iii) $Z(t) = \alpha t$

homogeni del: Je isti seveda $\hat{X}_H(t) = Ce^{-t/\tau}$

Partikularni del: $\tau \dot{\hat{X}} + \hat{X} = \alpha t$ Nastavek: $\hat{X}_p = At + B$

$$\tau A + At + B = \alpha t$$

$$A = \alpha$$

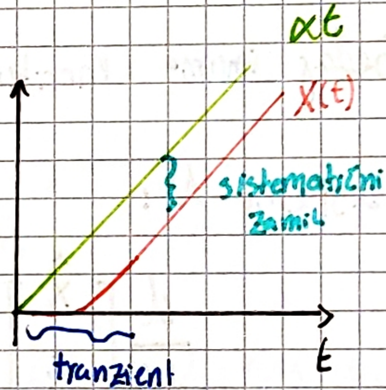
$$A\tau + B = 0$$

$$B = -\alpha\tau \Rightarrow X_p = \alpha t - \alpha\tau$$

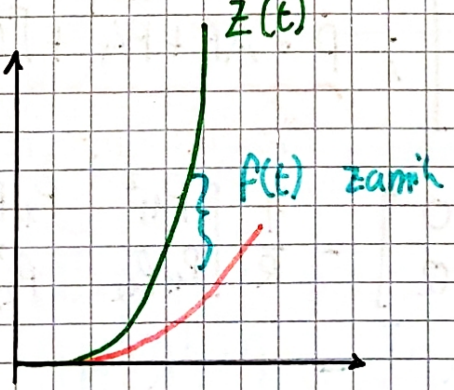
$$\Rightarrow X(t) = \alpha t - \alpha\tau + Ce^{-t/\tau}$$

Zahtevamo $X(0) = 0 \Rightarrow C = \alpha\tau$

$$\Rightarrow X(t) = \alpha(t - \tau) + \alpha\tau e^{-t/\tau}$$



Kaj če bi s tem senzorjem sledili $Z(t) = \beta t^2$



$\Rightarrow$ Ne sledi

Senzor 2 reda

V sistemu S:

Optimalen za $\frac{d^2x}{dt^2} = 0 + W$

Moramo prepisati v sistem linearnih enačb:

$$\begin{cases} \dot{x} = v \\ \dot{v} = 0 + W \end{cases} \qquad \langle W^2 \rangle = Q$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}\begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} W$$

$$\dot{x} = Ax + \Gamma x$$

Sedaj imamo meritev z samo prve komponente $\Rightarrow H = [1 \ 0]$

V sistemu M:

$$\frac{d}{dt}\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} = A\begin{bmatrix} \hat{x} \\ \hat{v} \end{bmatrix} + PH^TR^{-1}(z - H\hat{x})$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} ; \quad \dot{P} = AP + PA^T + \Gamma Q \Gamma^T - PH^TR^{-1}HP$$

To moramo rešiti

Torej:

$$\dot{P} = \begin{bmatrix} 2P_{12} & P_{22} \\ P_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} - \frac{1}{R}\begin{bmatrix} P_{11}^2 & P_{11}P_{12} \\ P_{11}P_{12} & P_{22}^2 \end{bmatrix}$$

$$\dot{P} = AP + PA^T + \Gamma Q \Gamma^T - PH^TR^{-1}HP$$

i) Stacionarne rešitve; $\dot{P} = 0$

$$\Rightarrow \quad 2P_{12} - \frac{1}{R}P_{11}^2 = 0 \Rightarrow P_{11}^2 = 2R\sqrt{QR}$$

$$P_{22} - \frac{1}{R}P_{11}P_{12} = 0 \Rightarrow P_{22} = \sqrt{2Q}\sqrt{\sqrt{QR}}$$

$$Q - \frac{1}{R}P_{12} = 0 \Rightarrow P_{12} = \sqrt{QR}$$

Poglejmo si Ojačevalne faktorje

$$K_\infty = P_\infty H^T R^{-1}$$

$$K_\infty = \frac{1}{R}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22}\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix} - \frac{1}{R}\begin{bmatrix}P_{11} \\ P_{12}\end{bmatrix}$$

$$\Rightarrow \quad \dot{\hat{x}} = \hat{v} + \frac{1}{R}P_{11}(z - \hat{x})$$

$$\dot{\hat{v}} = \frac{1}{R}P_{12}(z - \hat{x})$$

Spravimo to nazaj na enačbo samo ene spremenljivke

$$\ddot{\hat{x}} = \frac{1}{R}P_{12}(z - \hat{x}) + \frac{1}{R}P_{11}(\dot{z} - \dot{\hat{x}})$$

$$\Rightarrow \quad \ddot{\hat{x}} + \frac{P_{11}}{R}\dot{\hat{x}} + \frac{P_{12}}{R}\hat{x} = \frac{P_{11}}{R}\dot{z} + \frac{P_{12}}{R}z$$

Oz. v standardni obliki:

Dif. en. 2. reda za senzor 2. reda

Dušilni člen, ξ dušilni koef.

$$\boxed{\ddot{\hat{x}} + 2\xi\omega\dot{\hat{x}} + (\omega^2)\hat{x} = 2\xi\omega\dot{z} + \omega^2 z}$$

ξ je črka Zeta

$$\omega^2 = \frac{P_{12}}{R} = \frac{\sqrt{QR}}{R} = \sqrt{\frac{Q}{R}}$$

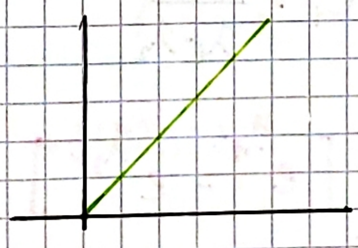
$$2\xi\sqrt{\frac{P_{12}}{R}} = \sqrt{2}\sqrt{\sqrt{QR}}$$

$$\xi = \frac{1}{\sqrt{2}}$$

Torej je dušilni koeficient $\xi = \frac{1}{\sqrt{2}}$ optimalen za Kalmanov filter II. reda

Sledilnik = Filter = Senzor

Ta je optimalen za sledenje:



$$\frac{d^2x}{dt^2} = 0$$

$$\hat{x} = x$$

Samo za hitrejše oznake če bo prof. pozabil

Tipični Vhodi $z(t)$

i) $z(t) = \delta(t)$ iščemo pa $x(t) = ? = G(t)$ Greenova Funkcija

Za RP:

$$\int_{-\varepsilon}^{\varepsilon} \ddot{x}\,dt + 2\xi\omega\int_{-\varepsilon}^{\varepsilon} \dot{x}\,dt + \omega^2\int_{-\varepsilon}^{\varepsilon} x\,dt = \omega^2\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = \omega^2$$

$$\lim_{\varepsilon\to 0}\ [\underbrace{\dot{x}(\varepsilon) - \dot{x}(-\varepsilon)}_{0}] + 2\xi\omega[x(\varepsilon) - \underbrace{x(-\varepsilon)}_{0}] + 0 = \omega^2$$

števec pri miru na začetku

$\Rightarrow 0$

$$\Rightarrow \dot{x}(0) + 2\xi\omega x(0) = \omega^2 \quad \Rightarrow \quad x(0) = 0, \quad \dot{x}(0) = \omega^2$$

Homogeni del: $x = e^{\lambda t}$ za karakteristični polinom

$$\lambda^2 + 2\xi\omega\lambda + \omega^2 = 0$$

$$\lambda_{1,2} = -\xi\omega \pm \frac{\sqrt{4\xi^2\omega^2 - 4\omega^2}}{2}$$

$$\lambda_{1,2} = -\omega\left[\xi \mp \sqrt{\xi^2 - 1}\right]$$

Splošna rešitev: $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$$x(0) = 0 = C_1 + C_2 \Rightarrow C_2 = -C_1$$

$$\dot{x}(0) = C_1\lambda_1 + C_2\lambda_2 = \omega^2 \Rightarrow C_1(\lambda_1 - \lambda_2) = \omega^2$$

$$C_1 = \frac{\omega^2}{\lambda_1 - \lambda_2}$$

$$\Rightarrow \quad x(t) = \frac{\omega^2}{\lambda_1 - \lambda_2}\left[e^{\lambda_1 t} - e^{\lambda_2 t}\right]$$

npr. Da je filter optimalen rabimo: $\xi = \frac{1}{\sqrt{2}}$

$$X(t) = -\omega\left[\xi \mp i\sqrt{1-\xi^2}\right]$$

$$\lambda_1 - \lambda_2 = $$

$$= 2\omega i\sqrt{1-\xi^2} = 2\omega i \frac{1}{\sqrt{2}}$$

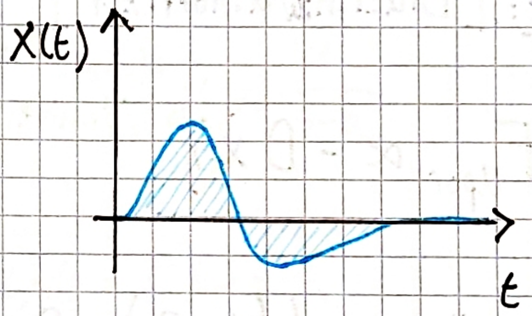
Torej:

$$X(t) = \frac{\omega^2\sqrt{2}}{2\omega i}\left[e^{i\frac{\omega}{\sqrt{2}}t} - e^{-i\frac{\omega}{\sqrt{2}}t}\right]e^{-\frac{\omega}{\sqrt{2}}t}$$

$$X(t) = \sqrt{2}\,\omega \sin\left(\frac{\omega}{\sqrt{2}}t\right)e^{-\frac{\omega}{\sqrt{2}}t}$$

Greenova funkcija

Za senzor 2. reda



Kaj pa če ξ zavzame neoptimalne vrednosti?

$$\lambda_{1,2} = -\omega\left[\xi \mp i\sqrt{1-\xi^2}\right]$$

$$\lambda_1 - \lambda_2 = 2\omega\sqrt{\xi^2-1}$$

$$X(t) = \frac{\omega^2}{\lambda_1 - \lambda_2}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) =$$

$$= \frac{\omega^2}{2\omega\sqrt{\xi^2-1}}\left[e^{\omega\sqrt{\xi^2-1}t} - e^{-\omega\sqrt{\xi^2-1}t}\right]e^{-\xi\omega t}$$

$\boxed{\xi = 0}$ brez dušenja

$$= \frac{\omega^2}{2i\sqrt{1-\xi^2}}\sin\left(\frac{\omega}{\sqrt{1-\xi^2}}t\right)e^{-\xi\omega t}$$

$$= \omega \sin\omega t$$

$\boxed{\xi \gg 1}$ predušen sistem

$$\frac{e^{x}-e^{-x}}{2} = \mathrm{Sh}\,x$$

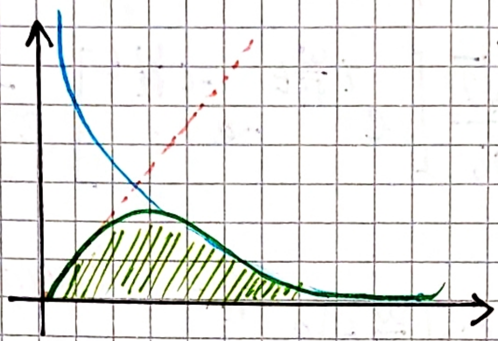
$$e^{x} = 1 + x + \frac{x^2}{2} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \dots$$

$$e^{x} - e^{-x} = 2x + \dots$$

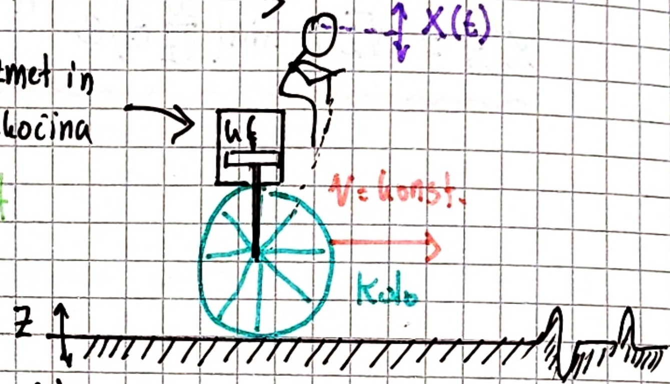
$$x(t) = \frac{\omega}{2\sqrt{\xi^2-1}}\left[2\omega\sqrt{\xi^2-1}\,t\right]e^{-\xi\omega t}$$

$$= \omega^2 t e^{-\xi\omega t}$$



Primer: [Blazilnik/amortizer]

Vzmet in tekočina

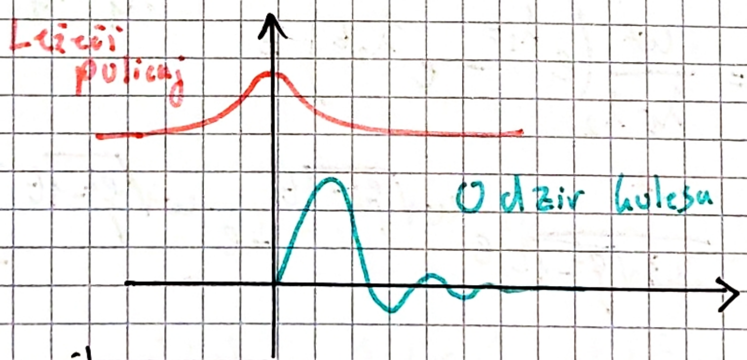


$$F_{upor} \propto -D\dot{x}\eta$$ Viskoznost

$$\sum F = m\ddot{x} = -k(x-z) - D\eta(\dot{x}-\dot{z})$$

$$\Rightarrow \quad \ddot{x} + \frac{D\eta}{m}\dot{x} + \frac{k}{m}x = \frac{D\eta}{m}\dot{z} + \frac{k}{m}z$$

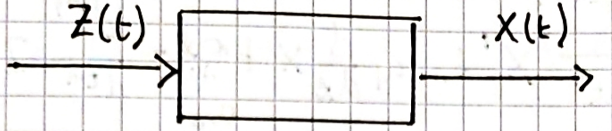
Torej je amortizer filter 2. reda.



