 Q_n}{T} - \frac{(\phi_n^4 P_n^2 + \cancel{\Gamma_n^4 Q_n^2} + \cancel{2\Gamma_n^2 Q_n \phi_n^2 P_n})}{\underbrace{M_{n+1}T}_{\to 0} + \sigma^2 T} =$$

$\uparrow 0$ podoben razlog

$$= 2AP(t) + \ldots =$$

$$\lim \frac{\Gamma^2 (Q_n \cdot T)}{T^2} \quad \begin{cases} \lim (Q_n \cdot T) \to Q(t) \\ \left(\frac{\Gamma}{T}\right)^2 \to \Gamma^2(t) \end{cases}$$

$$= 2AP(t) + \Gamma^2 Q - \frac{P^2(t)}{R}$$

Ostrenje, zaradi dotoka novih meritev

$$\Rightarrow \quad \underbrace{\dot{P}(t)}_{\text{sprememba}} = \underbrace{2AP}_{\text{kovariance zaradi znane dinamike}} + \underbrace{\Gamma^2 Q}_{\text{povečevanje dinamičnega}} - \underbrace{\frac{P^2}{R}}_{\text{zaradi šuma}}$$

Komentar k limitam:

$$\lim_{T \to 0} \frac{\Gamma_n}{T} = \Gamma(t)$$ → Bolj kot znatost zmanjšamo manjši je

$$\lim_{T \to 0} (Q_n T) = Q(t)$$ → Bolj ko je znato večje je negotovost večja kot šum.

⇓

Skupaj je njun produkt $\Gamma^2 Q$ konstan

Vektorske Spremenljivke

S $\vec{x}$ M

$\hat{x}_1$
$\hat{x}_2$
$\vdots$
$\hat{x}_N$

$$\sum F = m\ddot{x} \quad \Rightarrow \quad \begin{aligned} \dot{x} &= v \\ \dot{v} &= \sum F/m \end{aligned}$$

Dif. 2 reda — Sistem dif. 1. reda

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \vec{x}$$

$\hat{\vec{x}}$ – ocena k vektorju $\vec{x}$

$$\langle (\hat{x}_i - x_i)^2 \rangle = \sigma_i^2$$

Med komponentami so lahko korelacije

$$\langle (\hat{x}_i - x_i)(\hat{x}_j - x_j) \rangle$$

Zgled: $\bar{x} = x + m_x$ ← šum

$\bar{v} = v + m_v$ ← šum

$t \to t + T \Rightarrow x(t+T) = x(t) + vT$

Ostrenje vektorske spremenljivke

$z = x + r$ $\quad r$ – merilni šum $\quad$ (merimo samo lego) $\quad \vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$

$$\langle r^2 \rangle = \sigma_r^2$$

$\bar{x} = x + m_x$

$\bar{v} = v + m_v$

$\langle m_x r \rangle = \langle m_v r \rangle = 0$

$\langle m_x m_v \rangle \neq 0$

$$\hat{x} = x + \hat{p}_x = a_{xx}\,\underline{\underline{\bar{x}}} + a_{xv}\,\underline{\underline{\bar{v}}} + b_x\,\underline{\underline{z}}$$

$$\hat{v} = v + p_v = a_{vx}\,\underline{\underline{\bar{x}}} + a_{vv}\,\underline{\underline{\bar{v}}} + b_v\,\underline{\underline{z}}$$

$$\hat{x} = a_{xx}(x + m_x) + a_{xv}(v + m_v) + b_x(x + r) =$$

$$= x(a_{xx} + b_x) + v a_{xv} + \underbrace{a_{xx} m_x + a_{xv} m_v + b_x r}_{\hat{p}_x} =$$

$a_{xx} + b_x = 1$

$a_{xv} = 0$

$$\hat{p}_x = a_{xx} m_x + (1 - a_{xx}) \cdot r$$

$$\langle \hat{p}_x^2 \rangle = a_{xx}^2 \underbrace{\langle m_x^2 \rangle}_{\sigma_x^2} + (1 - a_{xx})^2 \underbrace{\langle r^2 \rangle}_{\sigma_r^2} + a_{xx}(1 - a_{xx}) \langle m_x \cdot r \rangle$$

$$\langle p_x^2 \rangle = a_{xx}^2 \sigma_x^2 + (1 - a_{xx})^2 \sigma_r^2 \quad \Big/ \frac{d}{d a_{xx}} = 0 \quad \text{min/mum}$$

$$0 = 2 a_{xx} \sigma_x^2 - (1 - a_{xx}) 2 \sigma_r^2$$

$$\Rightarrow a_{xx} = \frac{\sigma_r^2}{\sigma_x^2 + \sigma_r^2} \qquad b_x = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2}$$

Isto naredimo še za hitrost:

$$\hat{V} = a_{vx}(x+m_x) + a_{vv}(v+m_v) + b_v(x+r) =$$

$$= \underbrace{a_{vv}}_{=1} v + x\underbrace{(a_{vx}+b_v)}_{=0} + a_{vx}m_x + a_{vv}m_v + b_v r$$

