Uvod: Ocenjevanje natančnosti

$$u = f(x_1, x_2, \dots, x_N)$$

$$u + \Delta u = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \dots, x_N + \Delta x_N) \quad ; \quad \Delta x_i \ll x_i$$

Sprememba je mala

$$u + \Delta u = \underbrace{f(x_1, x_2, \dots, x_N)}_{u} + \left(\frac{\partial f}{\partial x_1}\right)\Delta x_1 + \dots + \left(\frac{\partial f}{\partial x_N}\right)\Delta x_N + \mathcal{O}(\Delta x^2)$$

$$\Rightarrow \quad \Delta u \simeq \left(\frac{\partial f}{\partial x_1}\right)\Delta x_1 + \dots + \left(\frac{\partial f}{\partial x_N}\right)\Delta x_N$$

Absolutna natančnost (občutljivost)

To lahko zapišemo vektorsko/matrično:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad \Delta\vec{x} = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_N \end{pmatrix} \quad \underset{\nabla f}{\underset{\parallel}{\left(\frac{\partial f}{\partial \vec{x}}\right)}} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \end{pmatrix}_{\vec{x}}$$

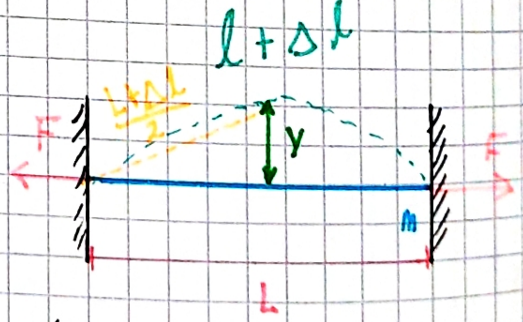
$$\Delta u = \Delta\vec{x}^T \cdot \left(\frac{\partial f}{\partial \vec{x}}\right)_{\vec{x}}$$

Relativna natančnost

$$\frac{\Delta u}{u} = \frac{1}{f}\Delta\vec{x}^T\left(\frac{\partial f}{\partial \vec{x}}\right)_{\vec{x}}$$

Primer: Oceni natančnost zvenenja strune

$$\left(\frac{\Delta \nu}{\nu}\right) = ?$$



$\nu = f(\dots)$ To funkcijo moramo poiskati;

Napnemo struno v prečni smeri. Spremeni se dolžina (in s tem μ) in F.

$$\Rightarrow \quad \nu = \frac{c}{\lambda} \quad ; \quad c = \sqrt{\frac{F}{\mu}} \quad \mu = \frac{m}{L}$$

$\lambda = 2l$ za osnovni nihajni način

$$\Rightarrow \quad \nu = \frac{1}{2l}\sqrt{\frac{F}{\mu}} = f(F,\mu)$$ l ostane fiksen

Torej:

$$\vec{X} = \begin{pmatrix} F \\ \mu \end{pmatrix}$$

$$\vec{\Delta X} = \begin{pmatrix} \Delta F \\ \Delta \mu \end{pmatrix} \qquad \left(\frac{\partial f}{\partial \vec{x}}\right) = \begin{pmatrix} \frac{1}{2l}\cdot\frac{1}{2}\sqrt{\frac{1}{\mu F}} \\ -\frac{1}{2l}\,\frac{1}{2}\sqrt{\frac{F}{\mu^3}} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2}\cdot\frac{\nu}{F} \\ -\frac{1}{2}\,\frac{\nu}{\mu} \end{pmatrix}$$

Vstavimo:

$$\Delta\nu = (\Delta F, \Delta\mu)\cdot\begin{pmatrix} \frac{1}{2}\,\frac{\nu}{F} \\ -\frac{1}{2}\,\frac{\nu}{\mu} \end{pmatrix}$$

$$\Delta\nu = \frac{\nu}{2}\,\frac{\Delta F}{F} - \frac{\nu}{2}\,\frac{\Delta\mu}{\mu} \;/:\nu \quad \Rightarrow \quad \left(\frac{\Delta\nu}{\nu}\right) = \frac{1}{2}\,\frac{\Delta F}{F} - \frac{1}{2}\,\frac{\Delta\mu}{\mu}$$

(i) $$\frac{\Delta F}{S} = E\,\frac{\Delta l}{l} \Rightarrow \Delta F = ES\cdot\frac{\Delta l}{l}$$

(ii) $$\mu = \frac{m}{l} \Rightarrow \Delta\mu = -\frac{m}{l^2}\,\Delta l = -\mu\,\frac{\Delta l}{l}$$

$$\Rightarrow \frac{\Delta\mu}{\mu} = -\frac{\Delta l}{l}$$

To oboje vstavimo:

$$\left(\frac{\Delta\nu}{\nu}\right) = \frac{1}{2}\,\frac{\Delta l}{l}\left(\frac{ES}{F} + 1\right)$$

Še aproksimacija za Δl

$$\left(\frac{l+\Delta l}{2}\right)^2 = y^2 + \left(\frac{l}{2}\right)^2$$

$$\cancel{l^2} + 2l\Delta l + \underbrace{\cancel{(\Delta l)^2}}_{\text{zanemarljiv ker } \Delta l \ll l} = 4y^2 + \cancel{l^2} \Rightarrow$$

$$l\Delta l = 2y^2$$

$$\frac{1}{2}\left(\frac{\Delta l}{l}\right) = \left(\frac{y}{l}\right)^2$$

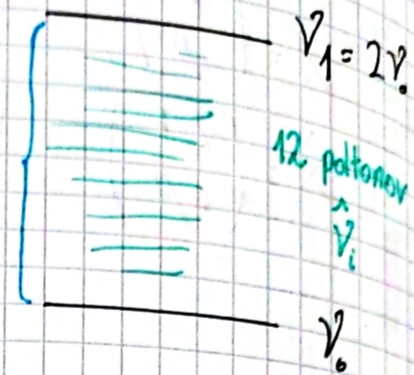
$$\Rightarrow \left(\frac{\Delta\nu}{\nu}\right) = \left(\frac{y}{l}\right)^2\left(\frac{ES}{F}+1\right)$$

Primer: Ton z neznano frekvenco

$\nu_0 = 440\ \text{Hz}$

$$\hat{\nu}_0 = \nu_0$$
$$\hat{\nu}_{12} = \nu_1$$

Oktava



Koliko časa moramo poslušati, da ocenimo ton na pol tona natančno?

Princip nedoločenosti (iz signalne teorije) $\underbrace{(\Delta\nu)}_{\text{pasovna širina}}\underbrace{(\Delta t)}_{\text{časovno okno}} \approx 1$

$$\Rightarrow (\Delta t) \approx \frac{1}{\Delta\nu}$$

$$\Delta\nu = \hat{\nu}_1 - \hat{\nu}_0 = x\hat{\nu}_0 - \hat{\nu}_0 = \nu_0(x-1)$$

$$\frac{\hat{\nu}_1}{\nu_0} = x; \quad \frac{\hat{\nu}_2}{\hat{\nu}_1} = x; \quad \Rightarrow \frac{\hat{\nu}_2}{\hat{\nu}_0} = x^2 \quad \Rightarrow \frac{\hat{\nu}_{12}}{\hat{\nu}_0} = x^{12} = 2$$

$$x = \sqrt[12]{2}$$

$$\Rightarrow \Delta\nu = \left(\sqrt[12]{2} - 1\right)\nu_0 \approx 26\ \text{Hz}$$

$$\Delta t = \frac{1}{\Delta\nu} \approx 40\ \text{ms}$$

Uvod v optimalno združevanje meritev

- Ocenjujemo neznano konst. količino X

- $A: Z_1^{(A)}, \dots, Z_M^{(A)} \Rightarrow (\bar{Z}_A, \sigma_A^2)$

izmerki / meritve

- $B: Z_1^{(B)}, \dots, Z_N^{(B)} \Rightarrow (\bar{Z}_B, \sigma_B^2)$

To dvoje združimo v novo oceno za X: $\boxed{\hat{X}}$

$(\hat{X}, \hat{\sigma}) = ?$

To hočemo optimalno

$\Downarrow$

minimalna varianca $\hat{\sigma}^2$

normalna distribucija)

$$\bar{Z}_A \sim N(x, \sigma_A^2)$$

$$\bar{Z}_B \sim N(x, \sigma_B^2)$$

Porazdeljeno normalno

naključni merilni šum

$$\bar{Z}_A = X + r_A \quad ; \quad r_A \sim N(0, \sigma_A^2)$$

$$\bar{Z}_B = X + r_B \quad ; \quad r_B \sim N(0, \sigma_B^2)$$

$$\hat{X} = X + \hat{r} \quad ; \quad \hat{r} \sim N(0, \hat{\sigma}^2)$$

- Nastavek:

$$\hat{X} = a \bar{Z}_A + b \cdot \bar{Z}_B$$

$$\hat{X} = a(X + r_A) + b(X + r_B) = X + \hat{r}$$

$$X(a+b) + a r_A + b r_B = X + \hat{r}$$

$$a + b = 1 \Rightarrow a = 1 - b$$

$$\hat{r} = (1-b) r_A + b r_B$$

$$\hat{X} = \bar{Z}_A + b(\bar{Z}_B - \bar{Z}_A)$$ → Ni še optimalna

Moramo izbrati b

Optimalen b bo dal najmanjšo $\hat{\sigma}^2$ $\left(\frac{\partial \hat{\sigma}^2}{\partial b} = 0\right)$

$$\bar{Z}_A \sim N(x, \sigma_A^2)$$

$$\hat{\sigma}^2 = \langle \hat{r}^2 \rangle = (1-b)^2 \sigma_A^2 + b^2 \sigma_B^2 + 2b(1-b)\langle r_A r_B \rangle$$

Kovarianca

$$E[\bar{Z}_A] = \langle \bar{Z}_A \rangle = X$$

$$Var[\bar{Z}_A] = \langle (\bar{Z}_A - x)^2 \rangle = \sigma_A^2$$

Odvisnost meritev (šumov)

$\langle w^2 \rangle = \sigma_w^2$

$$r_B = \alpha r_A + w \; ; \quad w \sim N(0, \sigma_w^2)$$

Odvisen

Neodvisen del $\Rightarrow \langle r_A w \rangle = 0$

$$\sigma_B^2 = \langle r_B^2 \rangle = \langle (\alpha r_A + w)^2 \rangle = \langle \alpha^2 \cdot r_A^2 + w^2 + 2\alpha r_A w \rangle =$$

$$= \alpha^2 \sigma_A^2 + \sigma_w^2$$

$$\Rightarrow \sigma_B^2 = \alpha^2 \sigma_A^2 + \sigma_w^2$$

$$1 = \alpha^2 \frac{\sigma_A^2}{\sigma_B^2} + \frac{\sigma_w^2}{\sigma_B^2}$$

ρ_{AB}^2 korelacijski koeficient

$$\rho_{AB} = \alpha \frac{\sigma_A}{\sigma_B} \; ; \quad 0 \leq \rho_{AB}^2 \leq 1$$

$$\downarrow$$

$$-1 \leq \rho_{AB} \leq 1$$

Polna antikorelacija ; Polna korelacija

↓

Meri odvisnost med dvema naključnima spremenljivkama

$\rho_{AB} = 0$ → Neodvisne naključne spremenljivke

Kovarianca

$$\sigma_{AB} = \langle r_A r_B \rangle$$

Tudi meri odvisnost med dvema spremenljivkama.

$$\sigma_{AB} = \langle r_A r_B \rangle = \langle \alpha r_A^2 + w r_A \rangle = \alpha \sigma_A^2$$

$$\sigma_{AB} = \sigma_{BA}$$

Izrazimo iz korelacijskega koeficienta:

$$\sigma_{AB} = \rho_{AB} \sigma_A \sigma_B$$

Kovarianca je pravzaprav nadpomenka variance. Npr. $A=B$ $\sigma_{AA} = \langle r_A r_A \rangle = \sigma_A^2$.
$\sigma_{AA} = \rho_{AA} \sigma_A \cdot \sigma_A \Rightarrow \rho_{AA} = 1$. Spremenljivka je (logično) popolno korelirana.