Analiziramo člene:

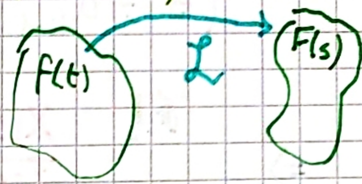
$$2\xi\omega = 2\xi\sqrt{\frac{k}{m}} = \frac{D\eta}{m} \Rightarrow \sqrt{2km} = D\eta$$

Prenosna funkcija senzorja



Definiramo Laplaceovo transformacijo

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad ; \; s \in \mathbb{C}$$



i) $\mathcal{L}(1) = ?$

$$\mathcal{L}(1) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st} = \frac{1}{s}$$

ii) $\mathcal{L}(e^{at}) = ?$

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}$$

iii) $\mathcal{L}(f(t)e^{at}) = ?$

$$\mathcal{L}(f(t)\,e^{at}) = \int e^{-(s-a)} f(t)\,dt = F(s-a)$$

iv) $\mathcal{L}\left(\frac{d}{dt} f(t)\right)$

Perpartes
$u = e^{-st}$
$du = -s e^{-st} dt$
$dv = \frac{\partial f}{\partial t} dt$
$v = f$

$$\mathcal{L}\left(\frac{d}{dt} f(t)\right) = \int \frac{d}{dt} f(t)\, e^{-st}\,dt =$$

$$= f e^{-st}\Big|_0^\infty + \int_0^\infty s e^{-st} f\,dt =$$

$$= \cancel{f(0)} + sF(s) = sF(s)$$

kazalec miruje <=> funkcija 0 na $t=0$

v) $\mathcal{L}(t) = ? \rightarrow 1/s^2$

vi) $\mathcal{L}(\delta) = 1$

vii) $\mathcal{L}(\cos \omega t), \mathcal{L}(\sin \omega t) = ?$

$$\cos \omega t + i \sin \omega t = e^{i\omega t}$$

$$\mathcal{L}(\cos t) + i\mathcal{L}(\sin \omega t) = \frac{1}{s - i\omega} = \frac{1}{s^2+\omega^2}(s + i\omega)$$

$$\Rightarrow \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2} \qquad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$$

n, m red diferencialne enačbe:

$$\frac{d^{(n)}}{dt^n} + a_{n-1}\frac{d^{(n-1)}}{dt^n}x' + \dots a_1\frac{d}{dt}x + a_0 x = \frac{d^{(m)}}{dt^m}z + \frac{d^{(m-1)}}{d^{(m-1)}}b_{m-1}z + \dots + b_0 z$$

Dajmo to transformirati z Laplaceovo ~~...~~ transformacijo:

$$(s^n + a_{n-1}s^{n-1} + \dots + a_0)X(s) = (s^m + b_{m-1}s^{m-1} + \dots + b_0)Z(s)$$

$$\Rightarrow \quad \frac{X(s)}{Z(s)} = H(s) = \frac{s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$$

Prenosna funkcija senzorja

Primer: [1. red]

$$\tau\dot{x} + x = z(t) \quad /\mathcal{L}$$

$$(\tau s + 1)X(s) = Z(s)$$

$$H(s) = \frac{X(t)}{Z(s)} = \frac{1}{1+\tau s}$$

Prenosna funkcija za 1. red

Primer: [2. red]

$$(s^2 + 2\xi\omega s + \omega^2)X(s) = \omega^2 Z(s)$$

$$H(s) = \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}$$

Prenosna funkcija za 2. red

Primer: [že znan problem]

$$Z(s) \cdot H(s) = X(s)$$

i) $z = \delta(t)$

$$H(s) = \frac{1}{1+\tau s}$$

$$\frac{1}{(1/\tau + 1)}\frac{1}{\tau} = X(s)$$

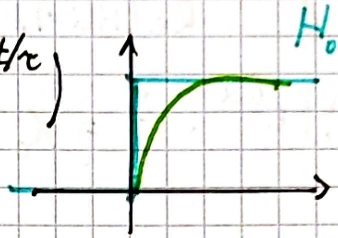
$$\Rightarrow \quad \frac{1}{\tau}e^{-t/\tau} = x(t)$$

(ii) $Z(t) = H_0(t) = \begin{cases} Z_0 ; & t>0 \\ 0 ; & \text{sicer} \end{cases}$

$$Z(s) = \frac{1}{s} Z_0 = \frac{1}{s}\frac{1}{1+\tau s} Z_0 = \qquad A + A\tau s + Bs = 1$$

$$= \frac{A}{2} + \frac{B}{1-\tau s} \qquad \Rightarrow A = 1 \quad B = 0$$

$$X(s) = Z_0\left[\frac{1}{s} - \frac{\tau}{1+\tau s}\right] = Z_0\left(1 - e^{-t/\tau}\right)$$



<u>Primer</u> [Senzor 2. reda]

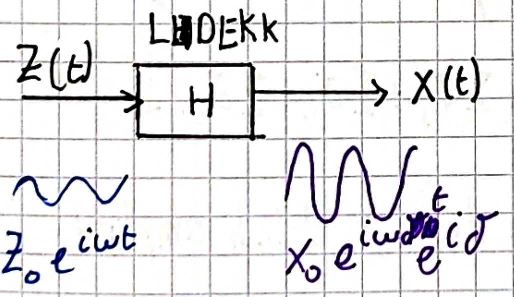
$Z(t) = \delta(t)$

$$Z(s)H(s) = \frac{\omega^2}{s^2+2\xi s\omega+\omega^2} = \frac{\omega^2}{(s+\xi\omega)^2+\omega^2(1-\xi^2)} =$$

$$= \frac{\omega\sqrt{1-\xi^2}\,\omega}{(s+\xi\omega)^2+\omega^2(1-\xi^2)}\left(\frac{1}{\sqrt{1-\xi^2}}\right) = X(t)$$

$$X(t) = \frac{\omega}{\sqrt{1-\xi^2}}\sin\left(\sqrt{1-\xi^2}\,\omega t\right) e^{-\xi\omega t}$$

Odziv senzorja na periodične signale (Bodejevi diagrami)



$$\frac{d}{dt}\left(e^{i\omega t}\right) = i\omega e^{i\omega t} \;/\; \mathcal{L}$$

$$\lambda\, \cancel{\mathcal{L}(e^{i\omega t})} = i\omega\, \cancel{\mathcal{L}(e^{i\omega t})}$$

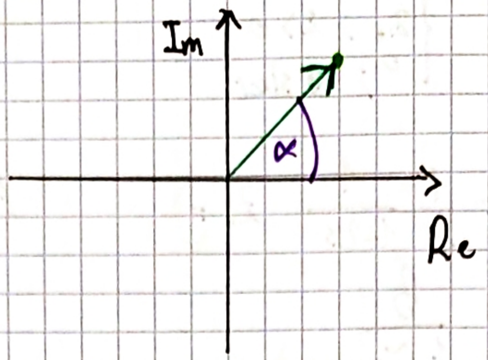
$\Rightarrow$ $s = \lambda = i\omega$; $H(s) = H(i\omega)$

Za periodične signale

$$\Rightarrow H(i\omega)\,Z(i\omega) = X(i\omega)$$

$$H(i\omega)\cdot Z\,\mathcal{L}(e^{i\omega t}) = X\,\mathcal{L}(e^{i\omega t})\,e^{i\delta} \;/\; |\;| \quad \Rightarrow |H(i\omega)| = \frac{X_0}{Z_0}$$

$H(i\omega) \in \mathbb{C}$



$$H(i\omega) = |H(i\omega)| e^{i\alpha}$$

$$tg\,\alpha = \frac{Im(H(i\omega))}{Re(H(i\omega))}$$

$$|H(i\omega)| e^{i\alpha} \cdot Z = x_0 e^{i\delta} \Rightarrow e^{i\delta} = e^{i\alpha} \quad tg\,\delta(i\omega) = \frac{Im\,H(i\omega)}{Re\,H(i\omega)}$$

V splošnem lahko kakršenkoli $H(i\omega)$ zapišemo kot kombinacijo prenosnih funkcij 1. in 2. reda

$$H(i\omega) = \frac{\prod_i (1 + i\omega \tau_i) \cdot \prod_j \left(\left(\frac{i\omega}{\omega_j}\right)^2 + \frac{2\xi_j i\omega}{\omega_j} + 1 \right)}{\prod_k (1 + i\omega \tau_k) \cdot \prod_l \left(\left(\frac{i\omega}{\omega_l}\right)^2 + \frac{2\xi_l i\omega}{\omega_l} + 1 \right)}$$

Lahko pišemo kot:

$$H(i\omega) = \frac{\prod_i (|1 + i\omega \tau_i|) \cdot \prod_j \left(\left| \left(\frac{i\omega}{\omega_j}\right)^2 + \frac{2\xi_j i\omega}{\omega_j} + 1 \right| \right) e^{i\delta_i} e^{i\delta_j}}{\prod_k (|1 + i\omega \tau_k|) \cdot \prod_l \left(\left| \left(\frac{i\omega}{\omega_l}\right)^2 + \frac{2\xi_l i\omega}{\omega_l} + 1 \right| \right) e^{i\delta_k} e^{i\delta_l}}$$

Fazni faktor je torej: $e^{i\left[\sum_i \delta_i + \sum_j \delta_j + \sum \delta_k + \sum \delta_l\right]}$

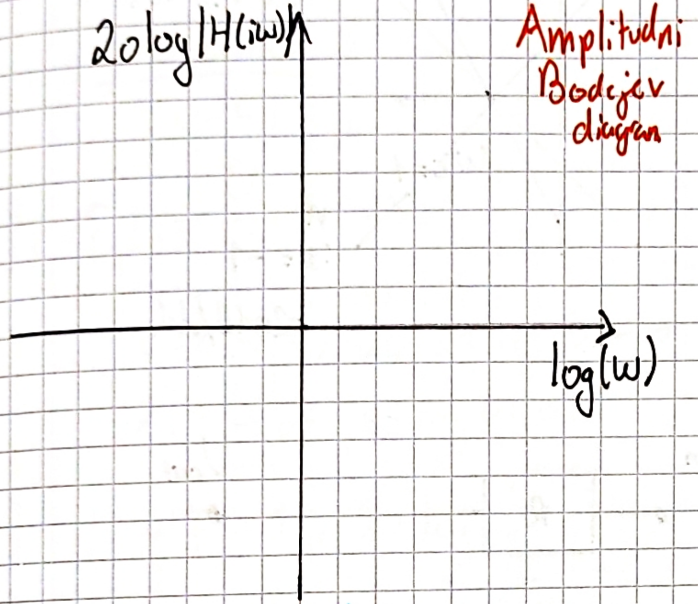
Razmerje amplitud: množenje in deljenje delnih amplitud

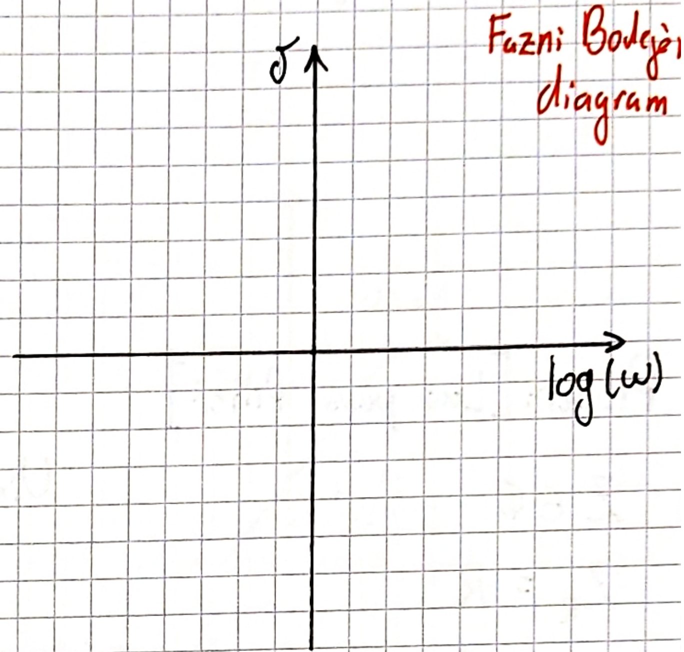
Fazni premik: seštevanje in odštevanje faznih zamikov

Definiramo:

$$\underbrace{20\log}_{dB}|H(i\omega)| = 10\log|H(i\omega)|^2$$

Decibel je definiran na moč, ne amplitude



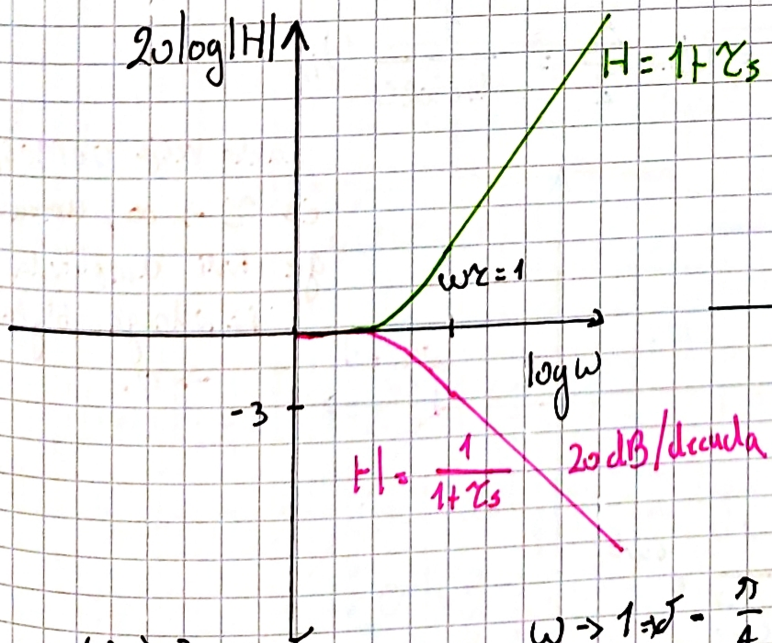


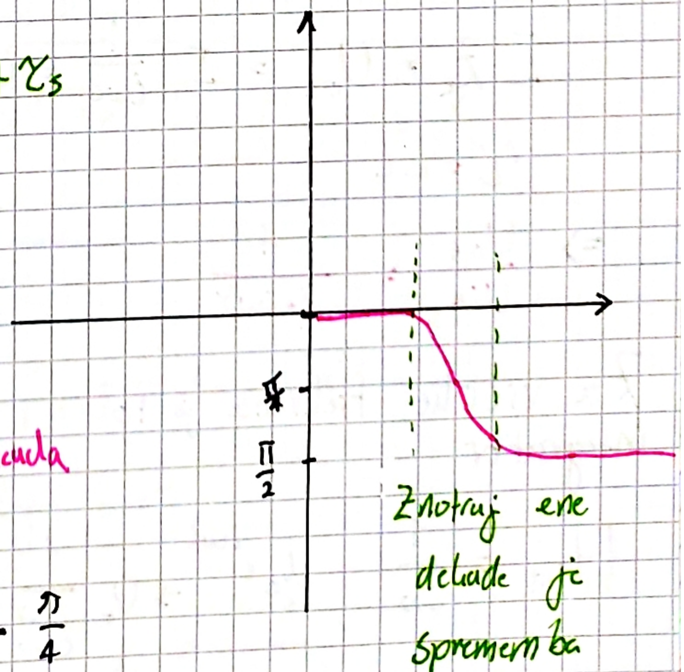
Za sistem 1. reda:

$$H(\underset{i\omega}{\overset{\uparrow}{s}}) = \frac{1}{1+i\omega\tau} \qquad |H(i\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$\omega \to 0 \quad 20\log|H| \Rightarrow 0$ $\qquad \omega\tau = 1 \Rightarrow 10\log\frac{1}{\sqrt{2}} = -3$

$\omega \to \infty \quad 20\log\frac{1}{\omega\tau} = -2\log\omega\tau$

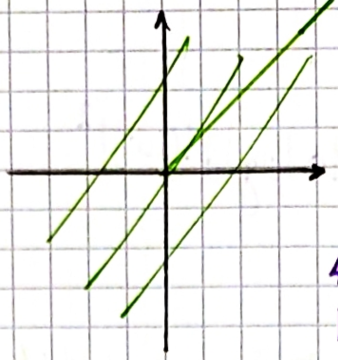




$\omega \to 0 \Rightarrow \varphi$ $\qquad \omega \to 1 \Rightarrow \varphi = -\frac{\pi}{4}$