$a_{vv} = 1$

$a_{vx} = -b_v$

$$= v + \underbrace{a_{vx}m_x + m_v + (-a_{vx})r}_{\hat{p}_v}$$

$$\langle \hat{p}_v^2 \rangle = a_{vx}^2 \langle m_x^2 \rangle + \langle m_v^2 \rangle + 2a_{vx}\langle m_x m_v \rangle + (a_{vx})^2 \langle r^2 \rangle \Big/ \frac{d}{da_{vx}}$$

$$0 = 2a_{vx}\sigma_x^2 + 2\langle m_x m_v \rangle + 2(a_{vx})\sigma_r^2$$

$$a_{vx}(\sigma_x^2 + \sigma_r^2) = -\langle m_x m_v \rangle$$

$$a_{vx} = -\frac{\langle m_x m_v \rangle}{\sigma_x^2 + \sigma_r^2} \qquad b_v = -a_{vx}$$

$$\hat{x} = \frac{\sigma_r^2 + \sigma_x^2 - \sigma_x^2}{\sigma_x^2 + \sigma_r^2}\bar{x} + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2} z$$

$$\hat{x} = \bar{x} + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_r^2}(z - \bar{x})$$

korelacija da odziv tudi komponent ki jih ne merimo

$$\hat{v} = \bar{v} - \frac{\langle m_x m_v \rangle}{\sigma_x^2 + \sigma_r^2}\bar{x} + \frac{\langle m_x m_v \rangle}{\sigma_x^2 + \sigma_r^2} z = \bar{v} + \frac{\langle m_x m_v \rangle}{\sigma_x^2 + \sigma_r^2}(z - \bar{x})$$

Kovariančna matrika ocene

Def:

$$M = \langle (\bar{x}-x)(\bar{x}-x)^T \rangle$$

$\bar{x} = \vec{\bar{x}}$

$x = \vec{x}^0$

Kovariančna matrika izostrek ocene

$$P = \langle (\hat{x}-x)(\hat{x}-x)^T \rangle$$

Za verjetnostno gostoto po prevzamemo večrazsežno Gaussovo porazdelitev

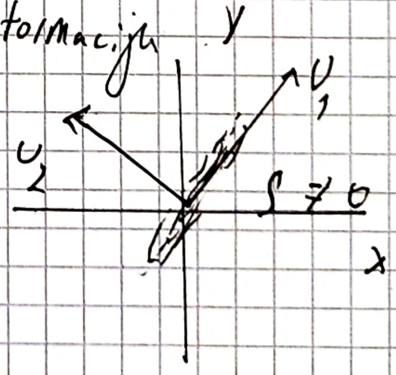
$$p(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \det M}} \exp\left[-\frac{1}{2}(\bar{x}-x)^T M^{-1}(\bar{x}-x)\right]$$

Primer:

$$p(\bar{x},\bar{y}) = \frac{1}{2\pi}\frac{1}{\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[\frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y}\right]$$

$$\bar{x},\bar{y} \to u_1, u_2 \qquad \vec{u} = \Omega\vec{x}$$

ortogonalna transformacija



$$M_0 = \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$$

$$P(u_1,u_2) = \prod_{i=1,2} \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma} e^{-\frac{1}{2}(u_i-\bar{u}_i)^2/\sigma_i^2}$$

$\Updownarrow$

produkt dveh Gaussovk.

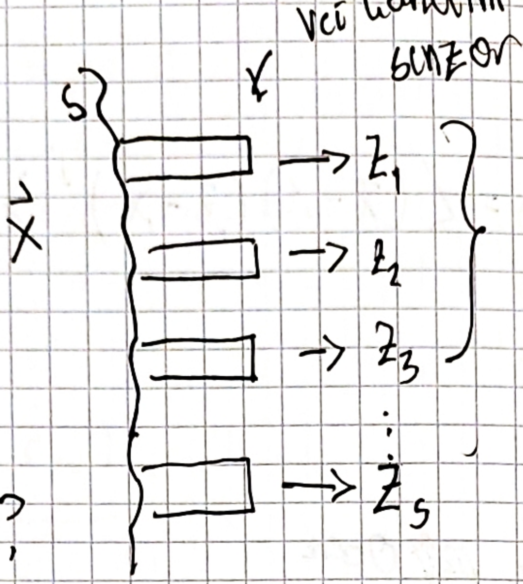
$$M_u = \Omega M \Omega^T$$

Merjenje več spremenljivk

$$\vec{z} = H\vec{x} + \vec{r}$$

↓ merilni šum

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix} \quad s?$$



$$\vec{z} \to \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Povezava je matrika senzorjev H (okvirno)

$$R = \langle \vec{r} \cdot \vec{r}^{\,T} \rangle$$

kovariančna matrika senzorskega šuma

Primer:

$$\begin{bmatrix} x \\ v \end{bmatrix} \quad H = [1, 0]$$

$$z = Hx + r$$
$$= [1, 0]\begin{bmatrix} x \\ v \end{bmatrix} = x$$

$H = [0, 1]$ → sum hitrost

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ 2 senzorja → lega → hitrost

$$H = \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}$$ → slabo. načeloma (meri malo obojega)

Manjkal kar punika!

$\hat{\vec{x}}$ ... izostrena ocena

$\bar{\vec{x}}$ ... napoved

$$\langle(\hat{\vec{x}}-\vec{x})(\hat{\vec{x}}-\vec{x})^T\rangle = P$$ kovariančna matrika izostrene ocene

$$\langle(\bar{\vec{x}}-\vec{x})(\bar{\vec{x}}-\vec{x})^T\rangle = M$$ kovariančna matrika napovedi

$$\hat{\vec{x}} = \bar{\vec{x}} + PH^TR^{-1}(\vec{z} - H\bar{\vec{x}})$$

Počasi spuščamo velikosti znak $\hat{\vec{x}} \to x$

$$P^{-1} = M^{-1} + H^TR^{-1}H \to P = M - MH^T(R + HMH^T)^{-1}HM$$

analog ostrenja:

$$\hat{\sigma}_x^{-2} = \bar{\sigma}_x^{2} - \sigma^{-2}$$

$P = M - MH^T(R + HMH^T)^{-1} HM$ ~~$/\cdot P^{-1}$ iz leve~~

$P^{-1} = M^{-1} + H^T R^{-1} H \quad /\cdot P$ iz leve, M iz desne $\rightarrow P = M - PH^T R^{-1} HM$

$$PP^{-1} = I = PM^{-1} + PH^T R^{-1} H$$

$$M = P + PH^T R^{-1} HM$$

$$= P(I + H^T R^{-1} HM) \quad /\cdot H^T$$

$$MH^T = P(H^T + H^T R^{-1} HMH^T)$$

$$= PH^T(\underbrace{I}_{R^{-1}R} + R^{-1} HMH^T)$$

$$= PH^T R^{-1}(R + HMH^T)$$

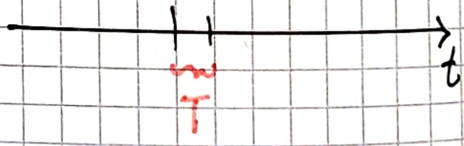
$$\Rightarrow \quad PH^T R^{-1} = MH^T(R + HMH^T)^{-1}$$

$$\Rightarrow \quad P = M - \underbrace{MH^T(R + HMH^T)^{-1} HM}$$

Zaradi meritve ($R < \infty$) se kovarianca zmanjšuje $\equiv$ Ostrenje

Dinamika + dinamični šum:

- Diskreten primer



$(nT) \rightarrow (n+1)T$; T... čas vzorčenja

Sistem:

Vsi vektorji

$$S; \quad \underline{x}_{n+1} = \Phi_n \underline{x}_n + \underline{c}_n + \Gamma_n \underline{w}_n$$

dinamični šum

$$M; \quad \underline{\bar{x}}_{n+1} = \Phi_n \underline{\hat{x}}_n + \underline{c}_n$$

$$M_{n+1} = \langle (\bar{x}_{n+1} - x_{n+1})(\bar{x}_{n+1} - x_{n+1})^T \rangle$$

Vstavimo predpise:

istо

$$= \langle (\Phi_n \hat{x}_n c_n - \Phi_n x_n - c_n - \Gamma_n w_n)(\Phi_n(\hat{x}_n - x_n) - \Gamma_n w_n)^T \rangle =$$

$$= \langle (\Phi_n(\hat{x}_n - x_n) - \Gamma_n w_n)((\hat{x}_n - x_n)^T \Phi_n^T - w_n^T \Gamma_n^T) \rangle =$$

$\rightarrow \langle w_n w_{n'} \rangle = \delta_{nn'} Q_n$

$$= \Phi_n \underbrace{\langle (\hat{x}_n - x_n)(\hat{x}_n - x_n)^T \rangle}_{P_n} \Phi_n^T + \Gamma_n \underbrace{\langle w_n w_n^T \rangle}_{Q_n} \Gamma_n^T +$$

$$+ \underbrace{\Phi_n \langle (\hat{x}_n - x_n) w_n w_n^T \rangle \Phi_n^T}_{0} + \dots \underbrace{\langle \dots \rangle}_{0} \dots$$

$$\Rightarrow \quad M_{n+1} = \Phi_n P_n \Phi_n^T + \Gamma_n Q_n \Gamma_n^T$$

Kovariančna matrika napovedi

Od prej. šč.: $P_{n+1}^{-1} = M_{n+1}^{-1} + H^T R^{-1} H$

$$P_{n+1} = M_{n+1} - M_{n+1} H^T (R_{n+1} + H M_{n+1} H^T)^{-1} H M_{n+1}$$

To je Kalmanov optimalen filter za diskretno vektorsko spremenljivko.

Prehod v kontinuumsko (zvezno) sliko:

- dinamični šum w
- merilni šum r

} Obravnavamo kot nekorelirana

Ojačevalni faktor

$$\dot{\hat{x}} = \frac{\hat{x}_{n+1} - \hat{x}_n}{T} = \frac{\Phi_n \hat{x}_n + c_n - \hat{x}_n}{T} + K_{n+1}(z_{n+1} - H\bar{x}_{n+1}) \cdot \frac{1}{T} =$$

T diskret	$\lim T \to 0$
$\hat{x}_{n+1}$	$\hat{x}(t)$
$\bar{x}_{n+1}$	$\bar{x}(t)$
TR_n	$R(t)$
z_n	$z(t)$
P_n	$P(t)$
M_n	~~[illegible]~~ $P(t)$

$$= \frac{(\Phi_n - I)}{T}\hat{x}_n + \frac{c_n}{T} + \frac{P_{n+1} H^T R_{n+1}^{-1}}{T}(z_{n+1} - H\bar{x}_{n+1}) =$$

$\Phi_n = I + AT$

$$= A(t)\hat{x}(t) + c(t) + PH^T R^{-1}(t)\,(z(t) - Hx(t))$$

$\dot{x} = Ax + r$

$$\Rightarrow \quad \dot{\hat{x}} = A(t)\hat{x}(t) + c(t) + PH^T R^{-1}(t)(z(t) - H\hat{x}(t))$$

Kaj pa je $\dot{P}$?

$$P_{n+1} - P_n = \underbrace{\Phi_n P_n \Phi_n^T + \Gamma_n Q_n \Gamma_n^T}_{M_{n+1}} - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1} - P_n$$

$$= (I + AT) P_n (I + AT)^T + \Gamma_n Q_n \Gamma^T - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1} - P_n =$$

$$= P_n + ATP_n + P_n T A^{-1} + AP_n A^T T^2 + \Gamma_n Q_n \Gamma_n^T - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1} - P_n$$