Torej poskusimo sedaj optimizirati b

optimal b

$$\frac{\partial \hat{\sigma}^2}{\partial b_0} = -2(1-b_0)\sigma_A^2 + 2 b_0 \sigma_B^2 + 2(1-2b_0)\sigma_{AB} = 0$$

$$b_0 (\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}) = \sigma_A^2 - \sigma_{AB}$$

$$\boxed{b_0 = \frac{\sigma_A^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}}$$

Optimalen b

To lahko vstavimo nazaj in dobimo predpis za Optimalno združevanje:

$$\hat{X} = \bar{Z}_A + \underbrace{\frac{\sigma_A^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}}_{\text{Ojačevalni faktor / ojačanje}} \underbrace{(\bar{Z}_B - \bar{Z}_A)}_{\text{inovacija}}$$

Primer: Kdaj je prepletevanje opt. združ?

Rekli smo, da more biti $b_0 = 1/2$

$$b_0 = \frac{1}{2} = \frac{\sigma_A^2 - \sigma_{AB}}{\sigma_A^2+\sigma_B^2-2\sigma_{AB}} \Rightarrow \sigma_A^2+\sigma_B^2-2\sigma_{AB} = 2\sigma_A^2 - 2\sigma_{AB}$$

$$\Rightarrow \underline{\sigma_A^2 = \sigma_B^2}$$

Poglejmo si še varianco izostrene ocene:

$$\hat{\sigma}^2 = (1-b_0)^2\sigma_A^2 + b_0^2\sigma_B^2 + 2b_0(1-b_0)\sigma_{AB}$$

$$= (1-b_0)\Big[\underbrace{(1-b_0)\sigma_A^2 + b_0\sigma_{AB}}_{u}\Big] + b_0\Big[\underbrace{b_0\sigma_B^2 + (1-b_0)\sigma_{AB}}_{v}\Big] = (*)$$

Poglejmo vmes:

$$(1-b_0) = \frac{\sigma_B^2 - \sigma_{AB}}{\underbrace{\sigma_A^2+\sigma_B^2-2\sigma_{AB}}_{I}} = \frac{\sigma_B^2-\sigma_{AB}}{I}$$

$$u - v = \frac{(\sigma_B^2-\sigma_{AB})\sigma_A^2}{I} + \frac{(\sigma_A^2-\sigma_{AB})\sigma_{AB}}{I} - \left[\frac{(\sigma_A^2-\sigma_{AB})\sigma_B^2}{I} + \frac{(\sigma_B^2-\sigma_{AB})\sigma_{AB}}{I}\right] =$$

$$= \frac{1}{I}\left[\cancel{\sigma_A^2\sigma_B^2} - \cancel{\sigma_A^2\sigma_{AB}} + \cancel{\sigma_A^2\sigma_{AB}} - \cancel{\sigma_{AB}^2} - \cancel{\sigma_A^2\sigma_B^2} + \cancel{\sigma_{AB}\sigma_B^2} - \cancel{\sigma_B^2\sigma_{AB}} + \cancel{\sigma_{AB}^2}\right] = 0$$

Torej lahko enega izpostavimo, ker je $u = v$

$$(*) = \left[(1-b_0)\sigma_A^2 + b_0\sigma_{AB}\right](1-b_0+b_0) =$$

Vstavimo b_0

$$= \frac{1}{I}\left[\sigma_B^2\sigma_A^2 - \cancel{\sigma_{AB}\cdot\sigma_A^2} + \cancel{\sigma_A^2\sigma_{AB}} - \sigma_{AB}^2\right] =$$

$$\sigma_{AB}^2 = \rho_{AB}^2\sigma_A^2\sigma_B^2$$

$$= \frac{1}{I}\left(1-\rho_{AB}^2\right)\sigma_A^2\sigma_B^2$$

$$\Rightarrow \quad \hat{\sigma}^2 = \left(1-\rho_{AB}^2\right)\frac{\sigma_A^2\sigma_B^2}{\sigma_A^2+\sigma_B^2-2\sigma_{AB}} =$$

$$= \left(1-\rho_{AB}^2\right)\left(\frac{1}{\sigma_A^2}+\frac{1}{\sigma_B^2}-\frac{2\rho_{AB}}{\sigma_A\sigma_B}\right)^{-1}$$

V primeru neodvisnosti $\rho_{AB}=0$ pa dobimo:

← ojačevalni faktor

$$\hat{X} = \bar{Z}_A + \frac{\sigma_A^2}{\sigma_A^2+\sigma_B^2}\left(\bar{Z}_B-\bar{Z}_A\right) = \bar{Z}_A + \frac{\hat{\sigma}^2}{\sigma_B^2}\left(\bar{Z}_B-\bar{Z}_A\right)$$

$$\frac{1}{\hat{\sigma}^2} = \frac{1}{\sigma_A^2}+\frac{1}{\sigma_B^2} \quad \text{oz.} \quad \hat{\sigma}^2 = \frac{\sigma_A^2\sigma_B^2}{\sigma_A^2+\sigma_B^2}$$

Poglejmo, da se skupna varianca res manjša

$$\hat{\sigma}^2 = \sigma_A^2 - \frac{\sigma_A^4}{\sigma_A^2+\sigma_B^2}$$

ostrenje!

Kalmanov filter za sledenje konstanti X

- Imamo meritve $z_i \sim N(x, \sigma_i^2)$ $\quad z_i = x + r_i$
- Meritve so neodvisne $\sigma_{ij} = \langle (z_i - x)(z_j - x) \rangle = 0$

Predpostavke

- Recimo, da smo po n-meritvah izračunali izostreno oceno $(\hat{x}_n, \hat{\sigma}_n)$
- V $(n+1)$-trenutku dobimo novo meritev $(z_{n+1}, \sigma_{n+1}^2)$

$$\hat{x}_{n+1} = \hat{x}_n + \frac{\hat{\sigma}_n^2}{\hat{\sigma}_n^2 + \sigma_{n+1}^2}(z_{n+1} - \hat{x}_n)$$

$$\hat{\sigma}_{n+1}^{-2} = \hat{\sigma}_n^{-2} + \sigma_{n+1}^{-2}$$

Kalmanov filter

Gre za rekurzivni zapis. Poglejmo, kako ga inicializiramo:

$n = 0$:
$$\hat{x}_1 = \hat{x}_0 + \frac{\hat{\sigma}_0^2}{\hat{\sigma}_0^2 + \sigma_1^2}(z_1 - \hat{x}_0)$$
$$\hat{\sigma}_1^{-2} = \hat{\sigma}_0^{-2} + \sigma_1^{-2}$$

Pričakujemo:
$\hat{x}_1 = z_1$
$\hat{\sigma}_1^2 = \sigma_1^2$

inicializacija
$$\begin{cases} \hat{x}_0 \ldots \text{ je lahko karkoli} \\ \hat{\sigma}_0^2 \to \infty \end{cases}$$

Vzemimo to inicializacijo:
$$\hat{\sigma}_1^{-2} = \frac{1}{\sigma_1^2} + \frac{1}{\infty} = \sigma_1^{-2}$$
$$\hat{x}_1 = \hat{x}_0 + z_1 - \hat{x}_0 = z_1 \quad \text{(ojačevalni faktor v lim} \to 1)$$

"Druga" oblika opt. združevanja: Uteženo povprečevanje

Predpostavke:

- Imamo set N meritev $\{(z_i, \sigma_i^2)\}_N$
- Iščemo $(\hat{x}, \hat{\sigma}^2)$

Vpeljemo utež: $w_i = \frac{1}{\sigma_i^2}$

Ekvivalentno Kalmanovem filtru, če imamo v naprej podane podatke.

$$\hat{x} = \frac{\sum_{i=1}^{N} w_i z_i}{\sum_{i=1}^{N} w_i}$$

$$\hat{\sigma}^{-2} = \sum_{i=1}^{N} w_i \quad \text{oz.} \quad \hat{\sigma}^2 = \left(\sum_{i=1}^{N} w_i\right)^{-1}$$

Varianca povprečja odvisnih meritev

- Imamo set meritev $\{(z_i, \sigma_i^2)\}_N$ $\quad \langle z_i \rangle = x$, $\quad \langle (z_i - x)^2 \rangle = \sigma_i^2$
- Meritve so odvisne $\sigma_{ij} = \langle (z_i - x)(z_j - x) \rangle \neq 0$

1.) $$\bar{z} = \frac{1}{N} \sum_{i=1}^{N} z_i$$

$$\langle \bar{z} \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle z_i \rangle = \frac{1}{N} \sum_{i=1}^{N} x = \underline{\underline{x}} = \langle \bar{z} \rangle$$

2.) Koliko odstopa povprečje?

$$\sigma_\mu^2 = \langle (\bar{z} - x)^2 \rangle = \left\langle \left(\frac{1}{N} \sum_i z_i - x\right)^2 \right\rangle =$$

$$= \frac{1}{N^2} \left\langle \left(\sum_i z_i - Nx\right)^2 \right\rangle = \frac{1}{N^2} \left\langle \left(\sum (z_i - x)\right)^2 \right\rangle =$$

Torej:

$$\sigma_\mu^2 = \frac{1}{N^2}\left\langle\left(\sum_{i=1}^{N}(z_i-x)^2 + \sum_{i\neq j}\sum (z_i-x)(z_j-x)\right)\right\rangle =$$

$$= \frac{1}{N^2}\left[\sum_{i=1}^{N}\sigma_i^2 + \sum_{i\neq j}\sum \sigma_{ij}\right] =$$

$$= \cancel{\frac{1}{N^2}}$$

$$\sigma_\mu^2 = \frac{1}{N^2}\left[\sum_{i=1}^{N}\sigma_i^2 + 2\cdot\sum_{i<j}\sum \sigma_{ij}\right]$$

npr. kaj če so neodvisne $\sigma_{ij} = 0$

$$\sigma_\mu^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2$$

Če pa velja še $\sigma_i^2 = \sigma^2$ (vse so enake)

$$\sigma_\mu^2 = \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N} \Rightarrow \sigma_\mu = \frac{\sigma}{\sqrt{N}}$$

Primer: GPS meritev nadmorske višine gore

$$h_1 = (2139 \pm 12)\,m$$
$$h_2 = (\underbrace{2130}_{z_i} \pm \underbrace{6}_{\sigma_i})\,m$$

Primerjamo povprečevanje in optimalno združevanje.
Meritvi sta neodvisni

a) Povprečje:
$$\bar{h} = \frac{1}{2}(h_1+h_2) = 2134{,}5\ m$$

$$\sigma_\mu^2 = \frac{1}{4}(12^2+6^2)\,m^2 = \frac{180}{4}\,m^2$$

$$\Rightarrow \sigma_\mu = \sqrt{45}\,m \approx 6{,}7\,m$$

Primer: [Hitrost zvoka v akustičnem resonatorju]