$\omega \to \infty \Rightarrow \varphi = -\frac{\pi}{2}$;

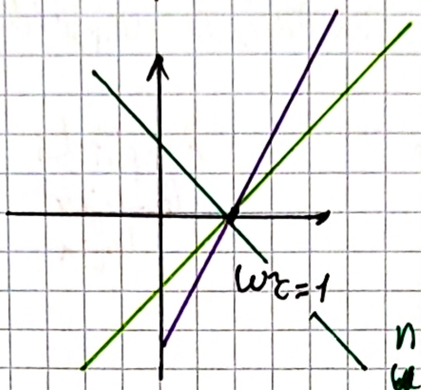
$H(i\omega) = H\, \tau_s$



- $H(i\omega) = (i\omega)^n; \quad n = 1,2,3$

$= \omega^n e^{in\frac{\pi}{2}}$

$20\log|H| = 20 n \log \omega$



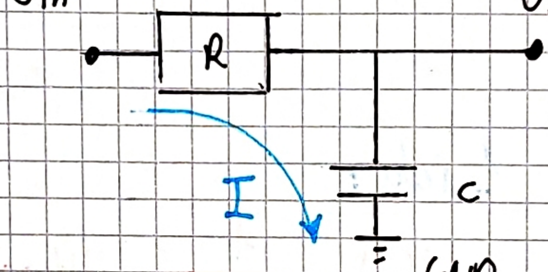
Primer: [Low pass filter]

$Z \in \mathbb{C}$

$Z_R = R$

$Z_C = \frac{1}{i\omega C}$

$Z_L = i\omega L$

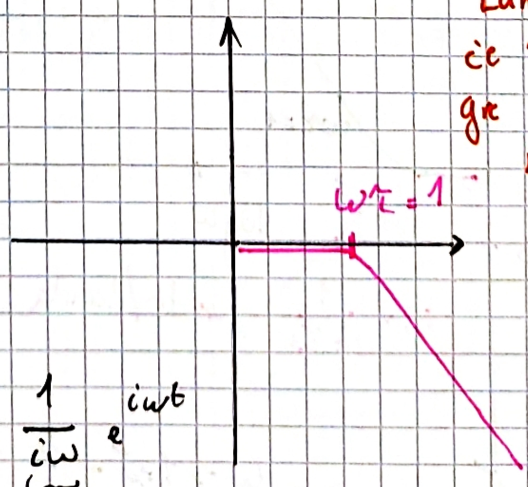


$$Z = Z + \frac{1}{i\omega C} = \frac{1 + i\omega RC}{i\omega C} \quad ; \quad \frac{U_{in}}{Z} = I$$

$$I Z_C = U_{out} = I \frac{1}{i\omega C} = U_{in} \frac{Z_C}{Z} = \frac{1}{1 + i\omega RC} U_{in}$$

$\tau = RC$

$$\Rightarrow \frac{U_{out}}{U_{in}} = \frac{1}{1 + i\omega\tau}$$



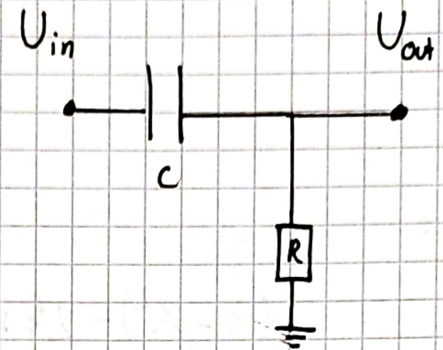
Lahko večje območje če $\tau \to \infty$, ampak gre tudi amplituda izhodnega signala $\to 0$.

Za visoke frekvence je to integrator

$$H_{int} = \frac{1}{i\omega} \qquad u_0 e^{i\omega t} \to U_0 \underbrace{\frac{1}{i\omega}}_{H_{int}} e^{i\omega t}$$

Torej za $\omega\tau \gg 1 = \frac{1}{i\omega\tau}$

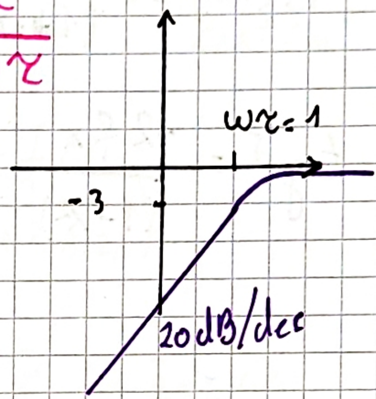
Primer: [High pass filter]



Z enak kot prej saj so iste komponente v vezju

$$Z = \frac{1+i\omega RC}{i\omega C}$$

$$U_{out} = RI = \frac{U_{in}}{Z}R \Rightarrow \frac{U_{out}}{U_{in}} = H = \frac{i\omega\tau}{1+i\omega\tau}$$



$\omega\tau \to 0,\ 20\log|H| \to -\infty$

$\omega\tau \to \infty,\ 20\log|H| \to 0$

$\omega\tau = 1,\ 20\log\frac{1}{\sqrt{2}} = -3$

Ta služi pa kot diferenciator v delu kjer duši

$$H = i\omega\tau = H_{Dif}\,\tau$$

Lahko večje frekvenčno območje če $\tau \to 0$, ampak gre tudi amplituda izhodnega signala proti 0.

za nižje

II. red:

$$H(s) = \frac{1}{s^2/\omega_0^2 + 2\xi/\omega_0 + 1} = \frac{1}{-\frac{\omega^2}{\omega_0^2} + \frac{2\xi\omega}{\omega_0} + 1} \qquad \left(\frac{\omega}{\omega_0}\right) = x$$

$$|H(i\omega)| = \frac{1}{\sqrt{\left(1-\omega^2/\omega_0^2\right)^2 + \left(\frac{2\xi\omega}{\omega_0}\right)^2}}$$

Torej je:

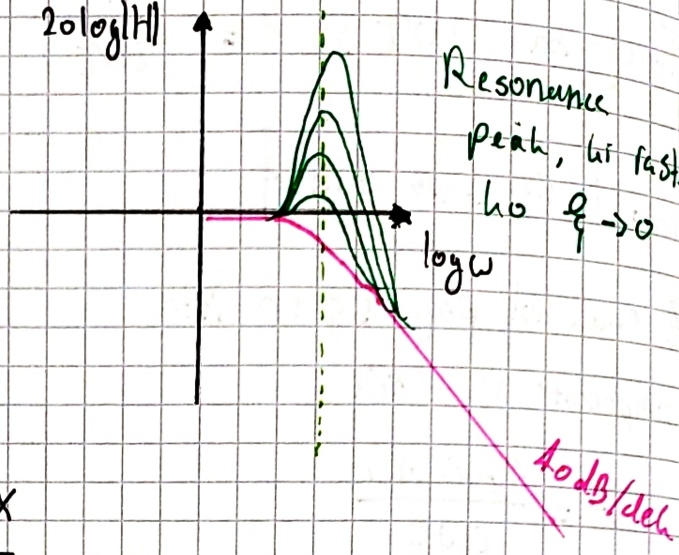
$$|H(i\omega)| = \left[(1-x^2)^2 + (2\xi x)^2\right]^{-1/2} = \left[1-2x^2+x^4+4\xi^2x^2\right]^{-1/2}$$

$x \to 0$; $20\log|H| = 20\log 1 = 0$

$x \gg 1$; $20\log\frac{1}{x^2} = -40\log x$

$x = 1$; $20\log\frac{1}{2\xi} = -3$; če $\xi = \frac{1}{\sqrt{2}}$

Kaj pa če ξ ni $\frac{1}{\sqrt{2}}$?



$$tg\delta = \frac{\mathrm{Im}H}{\mathrm{Re}H} =$$

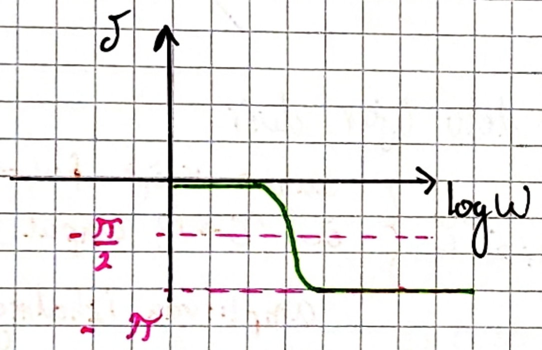
$$H(i\omega) = \frac{(1-x^2) - 2\xi x}{(1-x^2) + 4\xi^2 x^2}$$

$$\Rightarrow \quad tg\delta = \frac{-2\xi x}{(1-x^2)}$$

$\omega \to 0, \ \delta \to 0$

$\omega \to \infty \quad \delta \to 2\xi \frac{1}{x} \to -\pi$

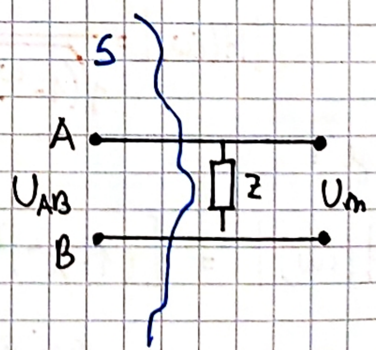
$\omega \to \omega_0 \quad \delta \to -\frac{\pi}{2}$
$(x \to 1)$



Vpliv senzorja na opazovani sistem

Iščemo prvo merilo, koliko senzor zmoti sistem S!

$$U_{AB} \neq U_m$$
$$? \ \ !?$$

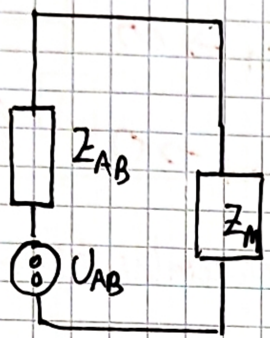


Senzor črpa iz sistema S: $P_m = \frac{U_m^2}{Z}$.

Merilo je torej moč (energija na enoto merjenja), ki jo črpamo iz S za delovanje senzorja.

Thevenin ov izrek: Vsako električno vezje, kar sestavljeno iz linearnih elementov (RCL) v poljubnih točka A in B lahko nadomestimo z generatorjem U_{AB} in notranjo upornostjo U_{AB}

Nadomestno vezje torej zgleda kot:



$$Z = Z_{AB} + Z_m$$

$$I = \frac{U_{AB}}{Z} = \frac{U_{AB}}{Z_{AB}+Z_m}$$

Sorry Future Marko :(

$$U_m = I Z_m = U_{AB}\left(\frac{Z_m}{Z_{AB}+Z_m}\right) = U_{AB}\left(\frac{1}{1+Z_{AB}/Z_m}\right)$$

Torej za $U_m < U_{AB}$; $\frac{Z_{AB}}{Z_m} \to 0 \Rightarrow Z_m \gg Z_{AB}$ če se hočemo približati pravi vrednosti

$$P_m = \frac{U_m^2}{Z_m} = \frac{U_{AB}^2 Z_m^2}{Z_m (Z_m + Z_{AB})^2} = U_{AB}^2 \frac{Z_m}{(Z_m+Z_{AB})^2}$$

Poglejmo kje je P_{max}:

$$\frac{dP_m}{dZ_m} = 0 = U_{AB}^2\left[\frac{(Z_m+Z_{AB})}{(Z_m+Z_{AB})^3} - \frac{2Z_m}{(Z_m+Z_{AB})^2}\right] \Rightarrow Z_{AB} = Z_A$$

Takrat največ moči porabljamo.

$$P_{max} = U_{AB}^2 \frac{Z_{AB}}{4Z_{AB}^2} = \frac{U_{AB}^2}{4Z_{AB}}$$

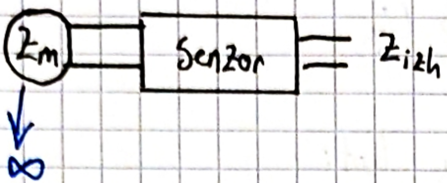
Želimo $P \ll P_{max}$.

$$P_m = \frac{U_{AB}^2 Z_m 4Z_{AB}}{4Z_{AB}(Z_m+Z_{AB})^2} = P_{max}\frac{4\, Z_{AB}/Z_m}{\left(\frac{Z_m}{Z_{AB}} + \frac{Z_{AB}}{Z_m}\right)^2} = P_{max}\, 4\,\frac{Z_{AB}}{Z_m}\left(\frac{1}{(1+\frac{Z_{AB}}{Z_m})^2}\right)$$

$$\approx \left(\frac{P}{P_{max}}\right) = 4\left(\frac{Z_{AB}}{Z_m}\right)$$

Torej $P \ll P_{max} \Leftrightarrow Z_m \gg Z_{AB}$

To merilo lahko posplošimo:

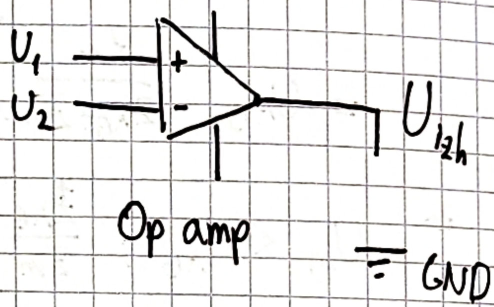


Pogoj: Vhodna impedanca $Z_{in} \to \infty$

Izhodna impedanca $Z_{out} \to 0$

Instrumentacijski ojačevalnik (diferencialni Preamp)

$A_{DC} \sim 10^6$

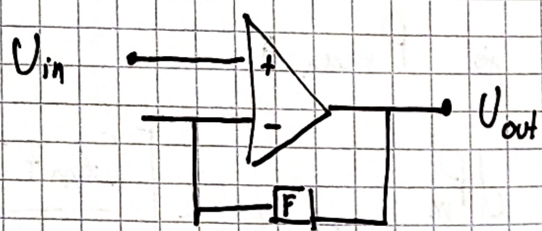


$$U_{izh} = A_{DC}(U_1 - U_2)$$

$$H = A_{DC} H_{LPF}$$ } v grobem

Instrumentacijski ojačevalnik je sestavljen iz treh opampov.