To delimo s časom T in $\lim T \to 0$:

$$\dot{P} = AP_n + P_n A^T + \emptyset + \left(\frac{\Gamma_n}{t}\right)(Q_n t)\left(\frac{\Gamma_n^T}{T}\right) - PH^T R^{-1}(t) HP$$

$\Gamma(t)$ $\quad Q(t)$ $\quad \Gamma^T(t)$

dinamičen razvoj kovariančne matrike v zveznem primeru

Riccatijeva enačba

$$\Rightarrow \quad \dot{P} = AP + PA^T + \Gamma Q \Gamma^T - PH^T R^{-1} HP$$

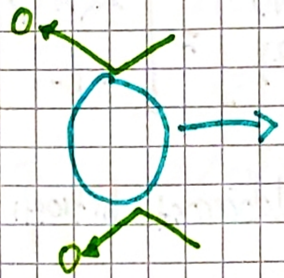
odvisno od primera (npr upor ima to negativno)

dinamični šum vedno veča

meritev vedno manjša

$R \to \infty$, nič ne merimo

Primer: [Brownovo gibanje koloidnega delca v raztopini]



Dinamika (1D) / Stokesov linearni zakon upora

$$m\ddot{x} = -6\pi r\eta \cdot \dot{x} + F_x(t)$$

naključne sile zaradi diskretnih trkov

$$\left\langle \frac{F_x(t)}{m} \cdot \frac{F_x(t')}{m} \right\rangle = Q\,\delta(t-t')$$ Opišemo z dinamičnim šumom

Kalmanov filter za sledenje delca: $\vec{x} = \begin{bmatrix} x \\ v \end{bmatrix}$

Ob $t=0$; $P(0) = P_0$

$\bar{x}(0) = 0$ → na začetku v izhodišču

Ob $t>0$;

Dinamika:
$$\dot{x} = v$$
$$\dot{v} = -\underbrace{\frac{1}{\tau}}_{6\pi r\eta} v + \frac{F_x(t)}{m}$$ } sistem 1. reda

Dinamični šum na 2. komponenti za $\vec{x}$

$\dot{\vec{x}} = A\vec{x} + \vec{c} + \Gamma w$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + 0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{F_x(t)}{m}$$

Kaj pa P?

$$\dot{P} = \begin{bmatrix} \dot{P}_{xx} & \dot{P}_{xv} \\ \dot{P}_{vx} & \dot{P}_{vv} \end{bmatrix} = AP + \underbrace{PA^T}_{(AP)^T} + \Gamma Q \Gamma^T$$

$$\dot{P} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle vx\rangle & \langle v^2\rangle \end{bmatrix} + PA^T + \begin{bmatrix} 0 \\ 1 \end{bmatrix} Q\,[0, 1] =$$

$$= \begin{bmatrix} \langle xv\rangle, & \langle v^2\rangle \\ -\frac{1}{\tau}\langle xv\rangle, & -\frac{1}{\tau}\langle v^2\rangle \end{bmatrix} + \begin{bmatrix} \langle xv\rangle & -\frac{1}{\tau}\langle xv\rangle \\ \langle v^2\rangle & -\frac{1}{\tau}\langle v^2\rangle \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} =$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle vx\rangle & \langle v^2\rangle \end{bmatrix} = \begin{bmatrix} 2\langle xv\rangle, & \langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle \\ -\text{II}-, & -\frac{2}{\tau}\langle v^2\rangle + Q \end{bmatrix}$$

Zanima nas a se nedoločenost hitrosti lje ustali (stacionarne rešitve):

$$\frac{d}{dt}\langle v^2\rangle = -\frac{2}{\tau}\langle v^2\rangle + Q$$

$$\frac{d}{dt}\langle v^2\rangle = 0\ ;\ t \to \infty$$

$$\Rightarrow \quad \langle v^2\rangle_\infty = \frac{Q\tau}{2}$$

Vbistu iščemo termodinamsko ravnovesje:

$$\frac{1}{2}m\langle v^2\rangle = \langle W_k\rangle = \frac{1}{2}k_BT$$

$$\Rightarrow \quad \langle v^2\rangle_\infty = \frac{kT}{m}$$

Torej:

$$\frac{Q\tau}{2} = \frac{kT}{m} \quad \Rightarrow \quad Q = \frac{2kT}{\tau m}$$

Korelacijski člen?

$$\frac{d}{dt}\langle xv\rangle = \langle v^2\rangle = \frac{1}{\tau}\langle xv\rangle$$

$$\Rightarrow \langle v^2\rangle_\infty = \frac{1}{\tau}\langle xv\rangle_\infty \Rightarrow \langle xv\rangle_\infty = \frac{\tau \cdot kT}{m}$$

Še za tretjo komponento:

$$\frac{d}{dt}\langle x^2\rangle = 2\langle xv\rangle \quad / \cdot \int dt$$

Ne obstaja stacionarna rešitev za lego.

$$\Rightarrow \quad \langle x^2(t)\rangle - \langle x^2(0)\rangle = \frac{2\tau kT}{m}t$$

D

$$\to \quad X^2 - X_0^2 = 2Dt$$

To je difuzijski zakon!