$c_1 = (342 \pm 2)\ \frac{m}{s}$ a) Kalmanov filter za sledenje konstanti (DN)

$c_2 = (343 \pm 4)\ \frac{m}{s}$

$c_3 = (346 \pm 6)\ \frac{m}{s}$ b) Uteženo povprečje

$(\hat{c}, \hat{\sigma}^2) = ?$

Uteži $w_i = \frac{1}{\sigma_i^2}$

$$\hat{c} = \frac{\sum w_i c_i}{\sum w_i}$$

$$\hat{\sigma}^2 = \left(\sum w_i\right)^{-1}$$

$$\hat{\sigma}^2 = \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36}\right)^{-1} \frac{m^2}{s^2} = \frac{144}{49}\left(\frac{m}{s}\right)^2 \approx 2{,}9\left(\frac{m}{s}\right)^2$$

$$\Rightarrow \hat{\sigma} = 1{,}7\ \frac{m}{s}$$ → Res smo izostrili meritev (skupna napaka je manjša od katerekoli posamezne)

$$\hat{c} = \frac{\frac{342}{4} + \frac{343}{16} + \frac{346}{36}}{\frac{1}{4} + \frac{1}{16} + \frac{1}{36}}\ \frac{m}{s} = 342{,}5\ \frac{m}{s}$$

Širjenje napak (po Gaussu)

$u = f(x,y)$

$x: \bar{x}, \sigma_{\bar{x}}^2$ $\quad \sigma_{\bar{x}\bar{y}} \neq 0$

$y: \bar{y}, \sigma_{\bar{y}}^2$

$\bar{u}, \sigma_{\bar{u}}^2 = ?$

$$\sigma_{\bar{u}}^2 = \langle (\bar{u} - u)^2 \rangle =$$

$$= \langle (\overline{f(x,y)} - f(x,y))^2 \rangle = (*)$$

- $f(x,y) = f(\bar{x},\bar{y}) + \left(\frac{\partial f}{\partial x}\right)_{(\bar{x},\bar{y})}(x-\bar{x}) + \left(\frac{\partial f}{\partial y}\right)_{(\bar{x},\bar{y})}(y-\bar{y}) +$

$+ \cancel{\mathcal{O}^2}$ Zanemarimo višje rede

- $$\overline{f(x,y)} = f(\bar{x},\bar{y}) + \left(\frac{\partial f}{\partial x}\right)_{(\bar{x},\bar{y})}\underbrace{\overline{(x-\bar{x})}}_{0} + \left(\frac{\partial f}{\partial y}\right)_{(\bar{x},\bar{y})}\underbrace{\overline{(y-\bar{y})}}_{0}$$

$$\Rightarrow \bar{u} = \overline{f(x,y)} = f(\bar{x},\bar{y})$$

Toraj:

$$(*) = \langle \left(\cancel{f(\bar{x},\bar{y})} - \left[\cancel{f(\bar{x},\bar{y})} + \frac{\partial f}{\partial x}(x-\bar{x}) + \left(\frac{\partial f}{\partial y}\right)(y-\bar{y}) \right] \right)^2 \rangle$$

$$= \langle \left(\frac{\partial f}{\partial x}\right)^2 (\bar{x}-x)^2 + \left(\frac{\partial f}{\partial y}\right)^2 (\bar{y}-y)^2 + 2\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)(\bar{x}-x)(\bar{y}-y) \rangle =$$

$$\Rightarrow \quad \sigma_{\bar{u}}^2 = \left(\frac{\partial f}{\partial x}\right)^2_{(\bar{x},\bar{y})} \sigma_{\bar{x}}^2 + \left(\frac{\partial f}{\partial y}\right)^2_{(\bar{x},\bar{y})} \sigma_{\bar{y}}^2 + 2\left(\frac{\partial f}{\partial x}\right)_{(\bar{x},\bar{y})} \left(\frac{\partial f}{\partial y}\right)_{(\bar{x},\bar{y})} \sigma_{\bar{x}\bar{y}}$$

Primer:

i) $u = f(x,y) = ax + by$ $\qquad \left(\frac{\partial f}{\partial x}\right) = a$

$\sigma_{\bar{x}\bar{y}} = 0$ neodvisni $\qquad \left(\frac{\partial f}{\partial y}\right) = b$

$$\Rightarrow \quad \sigma_{\bar{u}}^2 = a^2 \sigma_{\bar{x}}^2 + b^2 \sigma_{\bar{y}}^2$$

ii) $u = g(x,y) = A x^{\delta} y^{f}$; $\bar{u} = g(\bar{x},\bar{y}) = A \bar{x}^{\delta} \bar{y}^{f}$

$$\left(\frac{\partial g}{\partial x}\right)_{(\bar{x},\bar{y})} = A\delta \bar{x}^{\delta-1} \bar{y}^{f} = \delta \frac{\bar{u}}{\bar{x}}$$

$$\left(\frac{\partial g}{\partial y}\right)_{(\bar{x},\bar{y})} = A f \bar{x}^{\delta} \bar{y}^{f-1} = f \frac{\bar{u}}{\bar{y}}$$

$$\Rightarrow \quad \sigma_{\bar{u}}^2 = \delta^2 \frac{\bar{u}^2}{\bar{x}^2} \sigma_{\bar{x}}^2 + f^2 \frac{\bar{u}^2}{\bar{y}^2} \cdot \sigma_{\bar{y}}^2 \quad \Big/ \cdot \frac{1}{\bar{u}^2}$$

$$\Rightarrow \quad \left(\frac{\sigma_{\bar{u}}}{\bar{u}}\right)^2 = \delta^2 \underbrace{\left(\frac{\sigma_{\bar{x}}}{\bar{x}}\right)^2}_{\text{relativna napaka}} + f^2 \left(\frac{\sigma_{\bar{y}}}{\bar{y}}\right)^2$$

Normalna (Gaussova) porazdelitvena funkcija

$$z \sim N(\mu, \sigma^2) = \frac{dP}{dz}(z,\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

Zanima nas verjetnost P:

$$P(a<z<b) = \int_a^b \frac{dP}{dz}\,dz = \text{ni analitično}$$

Kumulativna f. normalne porazdelitve:

$$\tilde{F}(x,\mu,\sigma^2) = \int_{-\infty}^{x} \frac{dP}{dz}\,dz$$

Torej:

$$P(a<z<b) = \tilde{F}(b,\mu,\sigma^2) - \tilde{F}(a,\mu,\sigma^2)$$

To ni praktično ker ni tabelirano za vse vrednosti parametrov σ^2, μ.

Uvedemo standardizirano normalno porazdelitev

$$\cancel{\mu =}\ u = \frac{z-\mu}{\sigma}\ ;\ du = \frac{dz}{\sigma}$$ } Transformacijo, da standardizira katerokoli normalno porazdelitev

$$\frac{dP}{du} = \frac{dP}{dz}\frac{dz}{du} = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} = \underline{N(0,1)}$$

Standardizirana normalna porazdelitev

Tako je kumulativna funkcija standardizirano normalne porazdelitve:

$$F(x) = \int_{-\infty}^{x} \frac{dP}{du}\,du$$

$N(0,1)$

Vrednosti te funkcije so tabelirane

$F(-x) = 1 - F(x)$

$F(-\infty) = 0$

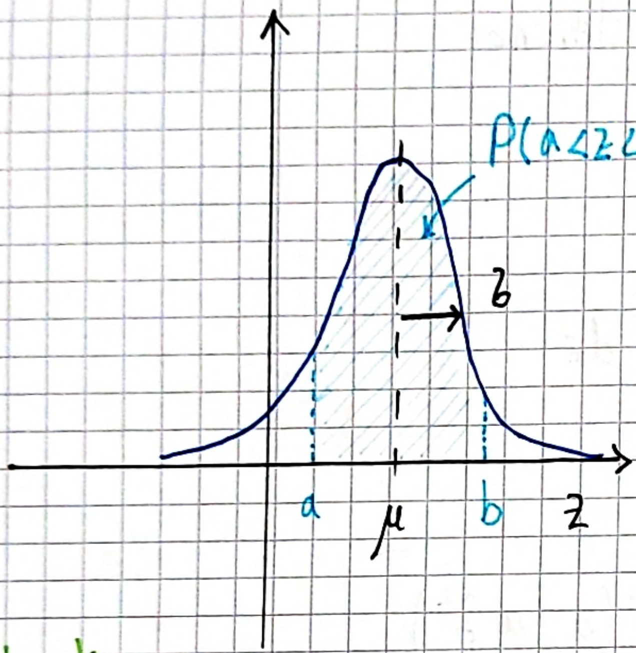
$F(+\infty) = 1$

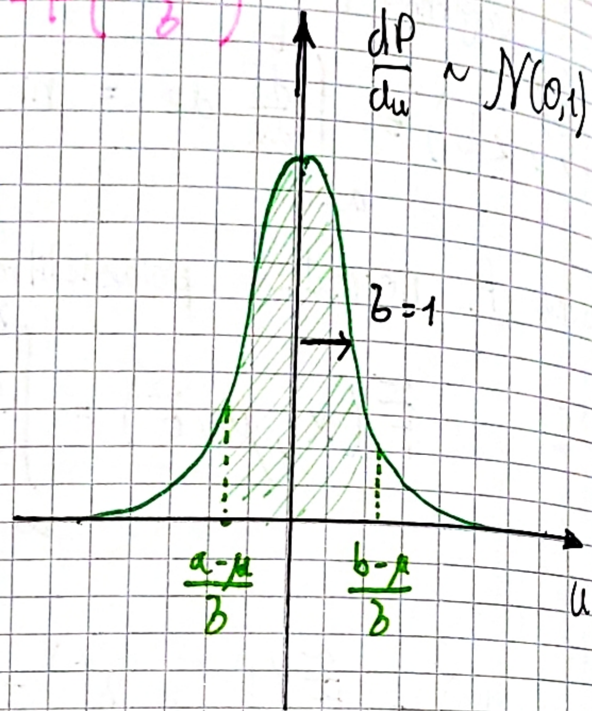
$F(0) = \frac{1}{2}$

Tako je torej:

$$P(a<z<b) = \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} \frac{dP}{du}\, du =$$

$$\Rightarrow P(a<z<b) = F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$





Intervali

$$P(\mu - n\sigma < z < \mu + n\sigma) = F(n) - F(-n) = 2F(n) - 1$$

a) $n = 1$ interval $\pm\sigma$

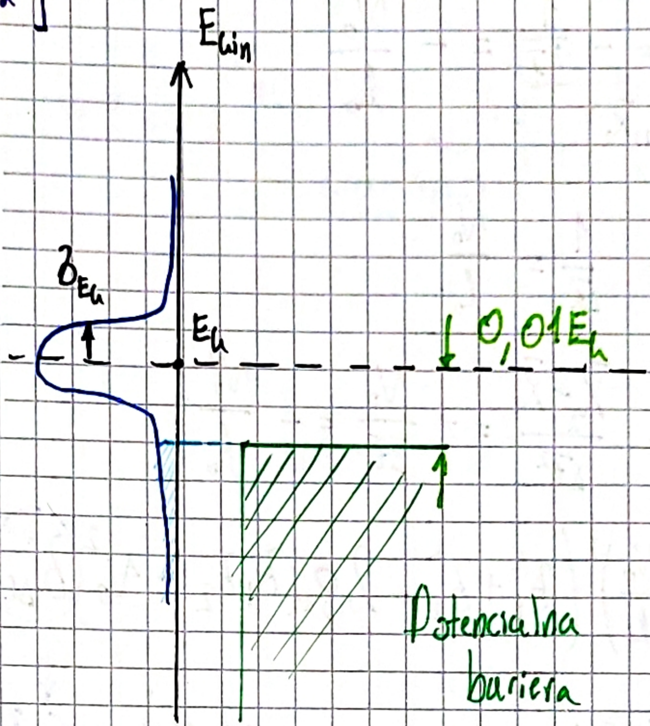
$$P(\mu-\sigma < z < \mu+\sigma) = 2F(1) - 1 = 0{,}68 = \underline{\underline{68\%}}$$

b) $n=2$ interval $\pm 2\sigma$

$$P(\mu-2\sigma < z < \mu+2\sigma) = 2F(2) - 1 = \underline{\underline{95\%}}$$

c) n 3 $P \approx \underline{\underline{99{,}7\%}}$

Primer: [Verjetnost za odboj delca]