Negativna povratna zanka



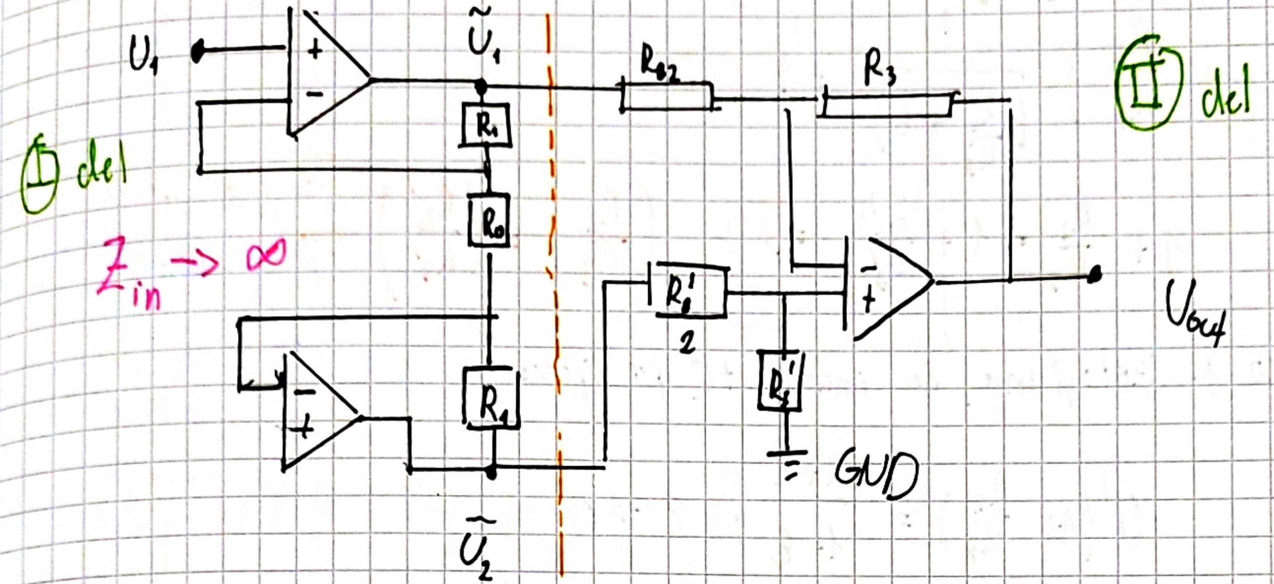
$$A(U_{in} - FU_{out}) = U_{out}$$

$$AU_{in} = AFU_{out} + U_{out}$$

$$\frac{U_{out}}{U_{in}} = \frac{A}{Af + 1} = \frac{1}{F + 1/A} = \frac{1}{F}$$ če $F = 1$ $H = \frac{U_{out}}{U_{in}} = 1$

V ~~praktičnem delu~~ To nam sledi vhodu, brez da bi od signala črpal moč.
Ima tudi $Z_{in} \to \infty$.

Torej instrumentacijski ojačevalnik



I.)

$$\frac{\tilde{U}_1 - U_1}{R_1} = \frac{U_2 - \tilde{U}_2}{R_1} = \frac{U_1 - U_2}{R_0}$$

Vsota se ohranja:

$$\tilde{U}_1 + \tilde{U}_2 = U_2 + U_1$$

$$\frac{(\tilde{U}_1 - \tilde{U}_2)(U_2 - U_1)}{\cancel{R_1}} = \frac{2(U_1 - U_2)}{R_0} R_1$$

$$(\tilde{U}_1 - \tilde{U}_2) = \left(2\frac{R_1}{R_0} + 1\right)(U_1 - U_2)$$

II.)

$$\Rightarrow \Delta\tilde{u} = \left(2\frac{R_1}{R_0} + 1\right)\Delta u$$

$$\frac{(\tilde{U}_2 - U)}{R_2'} = \frac{U}{R_3'} \to \frac{\tilde{U}_2}{R_2'} = U\left(\frac{1}{R_3'} + \frac{1}{R_2'}\right) = U\frac{R_2' + R_3'}{R_3' R_2'}$$

$$\Rightarrow \frac{R_3'}{R_2' + R_3'}\tilde{U}_2 = U$$

$$\frac{R_3}{R_2}\tilde{U}_1 - \frac{R_3}{R_2}U = U - U_{out} \to \cancel{-\frac{R_3}{R_2}}\tilde{U}_1$$

Lahko 1 ?

$$\Rightarrow U_{out} = -\frac{R_3}{R_2}\left[\tilde{U}_1 - \left(1 + \frac{R_2}{R_3}\right) k' \tilde{U}_2\right]$$

$$\left(1+\frac{R_2}{R_3}\right)\frac{R_1'}{R_2'+R_3'}=1$$

$$\left(1+\frac{R_2}{R_3}\right)\frac{1}{\left(\frac{R_2'}{R_3'}+1\right)}=1$$

$$1+\frac{R_2}{R_3}=1+\frac{R_2'}{R_3'} \Rightarrow \left(\frac{R_2}{R_3}\right)=\left(\frac{R_2'}{R_3'}\right)$$

To v praksi ni nujno korektno

Kolikšno je izstopanje, če imamo $+\varepsilon$ pri uporu:

$$u'\left(1+\frac{R_2}{R_3}\right)=\frac{1+\varepsilon}{1-\varepsilon};\quad \varepsilon\to 0$$

Takrat

$$U_{out}=-\frac{R_3}{R_2}\left[\frac{\tilde{U}_1}{1-\varepsilon}-\frac{1+\varepsilon}{1-\varepsilon}\tilde{U}_2\right]\frac{1-\varepsilon}{1-\varepsilon}$$

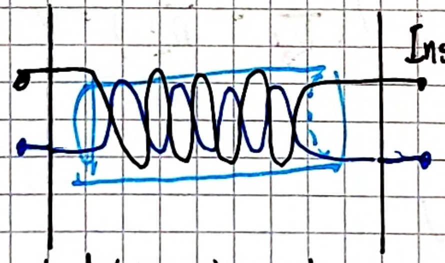
$$U_{out}=-\frac{R_3}{R_2(1-\varepsilon)}\left[\Delta\tilde{U}-\varepsilon(\tilde{U}_1+\tilde{U}_2)\right]$$

$\tilde{U}_1+\tilde{U}_2$ = common mode

CMRR (common mode rejection ratio)

$$CMRR = \frac{A(U_1-U_2)}{A(U_1+U_2)} = 10^6$$

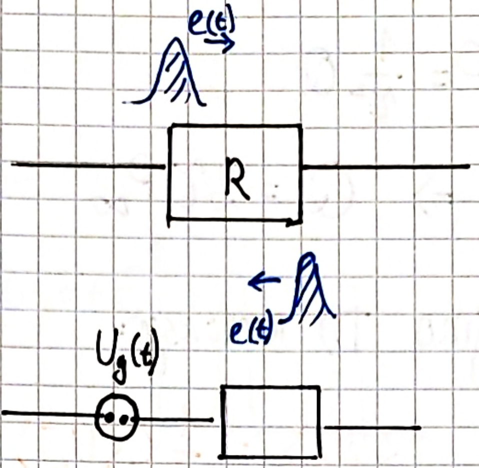
Za zelo dober senzor. Običajno podano v decibelih.



Inst. ojačevalnik

- Znebimo se induktivnih motenj
- Ojačevanje samo razliko med stranema
-

Termični šum na uporniku



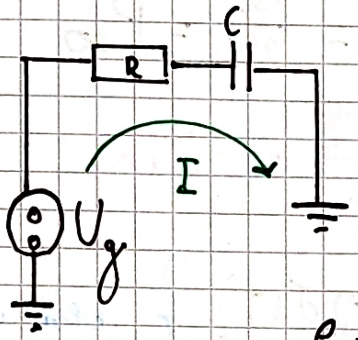
$$\frac{de}{dt} = I(t)$$

$$U_g(t) = IR$$

$$\langle U_g(t) \rangle = 0$$

$\langle U_g^2(t) \rangle \neq 0$ Kaj pa je potem?

Predstavljajmo si



$$U_g - IR - \frac{e}{C} = 0$$

$$e = C U_c$$

$$\frac{de}{dt} = C \dot{U}_c$$

$$RC = \tau$$

$$U_g - U_c = RC \dot{U}_c$$

$$\Rightarrow \dot{U}_c = -\frac{1}{\tau} U_c + \frac{U_g}{\tau}$$

Kalmanova dinamika za U_c, kjer je U_g dinamični šum:

$\hat{U}_c$; $\langle \hat{U}_c^2 \rangle = P$ kovarianca ; $\langle U_c \rangle = 0$

Poglejmo razvoj kovariance:

$$\dot{P} = 2AP + \Gamma Q \Gamma^T$$

$\dot{P} = -\frac{2}{\tau} P + \frac{1}{\tau^2} Q$, ko gre $t \to \infty$ $\dot{P} = 0$

$$\frac{2}{\tau}P = \frac{1}{\tau^2}Q$$

$$\Rightarrow \quad Q = 2\tau P_\infty$$

Sedaj poglejmo TD ravnovesje:

$$\langle W_c \rangle = \frac{1}{2}C\langle U_c^2 \rangle = \frac{1}{2}k_B T$$

samo ena prostostna stopnja (normalna smer na plošče kondenzatorja)

$$\Rightarrow \langle U_c^2 \rangle = \frac{kT}{c} = \frac{Q}{2\tau} = P_\infty$$

$$Q = \frac{2kT\tau}{c} = 2kTR$$

$$\langle U_g(t)U_g(t+\tau)\rangle = Q\underbrace{\delta(\tau)}_{\text{privzeto}} \Big\} \text{ beli šum (nekoreliran)}$$

Ocenimo koliko je ta šum:

$R = 1\,M\Omega$

$\tau = 1\,\mu s$

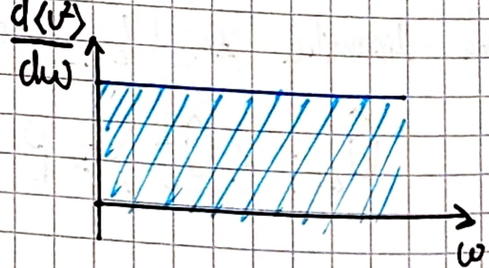
$T = 300K$

$$\sqrt{P} = \sqrt{\langle U_c^2 \rangle} = \sqrt{2 \cdot 1{,}38 \cdot 10^{-23} J/_K \; 300K \; \frac{1}{10^{-6}s} \; 10^6 \frac{V}{A}} =$$

$$= \sqrt{8 \cdot 10^{-9}\, V^2} = 9 \cdot 10^{-5} V \approx \boxed{10\mu V}$$

Spektralna gostota termičnega šuma

$$\langle U_g(t)U_g(t+\tau)\rangle = \text{konst.}\ \delta(\tau)$$



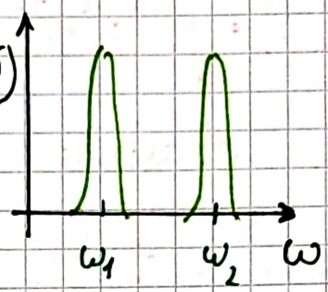
Če bi to veljalo bi moč, ki šumi bila neskončna:

$$\int_0^\infty \frac{d\langle U^2\rangle}{d\omega}\, d\omega \to \infty \Rightarrow \underset{\substack{\uparrow \\ \text{moč}}}{P} \to \infty \quad \cancel{\ }$$

$$P = \frac{\langle U^2\rangle}{R} \Rightarrow \frac{dP}{d\omega} = \frac{d\langle U^2\rangle}{R\,d\omega}$$

i) Posamezne frekvence obravnavamo neodvisno

$$U = U_0(\omega_1)\cos\omega_1 t + U_0(\omega_2)\cos(\omega_2 t + \delta)$$



$$\langle U^2\rangle = U_0^2(\omega_1)\cdot\frac{1}{2} + U_0^2(\omega_2)\frac{1}{2} +$$

$$+ \cancel{\langle U_0(\omega_1)U_0(\omega_2)\cos(\omega_1 t)\cos(\omega_2 t+\delta)\rangle}\ 0$$

⋮

$$\sum\langle U_i^2\rangle \rightarrow \int \frac{d\langle U^2\rangle}{d\omega}\,d\omega$$

Spektralna gostota šuma napetosti

Wiener Hinčinov izrek:

Spektralna gostota šuma = F.T. avtokorelacijske funkcije

Dokažimo na hitro ta izrek (smh):

$$C(\tau) = \int_{-\infty}^{\infty} U^*(t)\,U(t+\tau)\,dt$$

$$\text{F.T. } U(t) = \int U_\nu e^{-2\pi\nu i t}\,d\nu$$

$$U^*(t) = \int U^*_{\nu'} e^{+2\pi i\nu' t}\,d\nu'$$

ker $t+\tau$

$$\Rightarrow C(\tau) = \int_{-\infty}^{\infty}\int\int U^*_{\nu'} e^{2\pi i\nu' t}\,d\nu'\,U_\nu e^{-2\pi i t\nu}\,e^{-2\pi\nu i\tau}\,d\nu\,dt =$$

$$= \iint U_{\nu'}^{*} U_{\nu} \underbrace{\int e^{2\pi i(\nu'-\nu)t} dt}_{\delta(\nu-\nu')} e^{-2\pi i \nu' \tau} d\nu d\nu'$$

$$\Rightarrow \quad C(\tau) = \int |U_\nu|^2 e^{-2\pi i \nu \tau} d\nu$$

oz. obratno:

$$|U_\nu|^2 = \frac{1}{\pi} \int e^{-i\omega\tau} C(\tau) d\tau$$

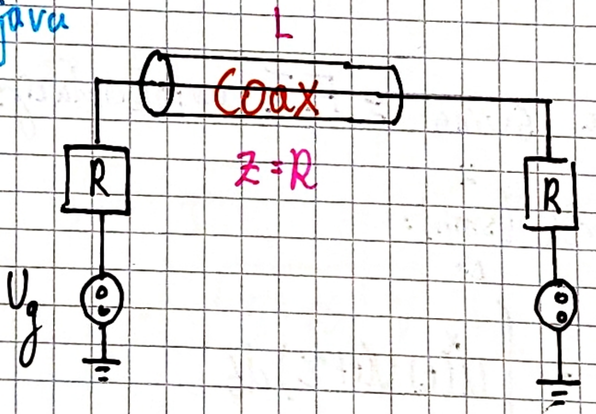
Za naš primer torej:

Nyquistova izpeljava

$$\frac{d\langle U^2\rangle}{d\omega} = \frac{1}{\pi} \int \underbrace{\langle U(t)U(t+\tau)\rangle}_{2kTR\,\delta(\tau)} e^{-i\omega\tau} d\tau$$

$$\Rightarrow \quad \frac{d\langle U^2\rangle}{d\omega} = \frac{2kTR}{\pi} = \text{konst.}$$

Nyquistova izpeljava



Št rodov v coax na enoto frekvence:

$$c = \lambda \nu$$

$$\lambda = \frac{c}{\nu} = \frac{2\pi}{\omega} c$$

$$L = n \frac{\lambda}{2} = n \frac{1}{2} \frac{2\pi}{\omega} c = \frac{n\pi c}{\omega}$$

$$n = \frac{\omega L}{\pi c} \qquad dn = d\omega\left(\frac{L}{\pi c}\right) \Rightarrow \frac{dn}{d\omega} = \frac{L}{\pi c}$$

Poglejmo energije na stanja:

$$\mathcal{E}(\omega) = \hbar\omega \underbrace{\frac{1}{e^{\hbar\omega/kT}-1}}_{\text{Bose-Einsten za zasedenost pri temperaturi } T}$$

$$dP = R\, d\langle I^2\rangle = \frac{1}{2}\,\frac{c}{\cancel{L}}\,\frac{\cancel{L}}{\pi c}\,d\omega \cdot \mathcal{E}(\omega)$$

energija potujočega je pol stoječega

$$I = \frac{U}{2R} \qquad \langle I^2\rangle = \frac{\langle U^2\rangle}{4R^2}$$

$$d\langle I^2\rangle = d\langle U^2\rangle \frac{1}{4R^2}$$

$$\Rightarrow \quad R\,\frac{d\langle U^2\rangle}{4R^2} = \frac{1}{2}\,\frac{1}{\pi}\,d\omega\,\hbar\omega\left(\frac{1}{e^{\hbar\omega/kT}-1}\right) =$$