Sedaj vključimo še meritev lege ($R < \infty$)

$$\dot{P} = \dots - PH^TR^{-1}HP$$

$$z = Hx + r$$

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$$

$\langle rr^T \rangle = R$ = skalar σ^2

$$-\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle xv\rangle & \langle v^2\rangle \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{R}\begin{bmatrix} 1, & 0 \end{bmatrix}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \\ \langle xv\rangle & \langle v^2\rangle \end{bmatrix} =$$

$$= -\frac{1}{R}\begin{bmatrix} \langle x^2\rangle \\ \langle xv\rangle \end{bmatrix}\begin{bmatrix} \langle x^2\rangle & \langle xv\rangle \end{bmatrix} = -\frac{1}{R}\begin{bmatrix} \langle x^2\rangle^2 & \langle x^2\rangle\langle xv\rangle \\ \langle xv\rangle\langle x^2\rangle & \langle xv\rangle^2 \end{bmatrix}$$

To moramo še dodati prejšnjemu $\dot{P}$. Iščemo spet stacionarne rešitve.

1.) $2\langle xv\rangle - \frac{1}{R}\langle x^2\rangle^2 = 0$

2.) $\langle v^2\rangle - \frac{1}{\tau}\langle xv\rangle - \frac{1}{R}\langle x^2\rangle\langle xv\rangle = 0$

3.) $-\frac{2}{\tau}\langle v^2\rangle - \frac{1}{R}\langle xv\rangle^2 + Q = 0$

$$y = \langle x^2\rangle \frac{\tau}{R}$$

Iz 3.) $\Rightarrow \langle v^2\rangle = \frac{\tau Q}{2} - \frac{\tau}{2R}\langle xv\rangle^2$

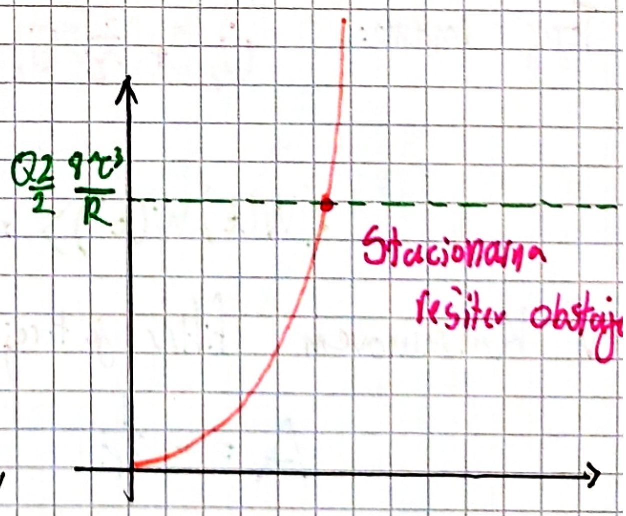
$$\frac{Q\tau}{2} - \frac{\tau}{2R}\langle xv\rangle^2 - \frac{1}{\tau}\langle xv\rangle - \frac{1}{R}\langle x^2\rangle\langle xv\rangle = 0$$

Uvedemo y

$$\frac{Q\tau}{2} - \frac{\tau}{2R}\langle x^2\rangle^4\frac{1}{4R^2} - \frac{1}{\tau}\frac{1}{2R}\langle x^2\rangle^2 - \frac{1}{R}\langle x^2\rangle\frac{1}{2R}\langle x^2\rangle^2$$

$$\frac{Q\tau}{2} - \frac{y^4 R}{\tau^3\cdot 8} - \frac{4y^2 R}{8\tau^3} - \frac{y^3 R\cdot 4}{8\tau^3}$$

$$\Rightarrow \frac{Q\tau}{2}\frac{8\tau^3}{R} = y^4 + 4y^2 + 4y^3$$



Kakršnakoli meritev vodi do omejevanja v prostoru.

Primer "grobe meritve"; R velik → y majhen

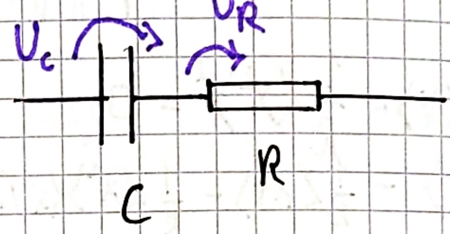
$$y \gg y^2 \gg y^3 \gg y^4$$

$$\Rightarrow \quad \frac{Q\tau}{2}\,\frac{8\tau^3}{R} = 4y^2 = \frac{4Q\tau^4}{R}$$

$$\Rightarrow \quad y = \sqrt{\frac{Q}{R}}\,\tau^2 = \langle x^2\rangle \cdot \frac{\tau}{R}$$

$$\Rightarrow \quad \langle x^2\rangle_\infty = \sqrt{\frac{Q}{R}}\,\tau R = \sqrt{QR}\cdot\tau$$

Primer: [Merjenje napetosti na RC členu]



Napišemo lahko Kirchoffov zakon:

$$\sum u_i = 0$$

$$U_R + u_c = 0$$

$$-IR - \frac{e}{C} = 0$$

$$-\frac{e}{C} = U_C = -U_R = IR$$

$$\dot{U}_c = -\frac{I}{C} = +\frac{U_R}{RC} = -\frac{U_c}{\tau} \; ; \; \tau = RC$$

Torej imamo: $\dot{U}_c + \frac{1}{\tau}U_c = W(t)$, Preimenujemo $U_c \to U$

dinamični šum

$$\langle W(t)W(t')\rangle = Q\,\delta(t-t')$$

V Kalmanovem filtru je torej

$$A = -\frac{1}{\tau} \qquad \Gamma = 1$$

Meritev napetosti na C:

$$z = u + r \quad ; \quad \langle r(t) r(t') \rangle \underline{\underline{R}} \, \delta(t-t')$$

V sistemu $\mathcal{M}$:

$\hat{u}$ ocena za U v sistemu S

Kovarianca ocene

$$\langle (\hat{U} - U)^2 \rangle = P$$

Toref je kalman:

$$\dot{P}(t) = \cancel{\#} 2AP + \Gamma^2 Q - P^2/R$$

i) Stacionarna rešitev $t \to \infty$; $P(t \to \infty) = P_\infty$

$$\dot{P} = 0 \quad \Rightarrow \quad -P^2/R + 2AP + \Gamma^2 Q = 0$$

Vstavimo A in Γ:

$$-\frac{P^2}{R} - \frac{2}{\tau} P + Q = 0$$

$$\frac{1}{R} (P - P_{1_\infty})(P - P_{2_\infty}) = 0$$

$$P_{1,2_\infty} = -\frac{2R}{\tau} \frac{1}{2} \pm \sqrt{\left(\frac{2R}{\tau}\right)^2 + 4QR} \cdot \frac{1}{2} =$$

$$= -\frac{R}{\tau} \pm \alpha \quad ; \qquad \alpha = \frac{R}{\tau} \sqrt{1 + \frac{QR\tau^2}{R^2}}$$

ii) Splošna rešitev

Za $\forall t$:

$$\frac{dp}{\left(P^2/R + 2P/\tau - Q\right)} = -dt$$

$$\frac{dp \cdot R}{\left(P + \frac{R}{\tau} - \alpha\right)\left(P + \frac{R}{\tau} + \alpha\right)} = -dt$$

Razbijmo na parcialne ulomke $\Rightarrow$ $$-dt = R\left[\frac{dp\, B}{(p+\frac{R}{\tau}+\alpha)} + \frac{dp\, D}{(p+\frac{R}{\tau}-\alpha)}\right]$$

$$Bp + B\frac{R}{\tau} - \alpha B + pD + D\frac{R}{\tau} + \alpha D = 1$$

$$p(B+D)=0 \Rightarrow B=-D$$

$$B\frac{R}{\tau} - \alpha B + \frac{R}{\tau}D + \alpha D = 1$$

$$\alpha(-B+D)=1$$

$$2D\alpha = 1 \Rightarrow D=\frac{1}{2\alpha} \quad B=-\frac{1}{2\alpha}$$

Tako imamo enačbo:

$$-dt = \frac{R}{2\alpha}\left[\frac{dp}{(p+\frac{R}{\tau}-\alpha)} - \frac{dp}{(p+R/\tau+\alpha)}\right] \Big/ \cdot \int$$

$$-\frac{2\alpha t}{R}\Bigg|_0^t = \ln\frac{(p+\frac{R}{\tau}-\alpha)}{(p+\frac{R}{\tau}+\alpha)}\Bigg|_{p_0}^{p(t)}$$

$$\Rightarrow \frac{(p+R/\tau-\alpha)}{(p+R/\tau+\alpha)} = \underbrace{\left(\frac{p_0+\frac{R}{\tau}-\alpha}{p_0+\frac{R}{\tau}+\alpha}\right)}_{p_0\to\infty} e^{-2\alpha t/R}$$

ko $(t=0)$ saj še nič ne vemo o sistemu $\Rightarrow (\ldots)=1$

Ostane:

$$(p+R/\tau-\alpha) = (p+R/\tau+\alpha)e^{-2\alpha t/R}$$

$$p(1-e^{-2\alpha t/R}) + \frac{R}{\tau}(1-e^{-2\alpha t/R}) = \alpha(1+e^{-2\alpha t/R})$$

Tako dobimo končno rešitev:

$$P(t) = -\frac{R}{\tau} + \alpha\left(\frac{1+e^{-2\alpha t/R}}{1-e^{-2\alpha t/R}}\right)$$

Limitni primer $t \to \infty$, da vidimo če se ujema

$$P_\infty = -\frac{R}{\tau} + \alpha$$
$$= -\frac{R}{\tau} + \sqrt{\left(\frac{R}{\tau}\right)^2 + QR}$$
$$P_\infty = -\frac{R}{\tau} + \frac{R}{\tau}\sqrt{1+\frac{Q\tau^2}{R}}$$
$$\approx -\frac{R}{\tau}\left(1-\left(1+\frac{1}{2}\frac{Q\tau^2}{R}\right)\right) \quad ; \quad Q \text{ majhen}$$

$$\Rightarrow P_\infty = +\frac{R}{\tau}\frac{Q\tau^2}{2R} \Rightarrow P_\infty = \frac{Q\tau}{2}$$

V sistemu M:

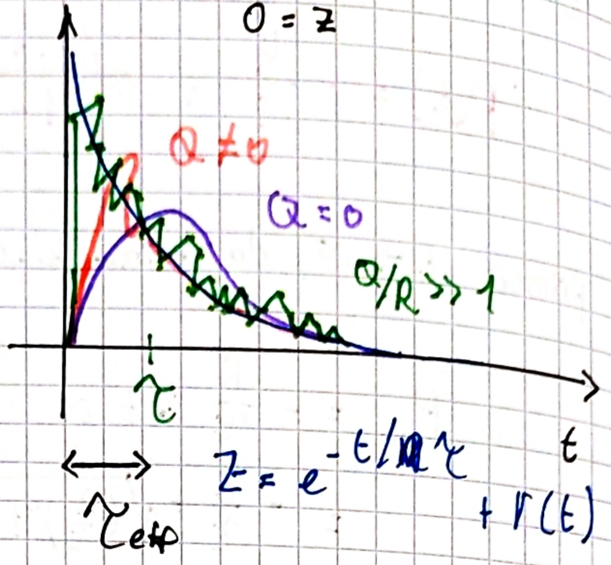
$$\dot{\hat{U}} = -\frac{1}{\tau}\hat{U} + \underset{\frac{P_\infty}{R} = \frac{Q\tau}{2R}}{\underset{\parallel}{K(t)}}[z-\hat{U}]$$