E_{kin}

$$\delta_{E_k} = \frac{\sigma_{E_k}}{E_k} = 4\%$$

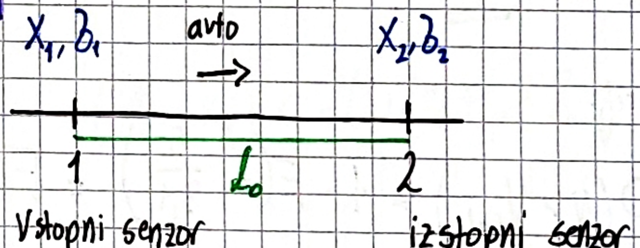
$$P = (E_{kin} < 0{,}99E_k) =$$

$$= F\left(\frac{0.99E_k - E_k}{0{,}04E_k}\right) =$$

$$= F\left(\frac{-0{,}01}{0{,}04}\right) = F(-0{,}25) =$$

$$= 1 - F(0{,}25) = 1 - 0{,}5987 \doteq 40\%$$

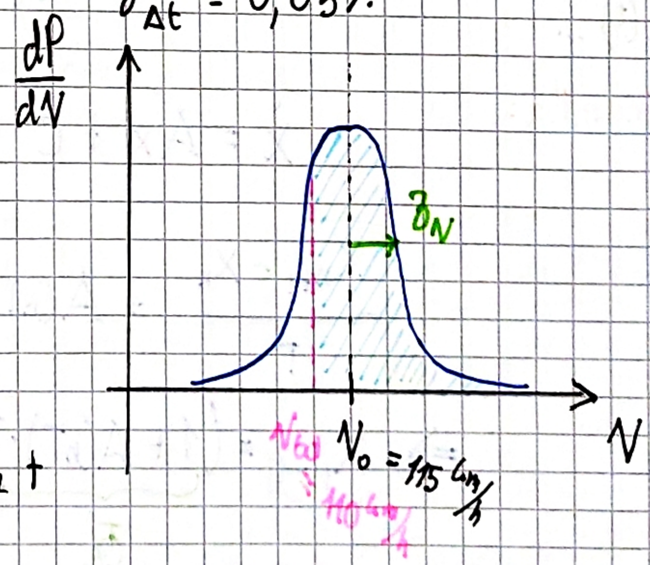
Koldvijska: [Sekcijsko merjenje hitrosti]



$l_0 = 1km$, $\sigma_1 = 10m$, $\sigma_2 = 20m$, $\rho_{12} = -0{,}6$

Kolikšen delež voznikov s hitrostjo $v_0 = 115\,km/h$ bodo "ujeli", če je toleranca hitrosti $v_{tol} = 110\,km/h$?

Merimo Δt z natančnostjo $\delta_{\Delta t} = 0{,}05\%$.



$$P(v > v_{tol}) = 1 - F\left(\frac{v_{tol} - v_0}{\sigma_v\,?}\right)$$

$$\sigma_v: \; v = \frac{x_2 - x_1}{\Delta t} = f(x_1, x_2, \Delta t)$$

$$l_0 = \bar{x}_2 - \bar{x}_1$$

$$\sigma_v^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_2^2 + 2\left(\frac{\partial f}{\partial x_1}\right)\left(\frac{\partial f}{\partial x_2}\right) \underbrace{\rho_{12}\sigma_1\sigma_2}_{\sigma_{1,2}} +$$

$$+ \left(\frac{\partial f}{\partial \Delta t}\right)^2 \sigma_{\Delta t}^2$$

$$\left(\frac{\partial f}{\partial x_1}\right)_{(\bar{x}_1,\bar{x}_2)} = -\frac{1}{\overline{\Delta t}} = -\frac{N_0}{l_0}$$

lahko izračunamo

$$\sigma_{\Delta t} = \left(\frac{\sigma_{\overline{\Delta t}}}{\overline{\Delta t}}\right) = 0{,}05\,\%$$

$$\sigma_{\Delta t} = \sigma_{\overline{\Delta t}}\,\overline{\Delta t}$$

$$\left(\frac{\partial f}{\partial x_2}\right) = \frac{1}{\overline{\Delta t}} = \frac{N_0}{l_0}$$

$$\left(\frac{\partial f}{\partial \Delta t}\right) = -\frac{\bar{X}_2-\bar{X}_1}{\overline{\Delta t}^2} = -\frac{N_0}{\overline{\Delta t}} = -\frac{N_0^2}{l_0}$$

$$\sigma_v^2 = \left(\frac{N_0}{l_0}\right)^2\left(\sigma_1^2+\sigma_2^2-2\rho_{12}\sigma_1\sigma_2+N_0^2\,\sigma_{\Delta t}^2\right) =$$

$$= \left(\frac{N_0}{l_0}\right)^2\left(\sigma_1^2+\sigma_2^2-2\rho_{12}\sigma_1\sigma_2+\delta_{\Delta t}^2\,l_0^2\right) \qquad \delta_{\Delta t}^2\left(\frac{l_0}{N_0}\right)^2$$

$$= \underline{\underline{3{,}1}}\ \mathrm{km/h}$$

Iz tabel

Torej:

$$P(v>v_{tol}) = 1-F\left(-\frac{5}{3{,}1}\right) = F\left(\frac{5}{3{,}1}\right) \simeq F(1{,}61) \approx 94{,}5\,\%$$

Kalmanov filter za sledenje skalarni spremenljivki

$$X_n = X(nT)$$

$X(t)$ ...

Dinamika:

$$\dot{x} = Ax + c \quad (\text{zvezni zapis})$$

$$\frac{x_{n+1}-x_n}{T} = A(nT)\,x_n + c(nT)$$

$$\Rightarrow\ x_{n+1} = \underbrace{(1+A(nT)\cdot T)}_{\Phi_n}\,x_n + \underbrace{c(nT)\cdot T}_{c_n}$$

$$x_{n+1} = \Phi_n\,x_n + c_n \quad (\text{diskretni zapis})$$

0) Poiščemo dinamiko za X

$$X_{n+1} = \Phi_n X_n + c_n + \Gamma_n \omega_n$$ (Kalmanova dinamika)

dinamičen šum

(ker Φ_n, c_n ne poznamo natančno)

Poznamo pa $(\hat{x}_n, \hat{\sigma}_n^2)$

Beli šum $\begin{cases} \langle W_n \rangle = 0 \\ \langle W_n W_{n'} \rangle = \underline{Q_n} \delta_{nn'} \end{cases}$

Varianca dinamičnega šuma

1.) V (n+1)-trenutki dobimo meritev

$(z_{n+1}, \sigma_{n+1}^2)$... "dobimo" meritev

Z dinamiko od 0) izračunajmo oceno:

ne popravljena → $$\bar{X}_{n+1} = \Phi_n \cdot \hat{X}_n + c_n$$

$$\bar{\sigma}_{n+1}^2 = \langle (\bar{X}_{n+1} - X_{n+1})^2 \rangle = \langle (\Phi_n \hat{x}_n + c_n - (\Phi_n x_n + c_n + \Gamma_n W_n))^2 \rangle =$$

$$= \Phi_n^2 \hat{\sigma}_n^2 + \Gamma_n^2 \cdot Q_n$$

2.) Ostrenje napovedi z meritvijo

$$\hat{X}_{n+1} = \bar{X}_{n+1} + \frac{\hat{\sigma}_{n+1}^2}{\sigma_{n+1}^2} (z_{n+1} - \bar{X}_{n+1})$$

$$\hat{\sigma}_{n+1}^2 = \bar{\sigma}_{n+1}^2 - \frac{(\bar{\sigma}_{n+1})^2}{\bar{\sigma}_{n+1}^2 + \sigma_{n+1}^2}$$

Nove oznake:

$$\hat{\sigma}_n^2 = P_n \qquad \bar{\sigma}_n^2 = M_n \qquad \sigma_n^2 = R_n$$

$$\Rightarrow M_{n+1} = \Phi_n^2 P_n + \Gamma_n^2 Q_n \qquad \hat{x}_{n+1} = \bar{x}_{n+1} + \frac{P_{n+1}}{R_{n+1}} (z_{n+1} - \bar{x}_{n+1}) ; \; K_{n+1} = \frac{P_{n+1}}{R_{n+1}}$$

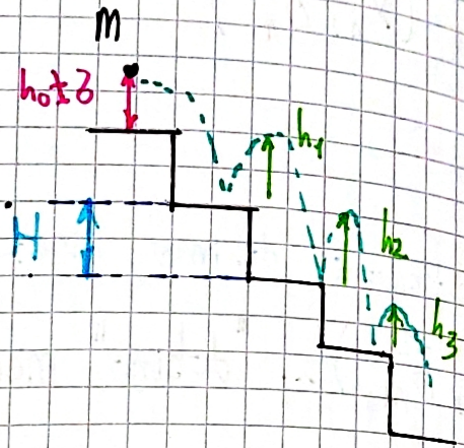
$$P_{n+1} = M_{n+1} - \frac{M_{n+1}^2}{M_{n+1} + R_{n+1}}$$

Npr. če ne merimo več:

Z_{n+1} ... kadkoli $\Rightarrow$ $\hat{X}_{n+1} = \bar{X}_{n+1}$

$R_{n+1} \to \infty$ $\qquad P_{n+1} = M_{n+1}$

Primer: [Kroglica na stopnicah]



Pri vsakem odboju se ohrani delež energije γ.