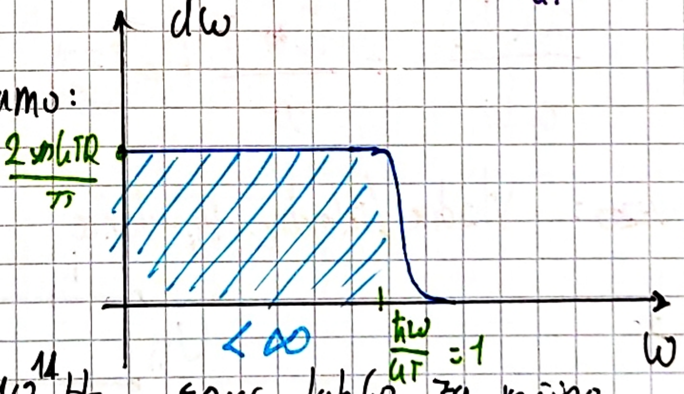
→ Za nižje frekvence $\hbar\omega \ll kT$

$$\frac{1}{1+\frac{\hbar\omega}{kT}-1}$$

nižje frekvence $= \frac{2}{\pi}\,\hbar\omega\,\frac{kT}{\hbar\omega}\,R = \frac{2kTR}{\pi}$

visoke frekvence $= 0$

Torej imamo:



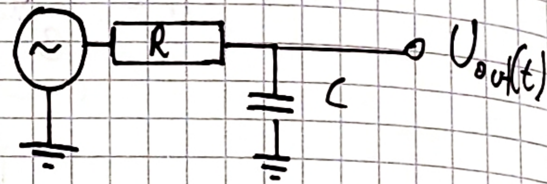
Pri $\hbar\omega = kT \Rightarrow$ cuttoff 10^{13}-10^{14} Hz, zato lahko za večino rečemo, da je konstanta kot zelo dober približek. Povzeto

$$\frac{d\langle U^2\rangle}{d\omega} = \frac{2kTR}{\pi} \quad \text{za } \hbar\omega \ll kT$$

$$= 0 \quad \text{za } \hbar\omega \gg kT$$

Širjenje termičnega šuma skozi linearno vezje

$U_{out} = H(s)U(s)$; $s = i\omega$



$U^2_{out}(i\omega) = |H(i\omega)|^2 U^2(i\omega)$

$$\frac{d\langle U^2_{out}(i\omega)\rangle}{d\omega} = |H(i\omega)|^2 \underbrace{\frac{d}{d\omega}\langle U^2(i\omega)\rangle}_{\text{če je upornik } = \frac{2kTR}{\pi}}$$

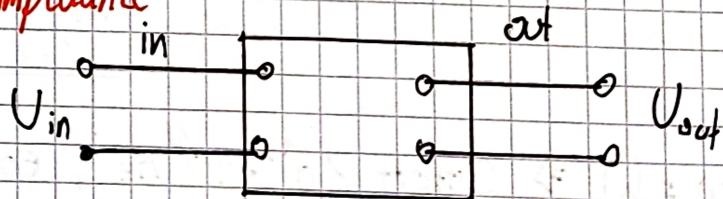
$$\Rightarrow \quad \frac{d\langle U^2_{out}(i\omega)\rangle}{d\omega} = \frac{2kTR}{\pi}|H(i\omega)|^2$$ Nyquistov izrek

Oz. $R \cdot |H(i\omega)|^2 = Re(Z_o)$

↳ Izhodna impedanca vezja

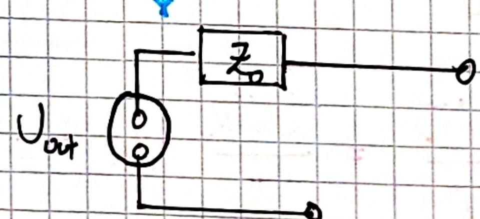
$$\Rightarrow \quad \frac{d\langle U^2\rangle}{d\omega} = \frac{2kT}{\pi} Re(Z_o)$$

Intermezzo: vhodne / izhodne impedance

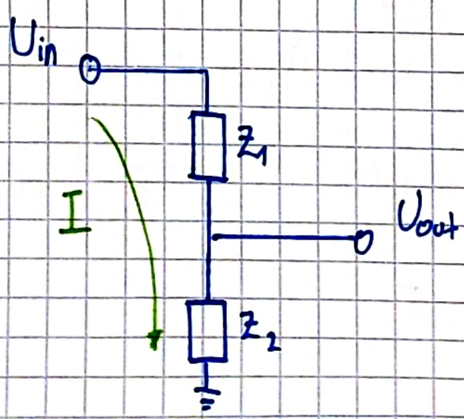


$Z_o = ?$

↓ Thevenin...



Napetostni delilnik:



OC, open circuit

SC, short circuit

$U_{in} = Z_{in} I_1$

$Z_{in} = \sum Z_i = Z_1 + Z_2$

$\Rightarrow U_{out} = I_1 Z_2 = \frac{U_{in} Z_2}{Z_1 + Z_2}$ (1)

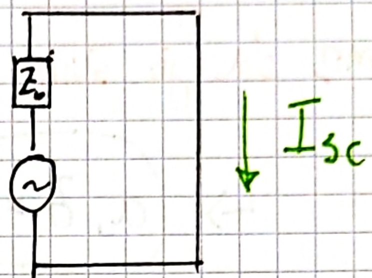
Kaj pa če imamo SC?

$Z_0 \cdot I_{SC} = U_{out}$

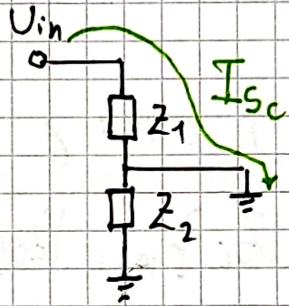
$Z_0 I_{SC} = U_{out}$ (2)

$I_{SC} = \frac{U_{in}}{Z_1}$ (3)

out:



in:



Damo skupaj enačbe (1), (2), (3):

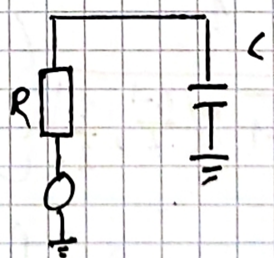
$$U_{out} = \left(\frac{Z_2}{Z_1 + Z_2}\right) Z_1 I_{SC}$$

$$U_{out} = \underbrace{\left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)^{-1}}_{Z_0} I_{SC}$$

$$Z_0^{-1} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

$$Z_{in} = \sum Z_i$$

RC- člen:



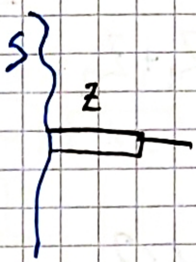
$$Z_{out} = \left(\sum Z_i^{-1}\right)^{-1} = \left(\frac{1}{R} + i\omega C\right)^{-1}$$

$$Z_{out} = \frac{R(1 - i\omega RC)}{1 + \omega^2 R^2 C^2}$$

$$\hookrightarrow Re(Z_{out}) = \frac{R}{1 + \omega^2 R^2 C^2} \; \textcircled{=} \; R|H(\omega)|^2 = \frac{R}{1 + \omega^2 R^2 C^2}$$

Meritve konstantnih količin / Statistika

$(z - Hx) = r$

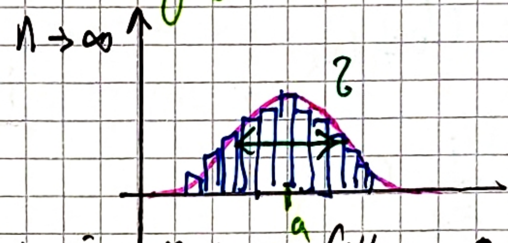


Merilni šum por. po Gaussovi porazdelitvi

$N(0, \sigma)$

Privzamemo, da je šum Gauss $\frac{dP}{dz} = N(a, \sigma)$

↳ $a, \sigma = ?$ To ocenjujemo



Zanima nas pri n meritev je omejen, če $n \to \infty$ fittamo Gaussovko.

Imamo vzorec $\{z_i\}_n$. Vzorčni statistiki:

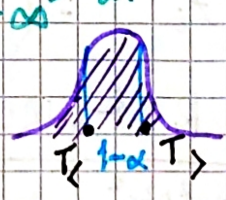
$$\bar{z} = \frac{1}{n}\sum z_i \quad \text{povprečje}$$

$$s^2 = \frac{1}{n-1}\sum (z_i - \bar{z})^2 \quad \text{varianca}$$

T- statistika:

$$T = \frac{\bar{z} - a}{\sqrt{s^2}}\sqrt{n} \quad \Big\} \text{ odvisna od izbire parametra } a$$

$$\frac{dP}{dT} = \frac{1}{\sqrt{n-1}\, B\left(\frac{n-1}{2}, \frac{1}{2}\right)} \underbrace{\left(1 + \frac{T^2}{n-1}\right)^{-n/2}}_{\text{Da zvonasto obliko}} \rightarrow \text{Je normirana} \int_{-\infty}^{\infty} \frac{dP}{dT}\, dT = 1$$



↓

$$\frac{dP}{dT} = S(n-1) \quad \text{Studentov zakon (studentova porazdelitev).}$$

$\alpha \to 1\%$ tipične vrednosti

$\alpha \to 5\%$

10%

Postopek: $\{z_i\}_n$

- Izberemo a : $N(a, b)$
- $\bar{z}, s^2$
- $T = \frac{\bar{z}-a}{s}\sqrt{n}$
- Tabele (n, izberemo α) $\quad P(|t| > t_>) = \alpha$

  iz tabele

- IF $T > t_>$:

  Na stopnji α zavrečemo parameter a

  ELSE:

  Na stopnji tveganja α ga ne moremo zavreči

Interval zaupanja:

$$\int_{t_<}^{t_>} \frac{dP}{dT}\, dT = (1-\alpha)$$

Torej $T \in [t_<, t_>]$ na stopnji $(1-\alpha)$. Bolj pravilno, izven tega intervala lahko pričakujemo T z gotovostjo α.

$a \in [a_<, a_>]$

Na stopnji $(1-\alpha)$ oz. a izven tega intervala lahko na stopnjo tveganja α zavržemo.

$$t_< = \frac{\bar{z}-a_>}{s}\sqrt{n} \qquad t_> = \frac{\bar{z}-a_<}{s}\sqrt{n}$$

Porazdelitev χ^2

$\{z_i\}_{1}^{n}$; Porazdeljeni (predpostavimo) $N(a, \sigma)$

↑ disperzija merilnega šuma

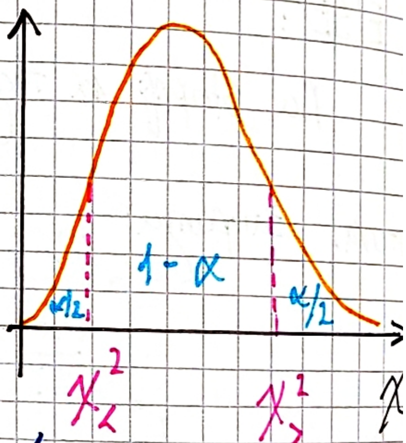
Sestavimo statistiko χ^2:

$$\chi^2 = (n-1)\frac{s^2}{\sigma^2} \quad \text{je odvisna od } \sigma$$

$\frac{dP}{d\chi^2}$ pa ni odvisna od izbranega σ in je univerzalna/tabelirana.

$$\chi^2(n-1) = \frac{dP}{d\chi^2} = \frac{1}{2^{\frac{n-1}{2}}\,\Gamma\left(\frac{n-1}{2}\right)} \underbrace{\left(\chi^2\right)^{\frac{n-3}{2}} e^{-\chi^2/2}}_{\text{da obliko}}$$

↑ zakon



Je tudi normirana:

$$\int_0^\infty \frac{dP}{d\chi^2}\, d\chi^2 = 1$$

Kar pomeni, da tako kot prej postavimo meje in ~~interval zaupanja~~ stopnjo tveganja.

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

Z verjetnostjo $(1-\alpha)$ pričakujemo, da bo $\chi^2 \in [\chi^2_<, \chi^2_>]$.

V nasprotnem če $\chi^2 \notin [\chi^2_<, \chi^2_>]$ lahko parameter σ na stopnji tveganja α zanemarimo.

DEF: Vsota kvadratov n neodvisnih (naključno porazdeljenih), standardizirano normalno porazdeljenih spremenljivk je porazdeljena po zakonu χ^2 z n prostostnimi stopnjami.

Dokaz, da je to isto kot prejšnja definicija:

x por. $N(0,1)$

$\sum x^2 = \chi^2$

$$s^2 = \frac{1}{n-1} \sum (z_i - \bar{z})^2$$

$\{z_i\}$, z por. po $N(\bar{z}, \sigma)$

$(z_i - \bar{z})$ por. po $N(0, \sigma)$

$\frac{z_i - \bar{z}}{\sigma}$ por po $N(0,1)$ ✓

$\Rightarrow$ Torej: $\sum \frac{(z_i - \bar{z})^2}{\sigma^2}$ po $\chi^2(n-1)$

$$\Rightarrow \chi^2 = (n-1)\frac{s^2}{\sigma^2}$$

DEF: $\cancel{X}$ X por. po $N(0,1)$

Y por. po $\chi^2(n)$

$$\Rightarrow T = \frac{X}{\sqrt{Y/n}} \text{ por. po } S(n)$$

Dokaz (Reeeeeeee):

$$T = \frac{\bar{z} - a}{s}\sqrt{n} = \frac{(\bar{z} - a)\sqrt{n}}{\sigma\sqrt{s^2/\sigma^2}} = \frac{(\bar{z} - a)}{\sigma/\sqrt{n}} \frac{1 \cdot \sqrt{n-1}}{\sqrt{\frac{s^2}{\sigma^2}(n-1)}}$$

por. po $\chi^2(n-1)$

por. po $S(n-1)$

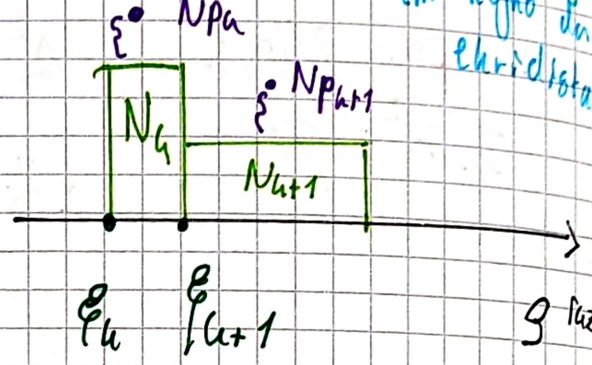
Oblikovni testi

$\{z_i\}$, n $\quad \frac{dP}{d\xi}$ - poljuben porazdelitveni zakon

Naredimo razrede:

- brez prekrivanja, brez "lukenj" sestavimo g razredov
- N_k izmerkov pade v k-ti razred
- $Np_k = N\int_{\xi_k}^{\xi_{k+1}} \frac{dP}{d\xi}\, d\xi$

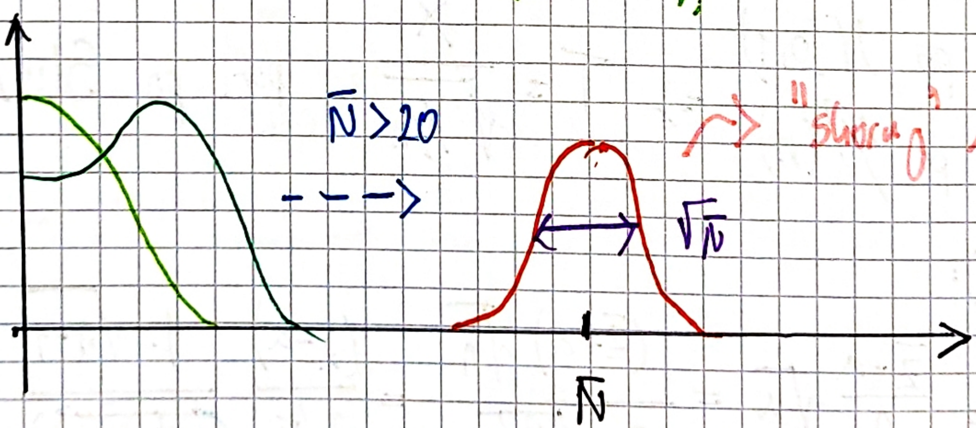
"kot histogram" (ni nujno da ekvidistančno)



$(N_k - Np_k)$; če smo si dobro izbrali $\left(\frac{dP}{d\xi}\right)$ bo to k statistični šum.