$$\dot{\hat{U}} = -\frac{1}{\tau}\hat{U} + \frac{Q\tau}{2R}(z-\hat{U}) = \underbrace{\left(-\frac{1}{\tau} - \frac{Q\tau}{2R}\right)}_{-1/\tau_{eff}}\hat{U} + \frac{Q\tau}{2R}z$$

$$\Rightarrow \frac{1}{\tau_{eff}} = \frac{1}{\tau} + \frac{Q\tau}{2R} \quad ; \quad \begin{array}{ll} Q=0 & \tau_{eff} = \tau \\ Q\to\infty & \tau_{eff} \to 0 \end{array}$$

Npr. eksponentno padajoč sunek

Temu sledimo diskretno



$$\dot{x} = Ax + c$$

$$-\frac{1}{\tau}\dot{x} = Ax + c$$

$$-\frac{1}{\tau} = A$$

$$\left(\Phi = 1 + AT = 1 - \frac{T}{\tau}\right)$$

i) $Q = 0$, $K_\infty = \frac{Q}{R}$

ii) $Q \neq 0$; $\tau_{eff} < \tau$

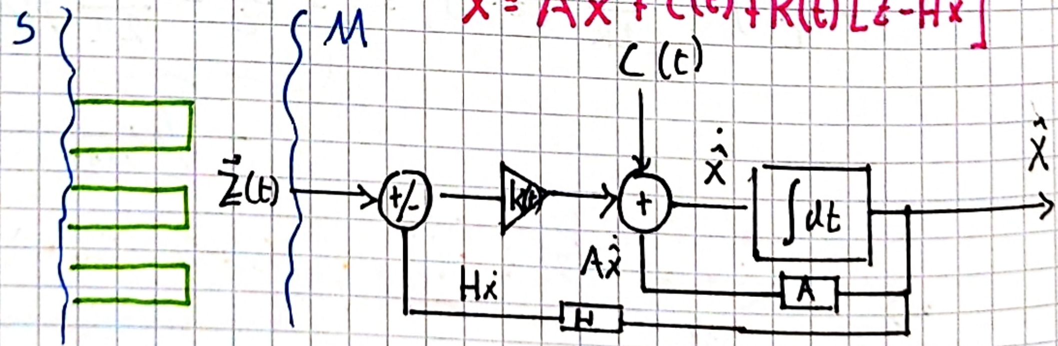
iii) $Q/R \gg 1$; $Q \neq 0$

Izkaže se, da je K_∞ dober za to če hočemo končen rezultat in nas sledenje začetnim tranzientom ne zanima.

(Spomni se temp. vode ko potopiš termometer in rabi nekaj časa, da se ustali. Samo ustaljeno nas zanima)

Poenostavitve kalmanove sheme in povratna zanka

$$\dot{\hat{x}} = A\hat{x} + c(t) + K(t)[z - H\hat{x}]$$



~~Korak~~ $K(t)[z - H\hat{x}]$ -> stopnja sinhronizacije med S in M

Kadar sta S in M usklajena je lahko K karkoli,

=> Tudi K_∞ bo dober

Poglejmo si senzor kot Univerzalen merilni sistem. Želimo si:

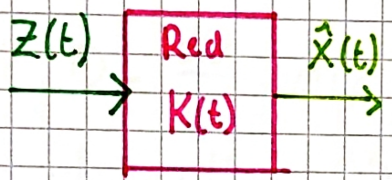
i) Na izhodu senzorja naj bo napetost $\hat{X} = U(t)$

ii) Odvisnost samo od ene količine (X)

iii) Senzor naj odpravi sam čim več merilnega šuma

iv) Senzor naj čim manj vpliva nazaj na opazovani sistem

v) $\hat{X}(t) = U(t)$; naj bo to berljiva količina



Senzor te dve povezuje preko diferencialne enačbe

Red senzorja;

Def = Red diferencialne enačbe, ki povezuje $Z(t)$ in $\hat{X}(t)$

Alter. Def = U-ti red senzorja $(\upsilon>0)$ obravnavamo kot idealen = Optimalen sledilni sistem za spremenljivke $\vec{X}(t)$ v sistemu s katerih katerih dinamike se spreminja kvečjemu:

$$\frac{d^{(\upsilon)}}{dt^{\upsilon}} X(t) = 0 + W(t)$$

Spomni se termometra pod pazduho in grejemo / ima večno zamik. Če bi bila temp non lin. odvisna recimo X^2, X^3 bi lahko T čisto pobegnila senzorju

Senzor 1. reda

V S: $\dot{X} = w(t)$ $\langle w^2 \rangle = Q$

$Z = x + r(t)$ $\langle r^2 \rangle = R$

Kalman za optimalno pravi:

$$\dot{X} = 0 - W$$

$$A = 0$$
$$C = 0$$
$$\Gamma = 1$$
$$H = 1$$

V sistemu $\mathcal{M}$ pa:

Ocena na izhodu senzorja

$$\dot{\hat{X}} = K(z - \hat{X})$$

$$\dot{P} = -P^2/R + Q$$

$$K = P/R$$

Če je ojačevalni faktor konstanten $K(t) \rightarrow K_\infty = \frac{P_\infty}{R}$

$$\dot{P} = 0 \,, \quad \frac{P_\infty^2}{R} = Q$$

$$\Rightarrow P_\infty = \sqrt{QR} \quad K_\infty = \sqrt{Q/R}$$

Vpeljemo še

$$\tau = \frac{1}{K_\infty} = \sqrt{\frac{R}{Q}}$$

$$\frac{1}{K_\infty}\dot{\hat{X}} + \hat{X} = Z(t)$$

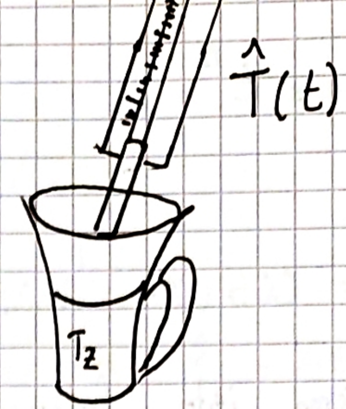
$$\boxed{\tau \dot{\hat{X}} + \hat{X} = Z}$$

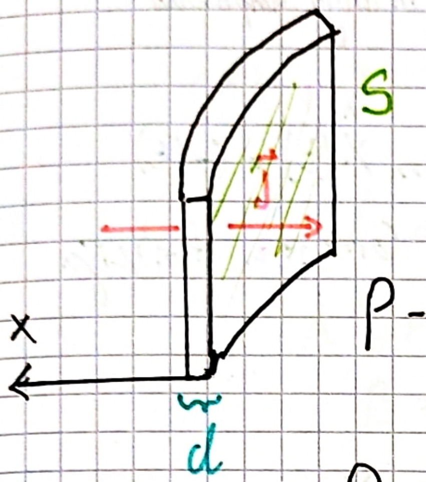
Dif. en. 1. reda za Senzor 1. reda

Je optimalni indikator

(menda za stalanje konstante)

Primer: [Termometer]





$$P - Sj = \frac{\lambda S(T_z - T)}{d}$$

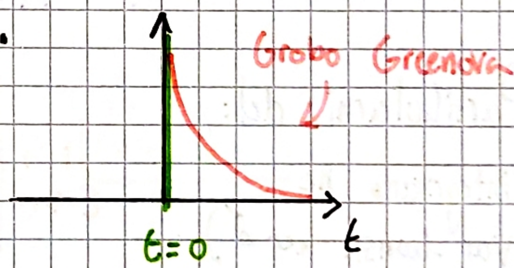
$$P = \frac{dQ}{dt} - mc_p \frac{dT}{dt}$$

$$\frac{d\,mc_p}{\lambda S}\frac{dT}{dt} = -\frac{\lambda S}{d}(T_z - T)$$

$$\Rightarrow \quad \underline{K\dot{T} + T = T_z(t)}$$

Enačba senzorja 1. reda!

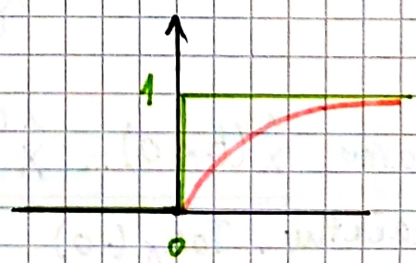
Zanima nas obnašanje senzorja 1. reda, ko $Z(t) \neq$ konst (in sistemske napake, prehodna obdobja...).



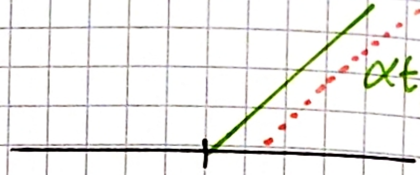
Tipični vhodi $Z(t)$: i) $Z(t) = \delta(t)$

Greenova funkcija nam pove vse od odziva senzorja in tipu ipd.

ii) $Z(t) = H_0(t)$



iii) $Z(t) = t^n$



iv) $Z(t) = \cos \omega t$ harmoničen ~~…~~ vhod

Pričakujemo isto frekvenco in fazni zamik in drugo amplitudo

1. red

$$\tau \dot{\hat{x}} + \hat{x} = Z(t) \cancel{= \delta(t)}$$

,, $\hat{X}(t) = G(t)$ Greenova funkcija

i) $Z(t) = \delta(t)$

Homogeni del: $\tau \dot{\hat{x}} + \hat{x} = 0 \qquad \tau \cdot \frac{d\hat{x}}{dt} = -x$

$$\hat{x} = Ce^{-\lambda t}$$

$$\tau(-\lambda) C e^{-\lambda t} + Ce^{-\lambda t} = 0$$

$$(-\lambda \tau + 1) e^{-\lambda t} = 0$$

$$\Rightarrow \lambda = 1/\tau$$

Homogeni del:

$$\hat{X}_H = C e^{-t/\tau}$$

Partikularni del: (integriramo ne var. konst. ker δ)

$$\lim_{\varepsilon \to 0} \left[\underbrace{\int_{-\varepsilon}^{\varepsilon} \tau \frac{d\hat{x}}{dt} dt}_{\downarrow} + \underbrace{\int_{-\varepsilon}^{\varepsilon} \hat{x}\, dt}_{\to 0} = \underbrace{\int_{-\varepsilon}^{\varepsilon} \delta(t) dt}_{1} \right]$$

$$\tau [\underset{\downarrow\, +0}{\hat{x}(\varepsilon)} - \underset{\downarrow\, -0}{\hat{x}(-\tau)}] = 1$$

Zahtevamo $\hat{x}(t<0) \ldots \hat{x}^{(n)}(t<0) = 0$, da je "števec" pri miru na začetku. Torej (-0) zanemarimo

$$\Rightarrow \hat{X}_p(0) = 1/\tau$$