$\hat{h}_5 = ?$

$\sigma_5^2 = ?$

$$\bar{H} = H + \eta$$

a) Natančno poznane stopnice $\langle \eta^2 \rangle = 0$

Sledimo maksimalnim višinam po posameznem odboju

$$R_{n+1} = \Phi_n \cdot h_n + C_n + \Gamma_n w_n$$

Za en odboj:

$n=0$
$$\gamma mg(h_0 + \bar{H}) = mgh_1$$
$$\to h_1 = \gamma h_0 + \gamma \bar{H}$$

$n=1$
$$\gamma mg(h_1 + \bar{H}) = mgh_2 \qquad h_{n+1} = \gamma h_n + \gamma \bar{H}$$
$$\to h_2 = \gamma h_1 + \gamma \bar{H} \quad \ldots$$

Razberemo koeficiente:

$$\Phi_n = \gamma \quad ; \quad C_n = \sqrt{H}$$

Dinamiko za ta problem znamo dobro določit, ker ni dinamičnega šuma.

b) $\bar{H} = H + \mu$; $\langle \mu^2 \rangle = \sigma_\mu^2$

$h_{n+1} = \delta h_n + \delta \bar{H}$

$= \delta h_n + \delta (H + \eta) = \delta h_n + \delta H \underbrace{(\delta \mu)}_{\text{šum}} \Rightarrow \Gamma_n \omega_n = \delta \cdot \eta$

$\langle \Gamma_n \cdot w_n \cdot \Gamma_{n'} w_{n'} \rangle = \cancel{F_n} \cancel{F_{n'}} \Gamma_{n} \Gamma_{n'} \langle w_n w_{n'} \rangle =$

$= \Gamma_{nn'} + Q\delta - \Gamma_n^2 Q$

$\mu = \eta$ $= \langle \delta \mu_n, \delta \mu_n \rangle = \delta^2 \langle \eta_{\mu} \eta_{\mu} \rangle = \boxed{\delta^2 \sigma_H^2 = \Gamma_n^2 Q_n}$

1.) Dinamiko smo poiskali. Dajmo napoved

$\bar{h}_{n+1} = \Phi_n \hat{h}_n + c_n = \delta \hat{h}_n + \delta H$; $M_{n+1} = \Phi_n^2 P_n + \Gamma_n^2 Q_n =$

$= \delta^2 P_n + \delta^2 \sigma_H$

2.) Oceniti naprej ne moremo, ker ne merimo

$$\left.\begin{aligned} \hat{h}_{n+1} &= \bar{h}_{n+1} \\ P_{n+1} &= M_{n+1} \end{aligned}\right\} \text{ker } \underline{\text{ne}} \text{ merimo}$$

Torej za a) Primer:

$\hat{h}_5 = ?$;

$\hat{h}_1 = \delta \hat{h}_0 + \delta H$

$\hat{h}_2 = \delta h_1 + \delta H = \delta(\delta \hat{h}_0 + \delta H) + \delta H$

$\hat{h}_3 = \delta \cdot \hat{h}_2 + \delta H = \delta(\delta(\delta \cdot \hat{h}_0 + \delta H) + \delta H) + \delta H$

$\vdots$

$\hat{h}_n = \delta^n \hat{h}_0 + H \sum_{i=1}^{n} \delta^i$

$$s_n = \sum_{i=1}^{n} \delta^i = \delta\Big(1 + \underbrace{\sum_{i=1}^{n-1} \delta^i}_{(s_n - \delta^n)}\Big) = \delta(1 + s_n - \delta^n)$$

$$\Rightarrow \quad s_n = \delta \frac{(1-\delta^n)}{(1-\delta)}$$ Samo lepotna poprava

$$\Rightarrow \quad \hat{h}_n = \delta^n \hat{h}_0 + H \frac{\delta}{1-\delta}(1-\delta^n)$$

$$\left.\begin{array}{l} \hat{h}_5 ; \; n = 5 \\ \delta = 1/2 \\ \hat{h}_0 = 0 \end{array}\right\} \Rightarrow \hat{h}_5 = H\left(1 - \frac{1}{2^5}\right) = H \cdot \frac{31}{32}$$

Še izračunajmo varianco P_5

$$P_1 = M_1 = \underline{\underline{\delta^2 \cdot P_0}}$$

$$P_2 = M_2 = \delta^2 P_1 = \delta^2(\delta^2 \cdot P_0)$$

$$P_n = \delta^{2n} P_0$$

Za $n = 5$:

$$P_5 = (\delta^2)^5 \cdot P_0 = \hat{\sigma}_5^2$$

$$\hat{\sigma}_5 = \sqrt{P_5} = \delta^5 \hat{\sigma}_0 \underset{\delta = 0.5}{=} \frac{\hat{\sigma}_0}{32}$$

kovarianca med šumoma

$\langle m_i \rangle = 0$

šum $\langle m_i m_j \rangle = \delta_{ij}$

Vektorske spremenljivke
(kovariančna matrika)

$x_1, \dots, x_N$

$$\bar{x}_i = x_i + m_i$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \dots \text{ ne poznamo}$$

$$\vec{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix}$$

$$\vec{\bar{x}} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_N \end{pmatrix} \dots \text{ Ocena za } \vec{x}$$

$$\Rightarrow \quad \vec{\bar{x}} = \vec{x} + \vec{m}$$

Uvedemo kovariančno matriko:

$$\underline{\underline{M}} = \langle \vec{m} \cdot \vec{m}^T \rangle = \langle (\vec{\bar{x}} - \vec{x})(\vec{\bar{x}} - \vec{x})^T \rangle$$

$M^T = M$ (simetrična)

npr. Kovariančna matrika izostrene ocene $\hat{\vec{x}}$:

$$\underline{\underline{P}} = \langle \vec{p} \cdot \vec{p}^T \rangle = \langle (\hat{\vec{x}} - \vec{x})(\hat{\vec{x}} - \vec{x})^T \rangle$$

$$(M)_{ij} = \sigma_{ij} = \langle m_i m_j \rangle$$

$$(M)_{ii} = \sigma_{ii} = \langle m_i^2 \rangle = \sigma_i^2$$

Enačba za širjenje napak za vektorske spremenljivke

$$\vec{\bar{x}} = \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_N \end{pmatrix}$$; poznamo kovariančno matriko $\underline{\underline{M}}$

$$\vec{u} = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_k(\vec{x}) \end{pmatrix}$$

Zanima nas U, kovariančna matrika ocen $\vec{u}$.

$$U = \langle (\vec{\bar{u}} - \vec{u})(\vec{\bar{u}} - \vec{u})^T \rangle$$

$\vec{\bar{u}} - \vec{u}$:

$$\vec{\bar{u}} \approx \begin{pmatrix} f_1(\vec{\bar{x}}) \\ \vdots \\ f_k(\vec{\bar{x}}) \end{pmatrix} ; \quad \vec{u} = \vec{\bar{u}} + J_{\vec{u}}(\vec{\bar{x}}) \, (x - \vec{\bar{x}})$$

Jacobijeva matrika

Za vektor funkcij u, razvitih okoli $\vec{\bar{x}}$

$$\vec{\mu} - \vec{\bar{\mu}} = \vec{\mu} - \left(\vec{\bar{\mu}} + J_N(\vec{\bar{x}})(\vec{x} - \vec{\bar{x}})\right) =$$

$$= J_u(\vec{\bar{x}})(\vec{x} - \vec{\bar{x}})$$

$$\Rightarrow U = \langle J_{\vec{u}}(\vec{\bar{x}})(\vec{x} - \vec{\bar{x}})(\vec{x} - \vec{\bar{x}})^T J_{\vec{u}}^T(\vec{\bar{x}}) \rangle$$

$$\Rightarrow U = J_{\vec{u}}(\vec{\bar{x}}) \cdot M \cdot J_{\vec{u}}^T(\vec{\bar{x}})$$

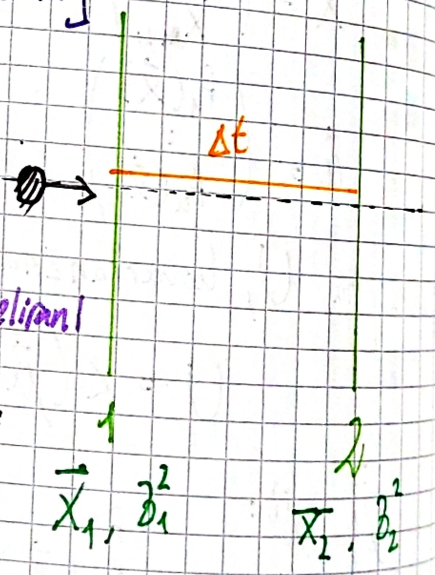
Kjer je Jacobijeva matrika:

$$J_{\vec{u}}(\vec{\bar{x}}) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1}, & \dots, & & \frac{\partial f_n}{\partial x_n} \end{pmatrix}\right|_{\vec{\bar{x}}}$$

Primer: [Določanje hitrosti delca s pomočjo časa preleta]

$$\vec{\bar{x}} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}; \quad M = \begin{pmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{pmatrix}$$

podano, da nista korelirani



Izračunaj kov. matriko, za primer:

$$\vec{u} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vec{u} \end{pmatrix}$$

$U = ?$

$$v = \frac{x_2 - x_1}{\Delta t}$$

$$\bar{v} = \frac{\bar{x}_2 - \bar{x}_1}{\Delta t}$$

$$\vec{u} = \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ f_3(\vec{x}) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \frac{x_2 - x_1}{\Delta t} \end{pmatrix}$$

Kalmanov filter za vektorske spremenljivke
(v diskretni sliki)

$x_1, x_2, \dots, x_N$:

$$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}$$

Dinamični šum

Dinamika: $$\vec{x}_{n+1} = \Phi_n \vec{x}_n + \vec{c}_n + \Gamma_n \vec{w}_n$$

Predpostavimo, da imamo $(\hat{\vec{X}}_n, P_n)$ kovariančna matrika

i) Napoved:

$$\bar{\vec{x}}_{n+1} = \Phi_n \hat{\vec{x}}_n + c_n$$

$$M_{n+1} = \Phi_n P_n \Phi_n^T + \Gamma_n Q_n \Gamma_n^T$$

→ Tu se konča če ne merimo!

ii) meritev $(\vec{z}_{n+1}, R_{n+1})$

$$\vec{z}_{n+1} = H \vec{x}_{n+1} + \vec{r}_{n+1}$$

Okenska matrika

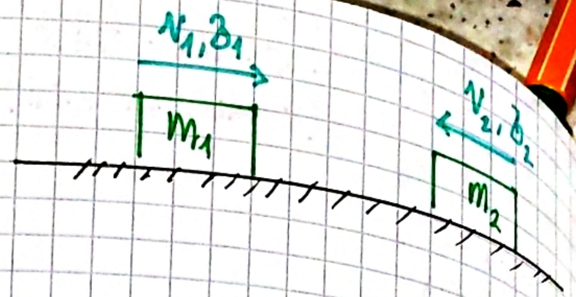
iii) Ostrenje

$$\hat{\vec{x}}_{n+1} = \bar{\vec{x}}_{n+1} + K_{n+1}(\vec{z}_{n+1} - H\bar{\vec{x}}_{n+1}) \quad ; \quad K_{n+1} = P_{n+1} H^T R_{n+1}^{-1}$$

$$P_{n+1}^{-1} = M_{n+1}^{-1} + H^T R_{n+1}^{-1} H$$

$$P_{n+1} = M_{n+1} - M_{n+1} H^T (H M_{n+1} H^T + R_{n+1})^{-1} H M_{n+1}$$

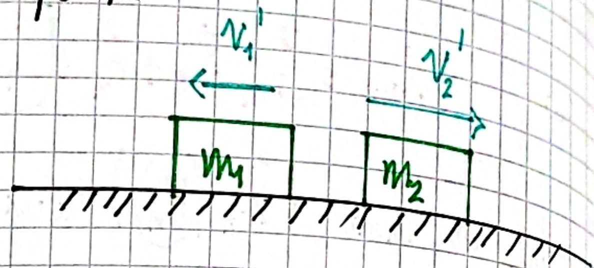
Primer: [Popolnoma prožni trk dveh teles]



Pred trkom recimo: $\sigma_{12} = 0$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad P = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

Zanima nas kovariančna matrika za hitrosti po trku.



$$\vec{v}\,' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}; \quad P' = ?$$

Tu nič ne merimo, samo dinamiko gledamo.