Štejemo dogodke; $\bar{N}$; $\frac{dP}{dN}$ = poisson

$$w_\nu = \frac{dP}{dN} = \frac{\bar{N}^\nu e^{-\bar{N}}}{\nu!}$$

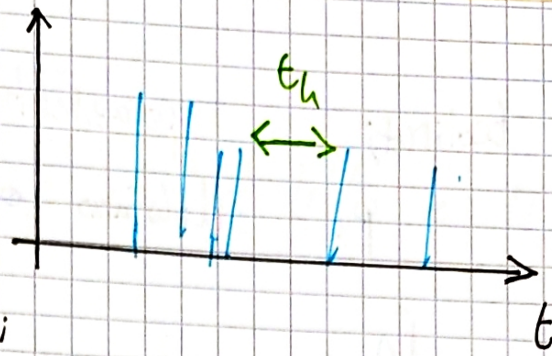


$$\chi^2 = \sum_{k=1}^{g} \frac{(N_k - Np_k)}{Np_k} \quad \text{por po } \chi^2(g-1)$$

Pearsonov χ^2 test

Primer: [Radioaktiven razpad]

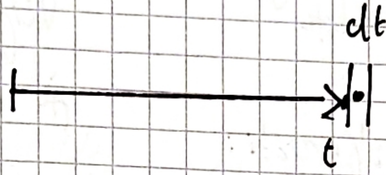
$\{t_k\}$



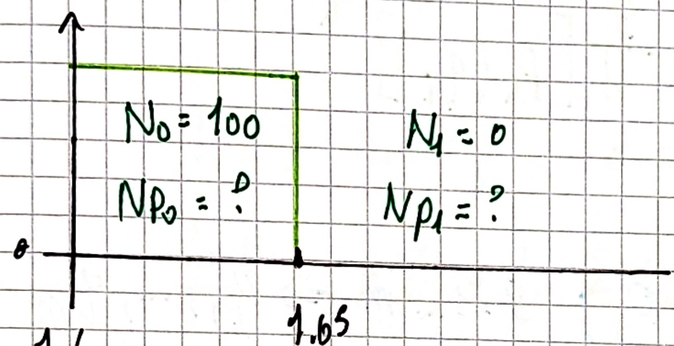
$N = 100$; ni parov kjer bi bil čas med zaporednimi sunki od 1.6 s

Ali lahko na stopnjo tveganja α ovržemo hipotezo $\tau = 1s$?

$$dp = \frac{1}{\tau} dt$$



$$e^{-t/\tau} \frac{1}{\tau} dt = dp$$



Imamo dva razreda

$g = 2$

$$P_0 = \int_0^{1.6} \frac{1}{\tau} e^{-t/\tau} dt = 1 - e^{-1.6/1} = 0.8$$

$$P_1 = \int_{1.6}^{\infty} \frac{1}{\tau} e^{-t/\tau} dt = 1 - 0.8 = 0.2$$

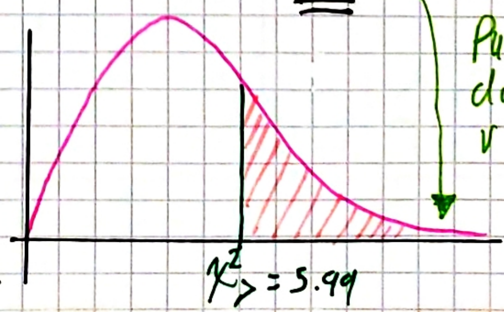
$$\Rightarrow \quad Np_0 = 80 \quad \Rightarrow \quad \chi^2 = \frac{(100-80)^2}{80} + \frac{(0-20)^2}{20}$$

$$Np_1 = 20$$

$$= 5 + 20 = \underline{\underline{25}}$$

Tabelirano:

$$\chi^2_>(2-1)^{5\%} = \underline{\underline{5.99}}$$



Pade daleč v rep

Torej lahko zavržemo na stopnjo tveganja α.

Fischer

Želimo testirati porazdelitveni zakon pri čemer ?

p_u določamo optimalno:

$$\frac{dP}{dz}(q_1, q_2, \dots, q_m)$$

m neznanih parametrov

testirana verjetnostna gostota

$$P_u = \int_{z_{u-1}}^{z_u} \frac{dP}{dz}(q_1, \dots, q_m)\,dz$$

$P_u(q_i, i=1, \dots, m)$

Verjetnost za u-ti razred je odvisna od parametrov

Sestavimo funkcija zanesljivosti:

$$L^* = \prod_{u}^{g} \left[P_u(q_i)\right]^{N_u}$$

$$\underbrace{\frac{\partial L^*}{\partial q_i} = 0}_{m \text{ enačb}} \; ; \quad i=1 \dots m \Rightarrow q_1, \dots, q_m$$

$$\{z_i\} \rightarrow m \text{ parametrov } \frac{dP}{dz}(q_i)$$

optimalen

$$P_u^*(q_i^*) \rightarrow \text{Pearson } \chi$$

Fischer

$$\chi^2 = \sum \frac{(N_u - Np_u^*)^2}{Np_u^*} \rightarrow \text{por. po zakonu } \chi^2(g-1-m)$$

Poenostavljena funkcija zanesljivosti

Če imamo zelo gosto razrede je v vsakem lahko le 0 ali pa 1 zadetkov.

$$\frac{\partial L}{\partial q_i} = 0$$

oz.

$$L = \prod \frac{dp}{dz}(z_k, q_1, \dots q_m)$$

$$\frac{\partial \log L}{\partial q_i} = 0$$

↑ v vrednostih meritev

Primer: [Radioaktivni razpad]

$$\{t_k\}_n \qquad \frac{dP}{dt} = \frac{1}{\tau} e^{-t/\tau}, \quad L = \prod_{k=1}^{n} \frac{1}{\tau} e^{-t_k/\tau} = \frac{1}{\tau^n} e^{-\sum t_k/\tau}$$

$$\ln L = -n \ln \tau - \frac{1}{\tau} \sum t_k$$

$$\frac{\partial \ln L}{\partial \tau} = -n \frac{1}{\tau} + \frac{1}{\tau^2} \sum t_k = 0$$

$$\frac{1}{\tau} \sum t_k = n \quad \Rightarrow \quad \tau = \frac{1}{n} \sum t_k$$

Povprečje

Test Kolmogorova (Kumulativni test)

$\{z_i\}, n, \left(\frac{dP}{d\xi}\right) = ?$ testiramo kumulativno porazdelitveno funkcijo:

$$F(z) = \int_{-\infty}^{z} \left(\frac{dp}{d\xi}\right) d\xi$$

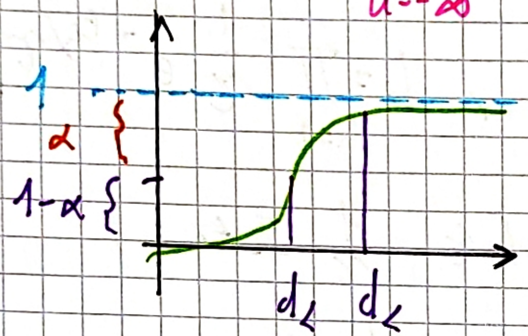
Sestavimo eksperimentalno kumulativno funkcijo

$$f(z) = \frac{\text{št. izmerkov} < z}{n}$$

Testiramo maksimalni odmik:

$$D = \sup |F(z) - f(z)|$$

Kolmogorov: $P(D\sqrt{n} < d) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 d^2}$ } Standardizirano



Tabele:

$d_\alpha^{5\%} = 1{,}36$

$d_\alpha^{1\%} = 1.65$

$d_\alpha^{0,1\%} = 1.96$

→ Manjkal 3 ure zaradi terapije/psihiatra

↳ Lokalna detekcija

↳ Izluščanje vrhov

↳ ...

Stabilnost povratne zanke

- Optimalen merilni sistem → Povratna zanka

$x(t) \longrightarrow \boxed{} \longrightarrow$

↳ $K(t)$

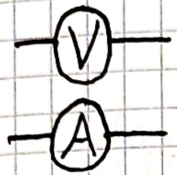
- Realen merilni sistem
  - Univerzalen (glede na $x(t)$)
  - ne-optimalen (tranzienti, offset-sistematične napake)

$K(\underbrace{z - Hx}_{\text{inovacija}})$

- Ohranjamo princip povratne zanke
- primerno dušenje

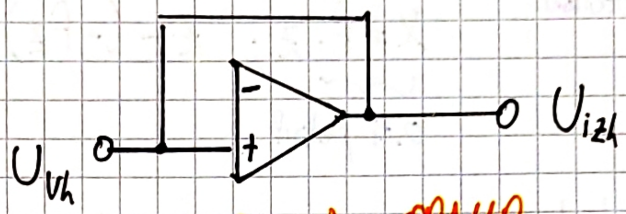
Zgled: [Električni merilni sistem]

$$H(s) = \frac{\alpha(1+2\xi s/\omega_0)}{1+2\xi s/\omega_0 + s^2/\omega_0^2}$$



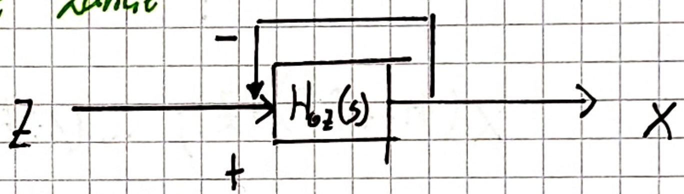
- stabilen
- dovolj hiter

=> Napetostni sledilnik



Unity gain OPAMP

$\Downarrow$



H_{oz}, prenosna funkcija odprte zanke

H_{zz}, prenosna funkcija zaključene zanke

$$H(s)(Z-X) = X$$

$$HZ = (H+1)X$$

$$\Rightarrow \quad \underbrace{H_{zz} = \frac{X}{Z} = \frac{H}{1+H}}$$

=> Nestabilno ko gre $H \to -1$

$$H(\omega) = \underbrace{|H(i\omega)|}_{1}\, e^{i\underbrace{\varphi(i\omega)}_{-1}}$$

} To nam bo dalo nestabilnost

$$\mathrm{tg}\,\varphi = \frac{\mathrm{Im}\, H(i\omega)}{\mathrm{Re}\, H(i\omega)} \Rightarrow \varphi \to \pm\pi$$

Naš zgled:

Unity gain $\left(\frac{X}{Z}\right)_{oz} \xrightarrow{\omega \to 0} A_{DC} \sim 10^6$, $\left(\frac{X}{Z}\right)_{oz} \xrightarrow[\omega \to \infty]{} 0$

$\left(\frac{X}{Z}\right)_{zz} \xrightarrow{\omega \to 0} 1$, $\left(\frac{X}{Z}\right)_{zz} \xrightarrow[\omega \to \infty]{} 0$

Stabilnost bomo preverili z delnimi valovi, ki potujejo skozi sistem z negativno povratno zanko.

Gledamo pri frekvencah, kjer je faza $\underline{\underline{\pm\pi}}$ $\Rightarrow H(i\omega) \to (-1)$

Rabimo še $f = |H(i\omega)| = ?$

1. prehod

$$\underbrace{x = f \cdot z}_{\text{0.-ti prehod}} ; \quad z-(-fz) = z+fz = \underline{(1+f)z}$$

2. prehod

$$\underbrace{x = (1+f)z}_{\text{1. prehod}} ; \quad z-((1+f))fz = z+fz+f^2z = (1+f+f^2)z$$

$\vdots$

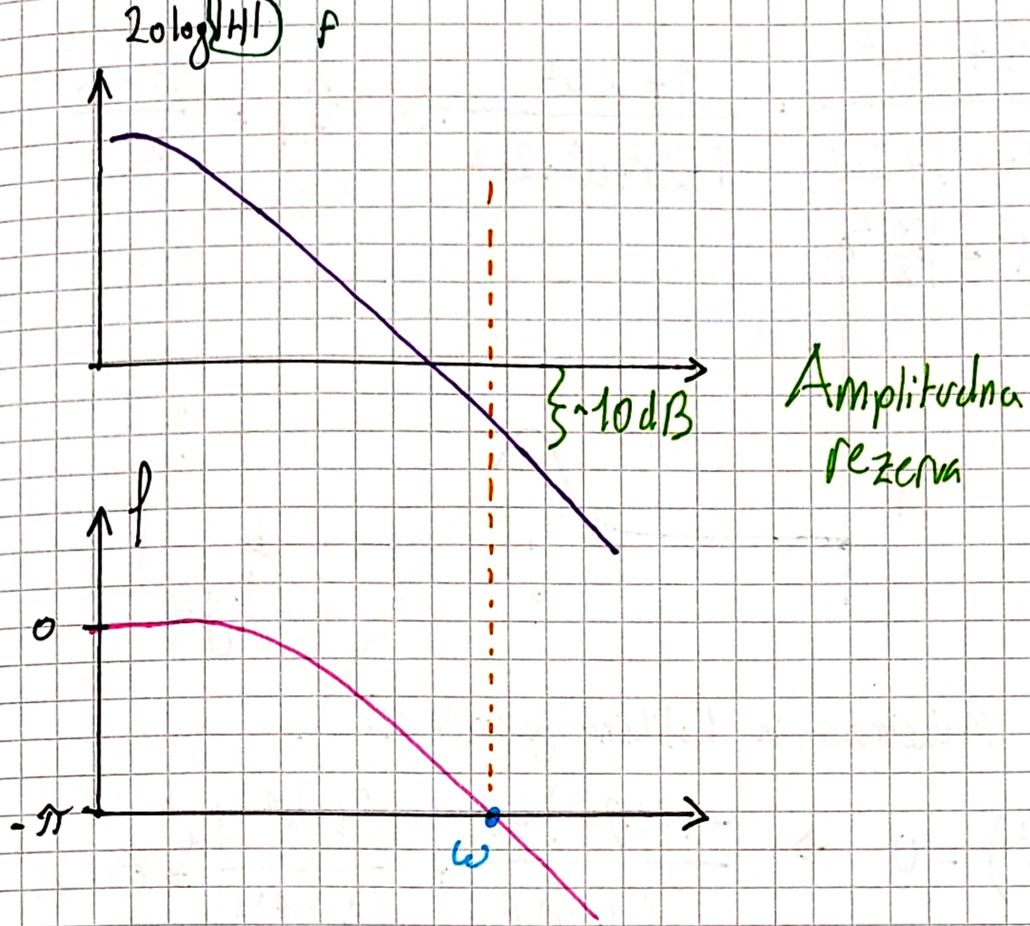
$$x(1+f+f^2)fz ; \quad \dots (1+f+f^2+\dots)z$$

n-ti red: $\underbrace{(f+f^2+\dots f^n)}\, z = x$

$$\sum_{k=1}^{n} f^k = \frac{1-f^n}{1-f} \; ; \text{ konvergira za } f<1$$

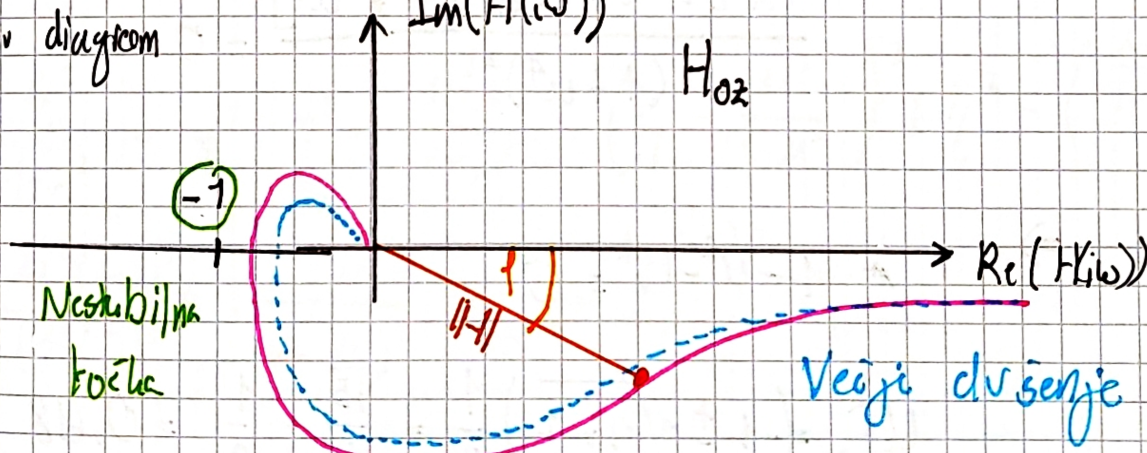
$\Rightarrow$ Stabilnostni kriterij: $f<1$, pri frekvencah, kjer je zamik ($\varphi = 180°$)

Ojačevalni faktor odprte zanke pri frekvenci, kjer $\varphi(\omega_0) = -\pi$



Potrebujemo še pogoj za primerno dušenje

Nyquistov diagram



Optimalen sistem II. reda → Nyquistov diagram?

$$u = \frac{\omega}{\omega_0}$$

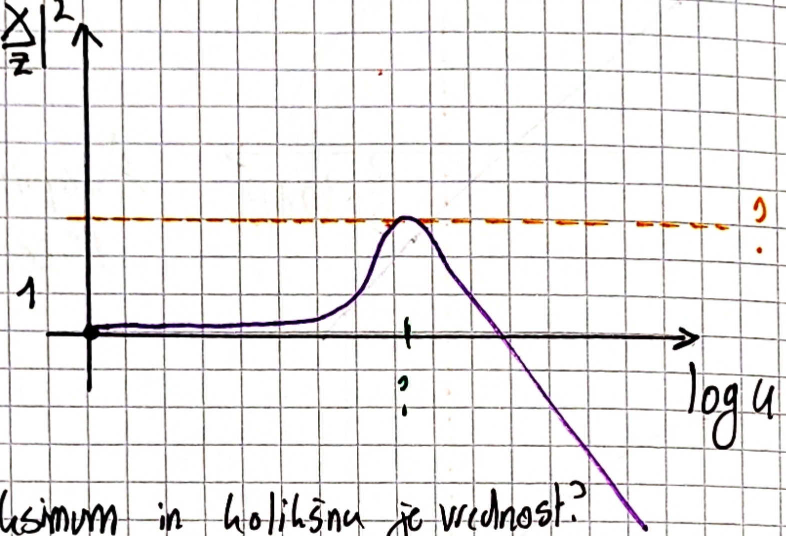
$$\left(\frac{X}{Z}\right)_{\substack{2.\text{red}\\ \text{optimalno}}} = \frac{1 + 2\xi s/\omega_0}{1 + 2\xi s/\omega_0 + s^2/\omega_0^2}$$

Optimalen: $\xi = \frac{1}{\sqrt{2}}$

$$\left|\frac{X}{Z}\right|^2 = \frac{1 + 4\xi^2(\omega/\omega_0)^2}{(1 - \omega^2/\omega_0^2)^2 + 4\xi^2(\omega/\omega_0)^2} = \to$$

$$= \frac{1+2u^2}{(1-u^2)^2+2u^2} = \frac{1+2u^2}{1-2u^2+u^4+2u^2} \Rightarrow \left|\frac{X}{Z}\right|^2 = \frac{1+2u^2}{1+u^4}$$

$M^2 = \left|\frac{X}{Z}\right|^2$



Kje je maksimum in kolikšna je vrednost?

$$\frac{d}{du}(M^2) = 0 \rightarrow \frac{-1(1+2u^2)4u^3}{(1+u^4)^2} + \frac{(1+u^4)4u}{(1+u^4)^2} = 0$$

$$\downarrow$$

$$\frac{4u\left[1+u^4-(1+2u^2)u^2 - u^4\right]}{(1+u^4)^2} = 0$$

$$\Rightarrow \quad 1-u^2-u^4 = 0$$

$$u_{1,2}^2 = \left(-1 \pm \sqrt{1+4}\right)\frac{1}{2} = \frac{\sqrt{5}}{2} - \frac{1}{2} = 0{,}618$$

Torej je maksimum pri $u = \sqrt{0{,}618} = 0{,}786$ **Pogoj**

Ojačanje pa je:

$M^2 < 1.62$

$M < 1.3$

$$M^2(u = 0{,}786) = 1.62 \Rightarrow$$

Za primerno državanje mora imeti $M_{oz} \leq 1.3$

Zaključena zanka:

$$\left(\frac{X}{Z}\right)^2_{ZZ} = M^2 = \left|\frac{H}{1+\underbrace{H}_{\xi}}\right|^2 \underbrace{\eta}$$

$$= \left|\frac{ReH + iImH}{1+ReH+iImH}\right|^2$$

$$M^2 = \frac{\xi^2+\eta}{(1+\xi)^2+\eta^2}$$

$$M^2(1+\xi)^2 + M^2\eta^2 = \xi^2+\eta^2$$

$$M^2(1+2\xi+\xi^2)+M^2\eta = \xi^2+\eta^2$$

$$M^2 + M^2 2\xi + M^2\xi^2 + M^2\eta = \xi^2+\eta^2$$

$$M^2 = \xi^2(1-M^2)+\eta^2(1-M^2) - M^2 2\xi \quad / :(1-M^2)$$

$$\frac{M^2}{1-M^2} = \xi^2+\eta^2 - \frac{M^2 2\xi}{1-M^2} = \left(\xi - \frac{M^2}{1-M^2}\right)^2+\eta^2 - \left(\frac{M^2}{1+M^2}\right)^2$$

$$\Rightarrow \frac{M^2(1-M^2)}{(1-M^2)^2} + \frac{M^4}{(1-M^2)^2} = \left(\xi - \frac{M^2}{1-M^2}\right)^2 + \eta$$

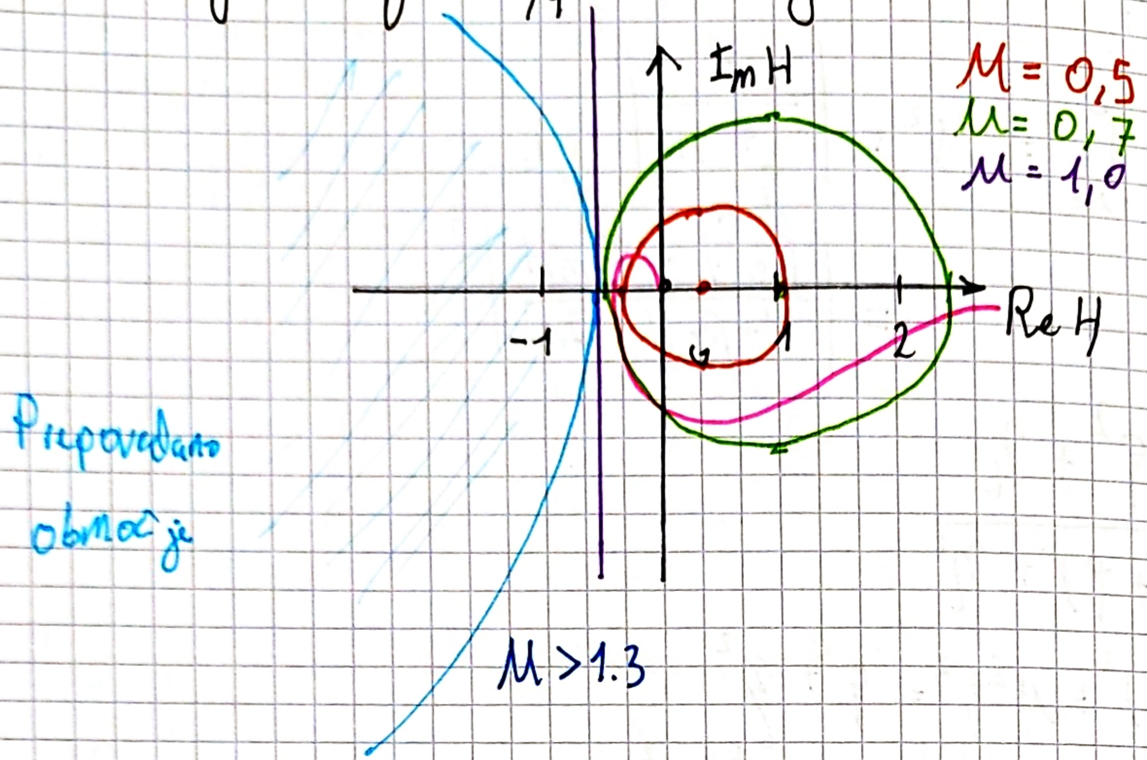
Torej:

$$\left(\frac{M}{1-M^2}\right)^2 = \left(\xi - \frac{M^2}{1-M^2}\right)^2+\eta^2$$

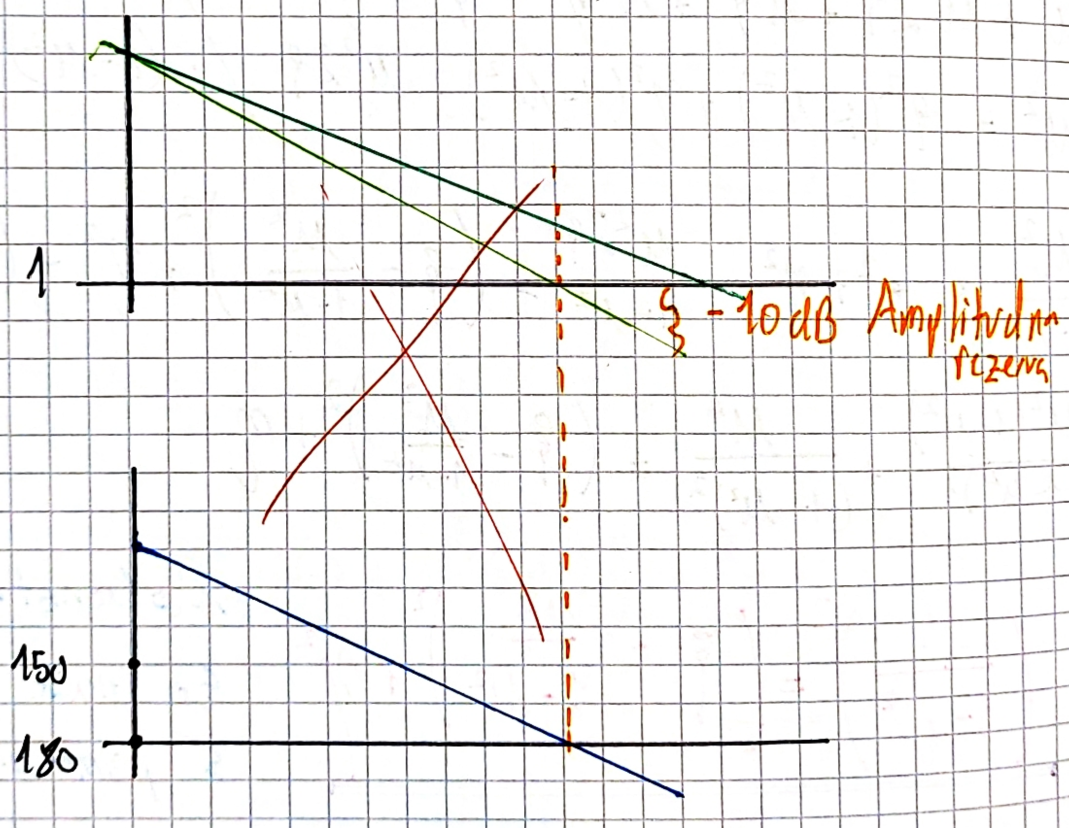
$M =$ konst.