$$\vec{v}\,' = \Phi \vec{v} + \vec{c} + 0$$

Dinamiko lahko natančno opišemo

Ker ne merimo:

$$\hat{\vec{v}}\,' = \tilde{\vec{v}}\,' = \Phi \hat{\vec{v}} + \vec{c}$$

napoved hitrosti po trku

$$P' = M' = \Phi P \Phi^T = ? \quad ; \quad \Phi = ?$$

Rešimo problem v težiščnem sistemu:

$$v_T = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$u_1 = v_1 - v_T$$
$$u_2 = v_2 - v_T$$

a) Ohranitev gibalne količine:

$$\sum p_i = 0 \Rightarrow m_1 u_1 + m_2 u_2 = 0 \quad \rightarrow \quad u_2 = -\frac{m_1}{m_2} u_1$$
$$m_1 u_1' + m_2 u_2' = 0 \qquad u_2' = -\frac{m_1}{m_2} u_1'$$

b) Ohranitev kinetične energije

$$\sum_i W_{k,i} = \sum_i W_{ki}' \Rightarrow \cancel{\frac{1}{2}} m u_1^2 + \cancel{\frac{1}{2}} m_2 u_2^2 = \cancel{\frac{1}{2}} m_1 u_1'^2 + \cancel{\frac{1}{2}} m u_2'^2$$

Vstavimo izraženo v b):

$$u_1^2\left(m_1 + \frac{m_1^2}{m_2}\right) = u_1'^2\left(m_1 + \frac{m_1^2}{m_2}\right)$$

$$u_1'^2 = u_1^2 \Rightarrow u_1' = \pm u_1 \to -u_1$$

Podobno

$$u_2'^2 = u_2^2 \Rightarrow u_2' = \pm u_2 \to -u_2$$

V laboratorijskem sistemu:

$$(v_1' - v_T) = -(v_1 - v_T)$$

$$v_1' = -v_1 + 2v_T =$$

$$= -v_1 + \frac{2m_1 v_1 + 2m_2 v_2}{m_1 + m_2} =$$

$$\Rightarrow v_1' = \frac{v_1(m_1 - m_2) + 2m_2 v_2}{m_1 + m_2}$$

$$\cancel{\vec{v}' = \Phi \vec{v} + \vec{c}}$$

$$\cancel{\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 2v_T \end{pmatrix}}$$

Narobe ker je v_T odvisen od v_1 in v_2

Za drugo telo pa je:

$$(v_2' - v_T) = -(v_2 - v_T)$$

$$v_2' = -v_2 + 2v_T$$

$$\vdots$$

$$v_2' = \frac{v_2(m_2 - m_1) + 2m_1 v_1}{m_2 + m_1}$$

Vpeljemo $\mu = \frac{m_2}{m_1}$:

$$v_1' = \frac{v_1(1-\mu) + 2\mu v_2}{1+\mu} \qquad v_2' = \frac{v_2(\mu - 1) + 2v_1}{1+\mu}$$

$$\vec{V}' = \Phi \vec{V} + \vec{c}$$

$$\begin{pmatrix} V_1' \\ V_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 1-\mu & 2\mu \\ 2 & -(1-\mu) \end{pmatrix} \frac{1}{1+\mu}}_{\Phi} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{c}}$$

Naslednji korak je:

$$P' = \Phi P \Phi^T \, ; \quad P = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

Začasno vpeljemo:

$$\Phi = A \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

$A = 1/_{1+\mu}$

$a = 1-\mu$

$b = 2\mu$

$c = 2$

$$P' = A^2 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} =$$

$$= A^2 \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} a\sigma_1^2 & c\sigma_1^2 \\ b\sigma_2^2 & -a\sigma_2^2 \end{pmatrix} =$$

$$= A^2 \begin{pmatrix} a^2\sigma_1^2 + b^2\sigma_2^2 & ac\sigma_1^2 - ab\sigma_2^2 \\ ac\sigma_1^2 - ab\sigma_2^2 & c^2\sigma_1^2 + a^2\sigma_2^2 \end{pmatrix}$$

Torej:

$$P' = \frac{1}{(1+\mu)^2} \begin{pmatrix} (1-\mu)^2\sigma_1^2 + 4\mu^2\sigma_2^2 & 2(1-\mu)[\sigma_1^2 - \mu\sigma_2^2] \\ 2(1-\mu)[\sigma_1^2 - \mu\sigma_2^2] & 4\sigma_1^2 + (1-\mu)^2\sigma_2^2 \end{pmatrix}$$

Posebni primeri

a) $b_1 = b_2 = b$

$$P' = \frac{b^2}{(1+\mu)^2}\begin{pmatrix} (1-\mu)^2 + 4\mu^2 & 2(1-\mu)^2 \\ 2(1-\mu)^2 & 4+(1-\mu)^2 \end{pmatrix}$$

a2) $m_1 = m_2$ še dodatno $\Rightarrow \mu = 1$

$$P' = \frac{b^2}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix} = P$$

b) Samo $m_1 = m_2$; $b_1 \neq b_2$

$\mu = 1$

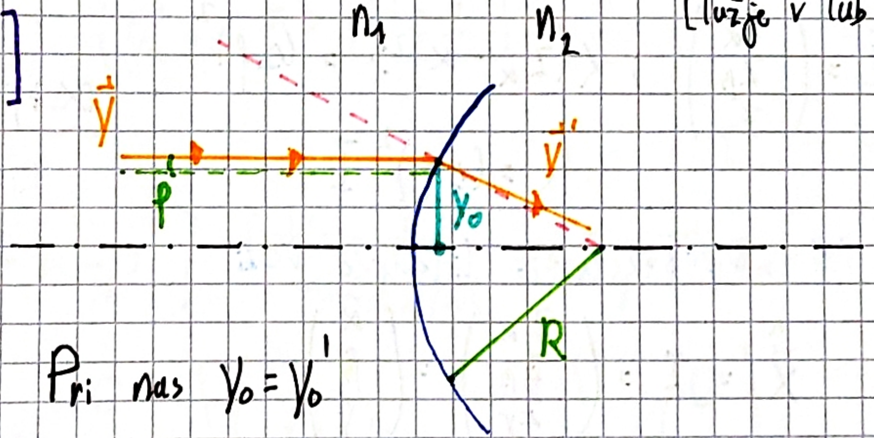
$$P' = \frac{1}{4}\begin{pmatrix} 4b_2^2 & 0 \\ 0 & 4b_1^2 \end{pmatrix} = \begin{pmatrix} b_2^2 & 0 \\ 0 & b_1^2 \end{pmatrix}$$

DN popdroma

Neprožni trk

(lužje v lab sist.)

Primer: [Geometrijska optika]



Žarek opišemo z dvema parametroma

Pri nas $y_0 = y_0'$

$$\vec{y} = \begin{pmatrix} y_0 \\ \operatorname{tg}\varphi \end{pmatrix} \approx \begin{pmatrix} y_0 \\ \varphi \end{pmatrix}$$

$$\vec{y}' = \begin{pmatrix} y_0' \\ \varphi' \end{pmatrix}$$

To lahko opišemo s prehodnimi matrikami $\vec{y}' = \underline{\underline{A}}\,\vec{y}$

$$A = \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(\frac{n_1}{n_2} - 1\right) & \frac{n_1}{n_2} \end{pmatrix}$$

Za konveksno mejo

Ravna meja $R \to \infty$:

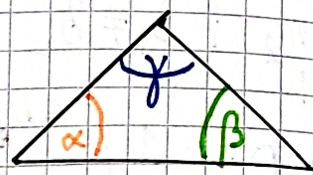
$$B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix}$$

Konkavna meja $R \to -R$

$$C = \begin{pmatrix} 1 & 0 \\ \frac{1}{R}\left(1 - \frac{n_1}{n_2}\right) & \frac{n_1}{n_2} \end{pmatrix}$$

Kalmanov filter z linearnimi vezmi

Primer [Trikotnik]



$$\vec{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\vec{z} = \begin{pmatrix} z_\alpha \\ z_\beta \\ z_\gamma \end{pmatrix} \; ; \; \langle (z_\alpha - \alpha)^2 \rangle = (\Delta\varphi)^2$$

$$R = (\Delta\varphi)^2 I$$

$$\underbrace{\alpha + \beta + \gamma = \pi}_{\text{Linearna vez}}$$

a) ne upoštevamo linearne vezi

$$\hat{\vec{x}} = \vec{z} = \begin{pmatrix} z_\alpha \\ z_\beta \\ z_\gamma \end{pmatrix} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix}$$

$$P = R = (\Delta\varphi)^2 I$$

b) Upoštevamo linearne vezi

$$\Phi = I \qquad \bar{\vec{x}}_{n+1} = \Phi \cdot \hat{\vec{x}}_n + \vec{c}_n$$

$$\vec{x}_{n+1} = I \vec{x}_n \qquad M_{n+1} = \Phi P_n \Phi^T$$

$$\hat{\vec{x}}_{n+1} = \bar{\vec{x}}_{n+1} + P H^T_{n+1} R^{-1}_{n+1} \left(\vec{z}_{n+1} - H \bar{\vec{x}}_{n+1} \right)$$

$$P^{-1}_{n+1} = M^{-1}_{n+1} + H^T R^{-1}_{n+1} H$$

Ne poznamo ničesar na začetku (ne poznamo napovedi)

$$M \to \infty$$

Pri nas je torej

$$P^{-1} = H^T R^{-1} H$$

$$P P^{-1} = I = P H^T R^{-1} H$$

$$\hat{\vec{X}} = \bar{\vec{X}} + P H^T R^{-1} \vec{Z} - \underbrace{P H^T R^{-1} H}_{I} \bar{\vec{X}}$$

$$\Rightarrow \hat{\vec{X}} = P H^T R^{-1} \vec{Z}$$

$$P^{-1} = H^T R^{-1} H$$

$$\vec{Z} = H \vec{X} + \vec{r}$$

$$\vec{Z} = \begin{pmatrix} z_\alpha \\ z_\beta \\ z_\gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{pmatrix} \Rightarrow \boxed{H = I}$$

Upoštevajmo sedaj še vez:

$$\gamma = \pi - \alpha - \beta \quad \Big\} \text{ Problem dveh spremenljivk} \to z_\gamma = \pi - z_\alpha - z_\beta$$

$$\underbrace{\begin{pmatrix} z_\alpha \\ z_\beta \\ z_\gamma \end{pmatrix}}_{\vec{Z}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}}_{H} \underbrace{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}_{\vec{X}} + \begin{pmatrix} r_\alpha \\ r_\beta \\ r_\gamma \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} z_\alpha \\ z_\beta \\ z_\gamma - \pi \end{pmatrix}}_{\tilde{\vec{Z}}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}}_{H} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} r_\alpha \\ r_\beta \\ r_\gamma \end{pmatrix}$$

$$P^{-1} = H^T R^{-1} H = \qquad R = (\Delta\varphi)^2 I \Rightarrow R^{-1} = (\Delta\varphi)^{-2} I$$

$$= (\Delta\varphi)^{-2} H^T H =$$

$$= (\Delta\varphi)^{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} = (\Delta\varphi)^2 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Za inverz 2×2 matriko:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Torej:

$$P = \frac{(\Delta\varphi)^2}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ldots \begin{pmatrix} \sigma^2_{\hat\alpha} & \sigma_{\hat\alpha\hat\beta} \\ \sigma_{\hat\alpha\hat\beta} & \sigma^2_{\hat\beta} \end{pmatrix}$$