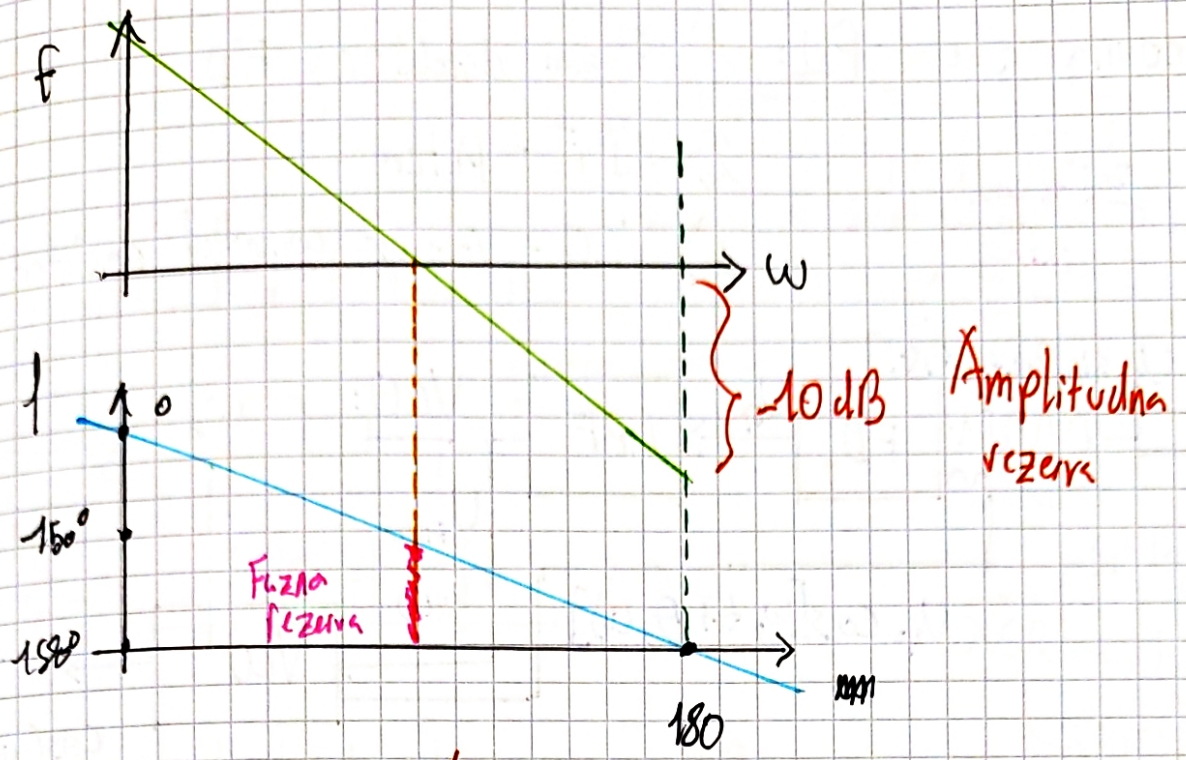
so enačbe krožnic s polmerom $\frac{M}{1-M^2}$ in premikom $\xi \to \frac{M}{1-M^2}$

Narišimo jih sedaj v Nyquistov diagram:

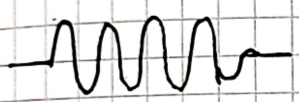


Pri $|H| = 1$ je lahko fazni zamik $\sim -150°$





Merjenje frekvence / PLL - fazno upeta zanka (phase locked loop)



Frekvence lahko dobro merimo ker lahko štejemo vrhove.

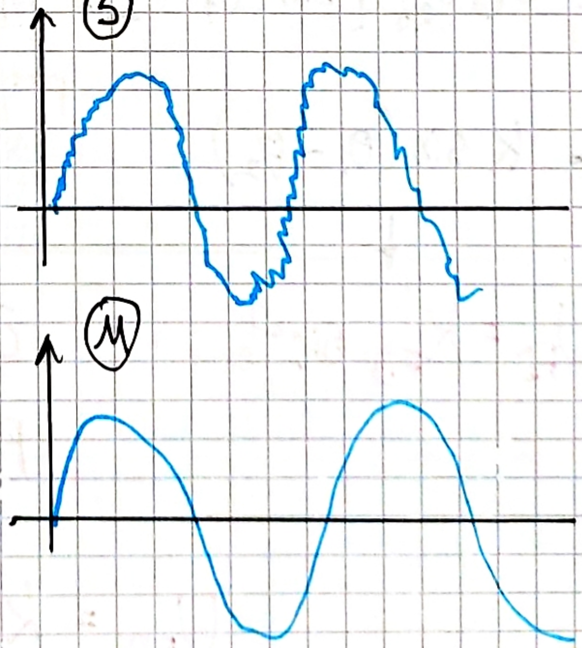
PLL:

S

Oscilator v sistem S

M

napetostno kontroliran modelski oscilator v sistemu M



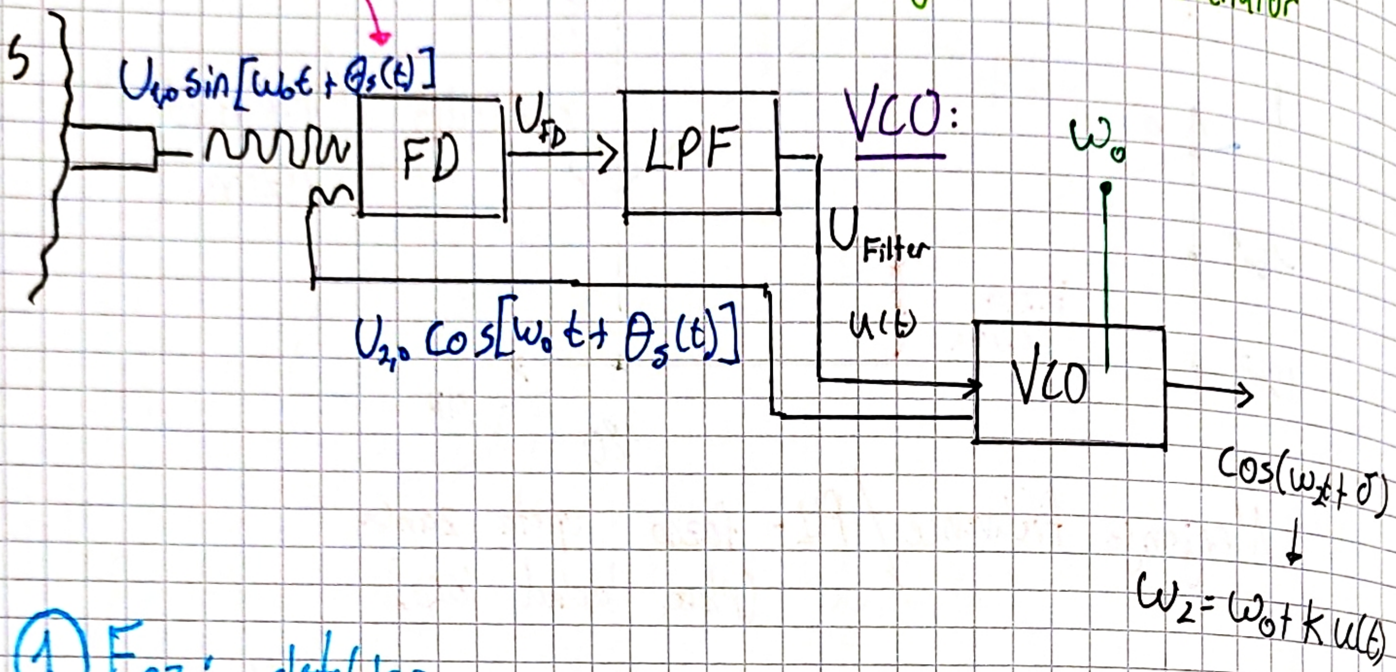
Želimo sinhronizirati oscilator v S z tistim v M.

Osnovna shema PLL: 1) Fazni detektor FD

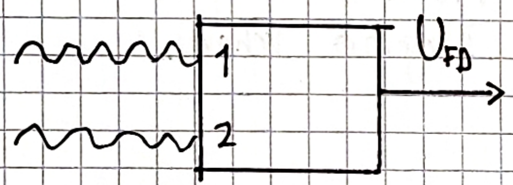
2) LPF - filter

3) VCO - Voltage controlled oscilator

Zašumljena faza



1. Fazni detektor



Šum

$$\theta_1 = \theta_s(t) + r(t)$$

$$U_{FD} = k \langle U_{1,0} U_{2,0} \sin(\omega_0 t + \theta_1) \cos(\omega_0 t + \theta_2) \rangle \quad ; \quad \theta_2 = \theta_x(t)$$

ni odvisno od časa

$$= U_{1,0} U_{2,0} \left[\sin(\theta_1 - \theta_2) + \sin(2\omega_0 t + \theta_1 + \theta_2) \right\rangle$$

povprečen po času = 0

$\theta_1 - \theta_2 = \theta_e$

razlika faz

$$= K \sin(\theta_1 - \theta_2)$$

↓

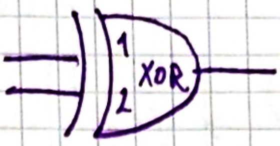
$$U_{FD} = K_{FD} \sin \theta_e$$

Če sta blizu skupaj potem $\sin\theta_e \approx \theta_e$

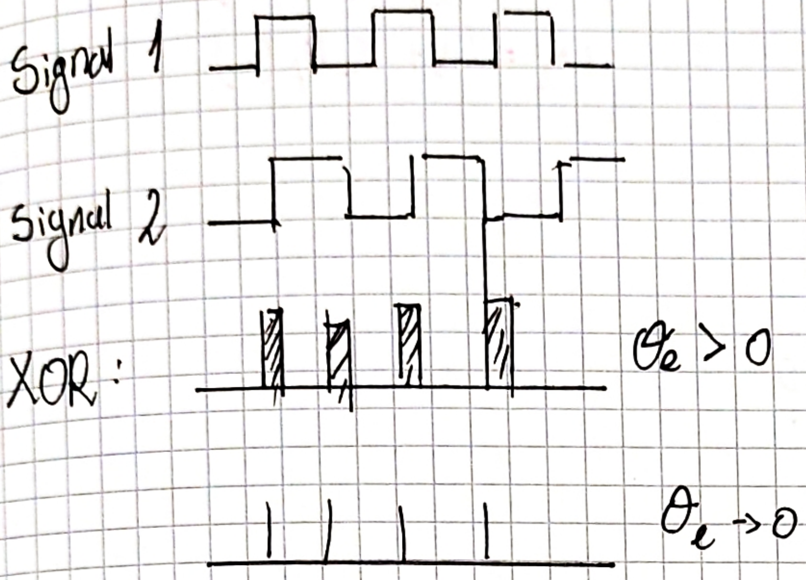
↓

$$U_{FD} = K_{FD}\, \theta_e$$

1.) type I ; XOR-gate



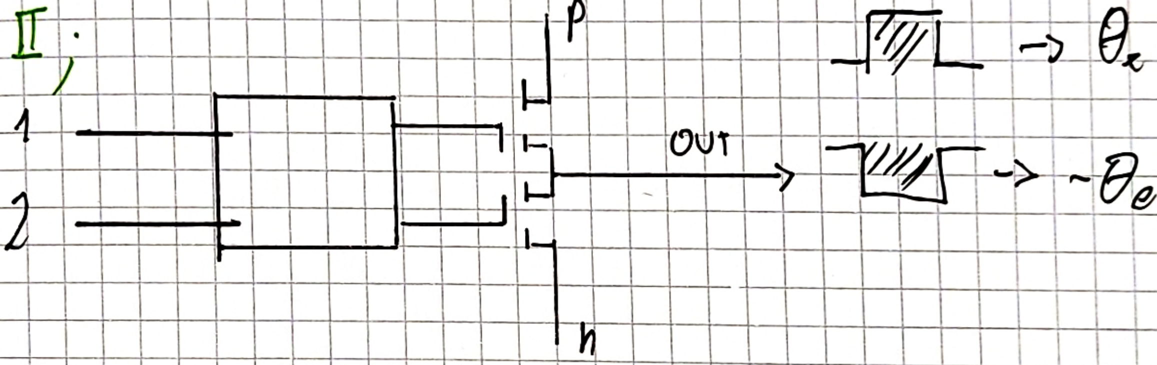
In tu še časovno povprečimo, ker je širina $\propto \theta_e$ in s tem tudi avg. voltage.

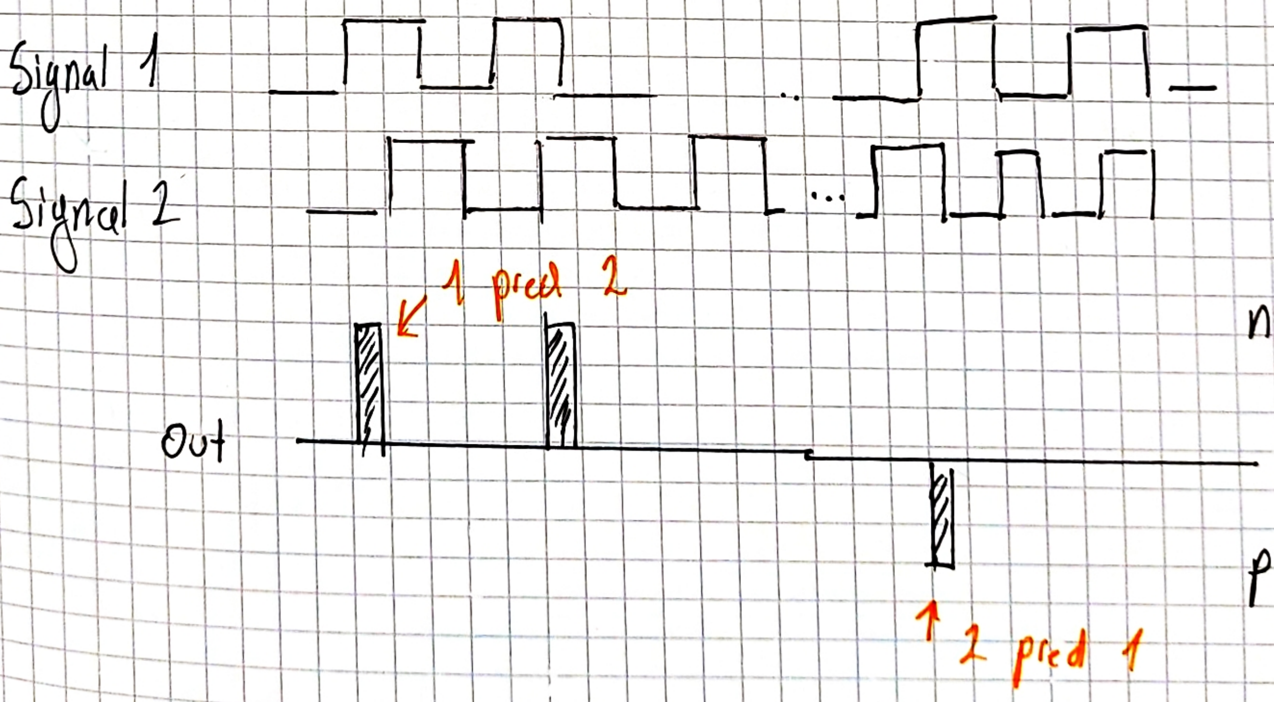


Problems:

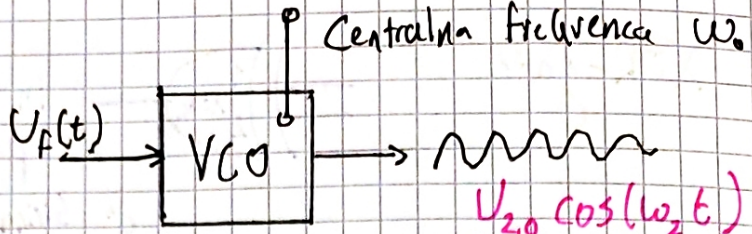
$\overline{XOR}$ → lagging
→ leading
→ vjetj. višjih harmonikov

2.) type II ;





2. VCO



$U_{2,0}\cos(\omega_2 t)$

$\omega_2 = \omega_0 + K_0 U_f(t)$

Zahtevamo:

- monotonost izhoda

3. Regulacijski filter

- LPF $\frac{1}{1+\tau s}$?