Ob upoštevanju vezi se kovarianca zmanjša

Ocenimo še varianco γ

$$\hat\gamma = \pi - \hat\alpha - \hat\beta$$

$$\gamma = \pi - \alpha - \beta$$

$$(\hat\gamma - \gamma) = -(\hat\alpha - \alpha) - (\hat\beta - \beta)$$

$$\sigma^2_\gamma = \langle (\hat\gamma - \gamma)^2 \rangle = \langle (\hat\alpha - \alpha)^2 + (\hat\beta - \beta)^2 + 2(\hat\alpha - \alpha)(\hat\beta - \beta) \rangle =$$

$$= \sigma^2_{\hat\alpha} + \sigma^2_{\hat\beta} + 2\sigma_{\hat\alpha\hat\beta} =$$

Te pa preberemo iz kovariančne matrike

$$= \frac{(\Delta\varphi)^2}{3} (2 + 2 - 2\cdot 1) = \frac{2}{3} (\Delta\varphi)^2$$

Izračunajmo še:

$$\hat{\vec{X}} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \frac{(\Delta\varphi)^2}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} (\Delta\varphi)^{-2} \begin{pmatrix} \tilde{z}_\alpha \\ \tilde{z}_\beta \\ \tilde{z}_\gamma \end{pmatrix} =$$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} z_\alpha \\ z_\beta \\ z_\gamma - \pi \end{pmatrix} =$$

$$\Rightarrow \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2z_\alpha - z_\beta - z_\gamma + \pi \\ -z_\alpha + 2z_\beta - z_\gamma + \pi \end{pmatrix}$$

$$\hat{\gamma} = \pi - \frac{1}{3}(z_\alpha + z_\beta - 2z_\gamma + 2\pi)$$

$$\Rightarrow \hat{\gamma} = \frac{1}{3}(2z_\gamma - z_\alpha - z_\beta + \pi)$$

V splošnem za linearne vezi:

- lin. vezi

$$\underline{\underline{A}}\,\vec{x} = \vec{b} \quad ; \text{ npr za prej } \underbrace{(1\ 1\ 1)}_{A} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \underbrace{(\pi)}_{\vec{b}}$$

$$\underline{\underline{G}} = A^T (AA^T)^{-1}$$

Optimal če ne uporabimo vezi

$$\tilde{\vec{X}} = \hat{\vec{X}} - G(A\hat{\vec{x}} - \vec{b}) = (I - GA)\hat{\vec{x}} + G\vec{b}$$

Zadostuje lin vezem.

$$\tilde{P} = (I - GA)P(I - GA)^T$$

V primeru, da je $P = \sigma^2 I$ potem je

$$\tilde{P} = \sigma^2 (I - GA)$$

Kalmanov filter v zvezni sliki

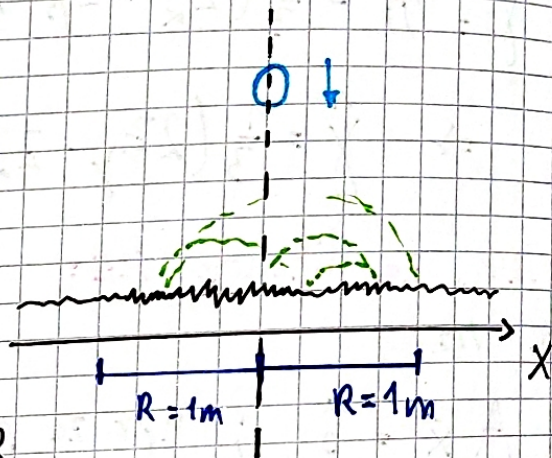
V zvezni sliki je sample rate $T_n \to 0$.

- Dinamika

$$\dot{\vec{x}} = A\vec{x} + \vec{c} + \Gamma\vec{\omega}$$

$$\dot{P} = \underbrace{AP + PA^T}_{\text{Sprememba zaradi dinamike}} + \underbrace{\Gamma Q \Gamma^T}_{\text{Dinamični šum}} - \underbrace{PH^T R^{-1} HP}_{\text{Ostrenje zaradi meritev}}$$

Primer: [Naključno hrapava plošča, ki se trese]



Ko pade natančno na sredino

a) $\to T = 5\,s$ najdemo $2/3$ ~~m~~ (približno) kroglic v $\pm R$.

b) Koliko časa $\tilde{T}$, da pridejo v interval $\pm R$ če je začetna nedoločenost lege v x-smeri $\sigma_x(t=0) = 0{,}2\,m$?

Vpliv hrapavosti površine opišemo z naključnimi silami v x smeri. Zahtevamo:

$$\langle F(t) \rangle = 0$$

$$m\langle a(t)\rangle = 0 \to \langle a(t)\rangle = 0$$

V Kalmanovem filtru jih opišemo z dinamičnim šumom.

$$\cancel{\langle F_i F_j \rangle = F_i}$$

$$\langle F(t) F(t') \rangle = \mathcal{F}(t)\,\delta(t-t')$$

Gibalna enačba za kroglice v x-smeri:

$$m\ddot{x} = F(t)$$

$$\ddot{x} = \frac{F}{m} = w(t)$$ Dinamični šum

$$\dot{x} = v$$
$$\dot{v} = \ddot{x} = w$$

Torej je dinamika:

$$\dot{\vec{x}} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{c}} + \underbrace{\begin{pmatrix} 0 \\ w \end{pmatrix}}_{\Gamma\vec{w}}$$

$$\dot{P} = AP + PA^T + \Gamma Q \Gamma^T$$

$$AP = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} P_{12} & P_{22} \\ 0 & 0 \end{pmatrix}$$

$$(AP)^T = P^T A^T = PA^T = \begin{pmatrix} P_{12} & 0 \\ P_{22} & 0 \end{pmatrix}$$!

$$\Gamma Q \Gamma^T = \langle \Gamma \vec{w} (\Gamma \vec{w})^T \rangle = \langle \Gamma \vec{w} \vec{w}^T \Gamma^T \rangle = \Gamma \underbrace{\langle \vec{w} \vec{w}^T \rangle}_{Q} \Gamma^T = \Gamma Q \Gamma^T$$

$$= \left\langle \begin{pmatrix} 0 \\ w \end{pmatrix} \begin{pmatrix} 0 & w \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 & 0 \\ 0 & w^2 \end{pmatrix} \right\rangle =$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$$ Nerodna oznaka!

Varianca dinamičnega šuma

V splošnem $Q(t)$ a predpostavimo konst.

Splošno glede oznak

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\Gamma Q \Gamma^T = \begin{pmatrix} Q_1 & \cdot \\ \cdot & Q_2 \end{pmatrix}$$

Ko to vse seštejemo

$$\dot{P} = \begin{pmatrix} 2p_{12} & p_{22} \\ p_{22} & Q \end{pmatrix} = \begin{pmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{21} & \dot{p}_{22} \end{pmatrix}$$

Rabimo še začetni pogoj

$$P^{(a)}(t=0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad P^{(b)}(t=0) = \begin{pmatrix} \sigma_x^2(0) & 0 \\ 0 & 0 \end{pmatrix}$$

Iščemo $p_{11}(t) = ?$

$$\dot{p}_{11} = 2p_{12} \quad \Rightarrow \quad p_{11}(t) = \frac{1}{3}Qt^3 + p_{22}(0)t^2 + 2p_{12}(0)t + p_{11}(0)$$

$$\dot{p}_{12} = p_{22} \quad \Rightarrow \quad p_{12}(t) = \frac{1}{2}Qt^2 + p_{12}(0) + p_{22}(0)$$

$$\dot{p}_{22} = Q \quad \Rightarrow \quad p_{22}(t) = Qt + p_{22}(0)$$

a) $\sigma_x^2(0) = 0 \Rightarrow P(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

ostali ništa napisani

torej privzemimo, da je 0.

$T = 5\,s$

$R = 1\,m$ $\Rightarrow \sigma_x(T) = R$

b) $\sigma_x(0) = 0{,}2\,m \Rightarrow P(0) = \begin{pmatrix} \sigma_x^2(0) & 0 \\ 0 & 0 \end{pmatrix}$

Zanima nas $\tilde{T} = ?$ da bo $\sigma_x(\tilde{T}) = R$

b) $\sigma_x^2(\tilde{T}) = \frac{1}{3}Q\tilde{T}^3 + \sigma_x^2(0) = R^2$

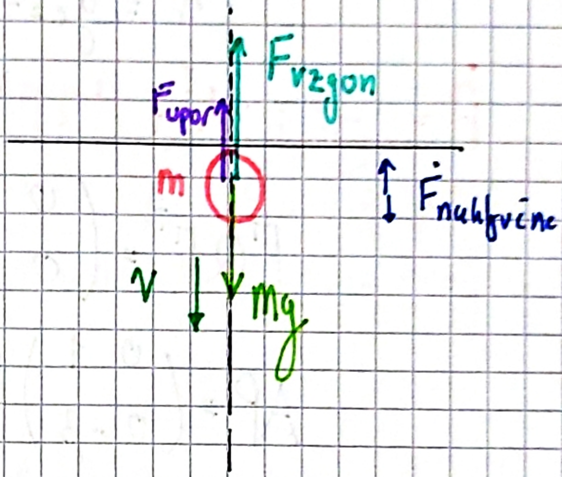
$$\tilde{T}^3 = \frac{3}{Q}\left(R^2 - \sigma_x^2(0)\right)$$; Q je še neznanka.

Dobimo jo iz a)

a) $\sigma_x^2(T) = R^2 = \rho_{11}(T) = \frac{1}{3}QT^3 \Rightarrow Q = \frac{3R^2}{T^3}$

Vstavimo to v našo rešitev za $\tilde{T}$:

$$\tilde{T}^3 = T^3\left(1 - \frac{\sigma_x^2(0)}{R^2}\right)$$



Primer: [Kroglice v tekočini]

$$\vec{X} = \begin{pmatrix} z \\ v \end{pmatrix}$$

Dinamika bo: $\sum F_i = m\ddot{z}$

Imamo še:

$F_u = -m\beta v$ ... sila upora

Naključne sile Γ

Podano še:

$$\rho_{22}(t \to \infty) = \frac{1}{4}\rho_{22}(0) \neq 0$$

Najdi $\rho_{22}(\tilde{t})$, če $\beta\tilde{t} = 1$

Dinamika $= A\vec{x} + \vec{c} + \Gamma\vec{w} = \dot{\vec{X}}$

Torej poglejmo prvo vse sile na kroglico. Da bo manj pisanja uvedemo efektivni težnostni pospešek, ki upošteva silo vzgona v navpično smer.

$$m\ddot{z} = mg_{ef} - m\beta v + F_n$$

Kot beli šum, porazdeljene po gauss

$$\ddot{z} = g_{ef} - \beta v + \frac{F_{null}}{m}$$ w dinamični šum

$$\dot{\vec{X}} = \begin{pmatrix} \dot{z} \\ \dot{v} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -\beta \end{pmatrix}}_{A} \begin{pmatrix} z \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ g_{ef} \end{pmatrix}}_{\vec{C}} + \underbrace{\begin{pmatrix} 0 \\ w \end{pmatrix}}_{\Gamma \vec{\omega}}$$

$$\dot{z} = v$$

$$\dot{v} = g_{ef} - \beta v + w$$

$$\dot{P} = AP + PA^T + \Gamma Q \Gamma^T$$

$$\Gamma Q \Gamma^T = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$$

$$AP = \begin{pmatrix} 0 & 1 \\ 0 & -\beta \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{12} & p_{22} \\ -\beta p_{12} & -\beta p_{22} \end{pmatrix}$$

$$(PA)^T = (AP)^T = \begin{pmatrix} p_{12} & -\beta p_{12} \\ p_{22} & -\beta p_{22} \end{pmatrix}$$

$$\dot{P} = \begin{pmatrix} 2p_{12} & p_{22} - \beta p_{12} \\ p_{22} - \beta p_{12} & Q - 2\beta p_{22} \end{pmatrix} = \begin{pmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{12} & \dot{p}_{22} \end{pmatrix} \; ; \; p_{22}(t) = ?$$

$$\dot{p}_{22} = Q - 2\beta p_{22} \quad / \quad u = Q - 2\beta p_{22}$$

$$\dot{u} = -2\beta \dot{p}_{22}$$

$$-2\beta \dot{p}_{22} = -2\beta u$$

$$\Rightarrow \dot{u} + 2\beta u = 0$$

$$u(t) = C e^{-2\beta t}$$

$$u(0) = C$$

Prevedemo nazaj

$$\left(Q - 2\beta P_{22}(t)\right) = \left(Q - 2\beta P_{22}(0)\right) e^{-2\beta t}$$

Potrebujemo še Q:

$$P_{22}(t \to \infty) = \frac{1}{4} P_{22}(0)$$

V lim $t \to \infty$

$$Q - 2\beta P_{22}(\infty) = 0$$

$$Q = 2\beta P_{22}(\infty)$$

Isto dobimo če iščemo stacionarno rešitev $\dot{P}_{22} = 0$. To je vrednost, ki jo zavzame po dolgem času. Dobljeno lahko vstavimo nazaj v enačbo za P_{22}:

$$P_{22}(\infty) - P_{22}(t) = \left(P_{22}(\infty) - P_{22}(0)\right) e^{-2\beta t}$$

$$P_{22}(t) = P_{22}(\infty) + \left(P_{22}(0) - P_{22}(\infty)\right) e^{-2\beta t}$$

Izrazimo kot funkcijo $P_{22}(0)$

$$P_{22}(t) = \frac{P_{22}(0)}{4} + \frac{3}{4} P_{22}(0) e^{-2\beta t}$$

$$\Rightarrow \quad P_{22}(t) = \frac{P_{22}(0)}{4}\left(1 + 3e^{-2\beta t}\right)$$

Pri $\beta\tilde{t} = 1$

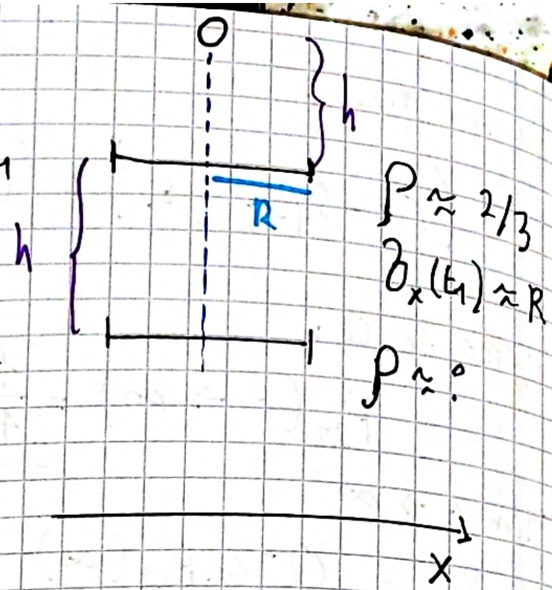
$$P_{22}(\tilde{t}) = \frac{P_{22}(0)}{4}\left(1 + 3e^{-2}\right) \cong 0{,}35\, P_{22}(0)$$

DN

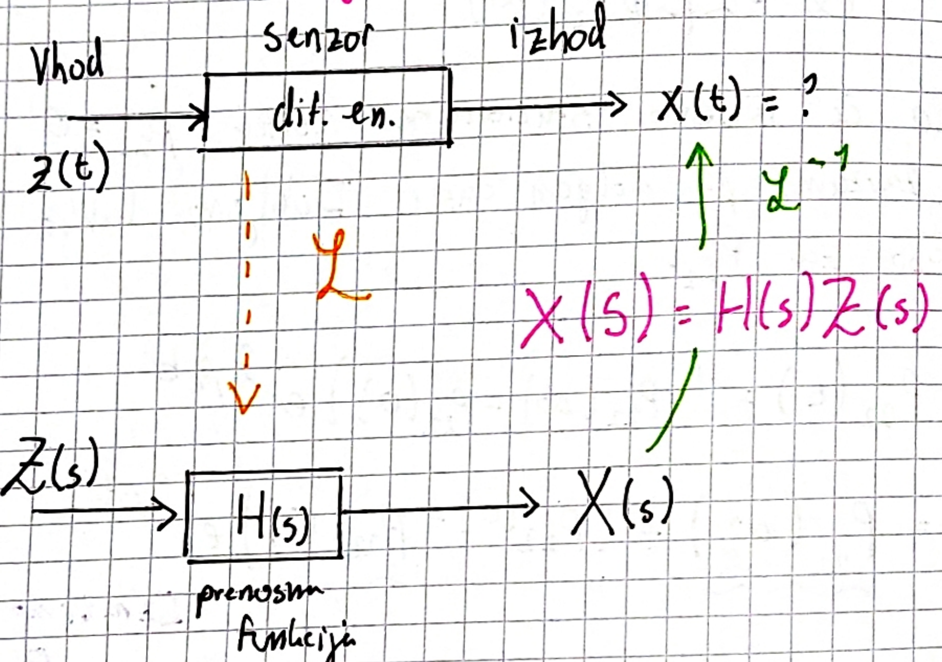
Če jih 2/3 pade skozi prva koliko jih bo skozi druga?

$\sigma_x(t) = ?$

$P_{11}(t) = ?$



Senzorji



$$\mathcal{L}: f(t) \mapsto F(s)$$

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t) e^{-st} \, dt$$

$f(t)$	$F(s)$
$\delta(t)$	1
e^{at}	$\frac{1}{s-a}$
1 oz. $\theta(t)$; Heaviside	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$f(t)e^{at}$	$F(s-a)$
$\theta(t-T) f(t-T)$	$F(s) e^{-sT}$

$\frac{d^n f}{dt^n}$	$s^n F(s)$

Za Prvi red:

$$\tau \dot{x} + x = z$$

$$\tau \cdot s X(s) + X(s) = Z(s)$$

$$\Rightarrow \quad X(s) = \frac{1}{1+\tau s} Z(s)$$

H(s) prenosna funkcija senzorja prvega reda

Primer: [1. red in $z(t) = kt$]

$$z(t) = kt \rightarrow Z(s) = k \frac{1}{s^2}$$

$$X(s) = \frac{1}{1+\tau s} Z(s) = \frac{1}{1+\tau s} \cdot \frac{k}{s^2} =$$

Razcep na parcialne ulomke

$$= k \left(\frac{A}{1+\tau s} + \frac{Bs + C}{s^2} \right) = (*)$$

eno stopnjo nižji polinom kot imenovalec

$$1 = s^2(A + \tau B) + s(B + \tau C) + C$$

$$\Rightarrow C = 1 \quad B + \tau C = 0 \Rightarrow B = -\tau$$

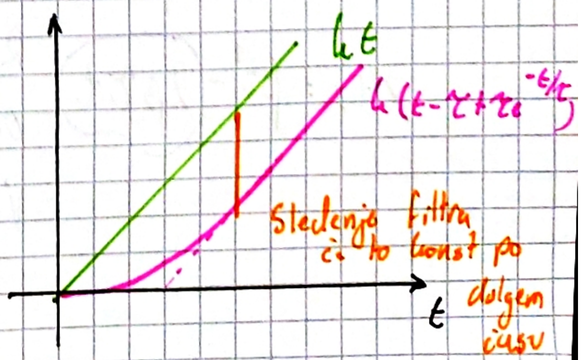
$$A + \tau B = 0 \Rightarrow A^2 = \tau^2$$

$$(*) = k \left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2 /:\tau}{1+\tau s /:\tau} \right) =$$

$$\hookrightarrow \frac{\tau}{s + 1/\tau}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

Vsak ulomek pretransformiramo nazaj s tabelo



$$x(t) = k\left(t - \tau + \tau e^{-t/\tau}\right)$$

$$x(t \gg \tau) = k(t - \tau)$$

Primer: [Kvadratna funkcija $Z(t)$]

$Z(t) = \beta t^2$

$X(s) = H(s) Z(s)$

1. red

$Z(s) = \beta \frac{2}{s^3}$ Iz tabel $\Rightarrow X(s) = \frac{1}{1+\tau s} \frac{2\beta}{s^3}$

$X(s) = \left(1 - \frac{\tau s}{1+\tau s}\right) \frac{2\beta}{s^3} = \underbrace{\frac{2\beta}{s^3}}_{\text{znamo poenostavit}} - \frac{2\beta\tau}{(1+\tau s)s^2} = (*)$

$\frac{2\beta\tau}{(1+\tau s)s^2} = 2\beta\tau\left(\frac{A}{1+\tau s} + \frac{Bs+C}{s^2}\right) =$

$1 = s^2(A + B\tau) + s(B + \tau C) + C$

$C = 1$

$B = -\tau C = -\tau$

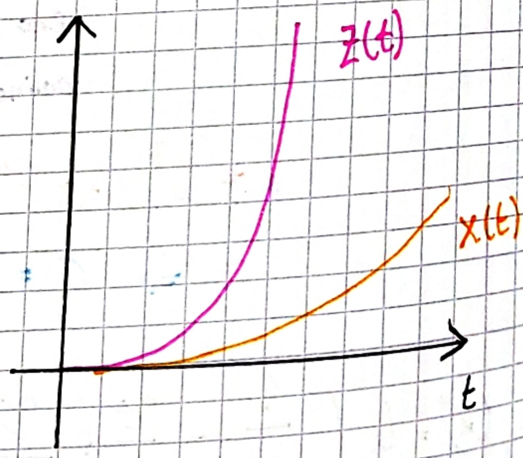
$A = -\tau B = \tau^2$

$= 2\beta\tau\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s+1/\tau}\right)$

$\Rightarrow (*) = \frac{2\beta}{s^3} - 2\beta\tau\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s+1/\tau}\right)$

Spet pretransformiramo:

$X(t) = \beta t^2 - 2\beta\tau\left(t - \tau + \tau e^{-t/\tau}\right)$

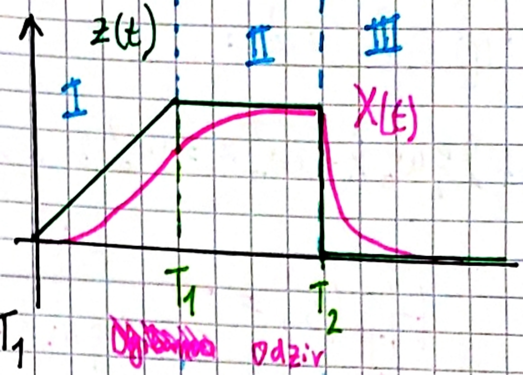


Ali senzor sledi pri $t \gg \tau$:

$X(t) = \beta t^2 - 2\beta\tau t$ Razlika med $Z(t)$ in $X(t)$ je odvisna od časa. Senzor 1. reda ne sledi vhodu $Z(t) = \beta t^2$

Primer: [1. red in sestavljen signal]

$X(t) = ?$



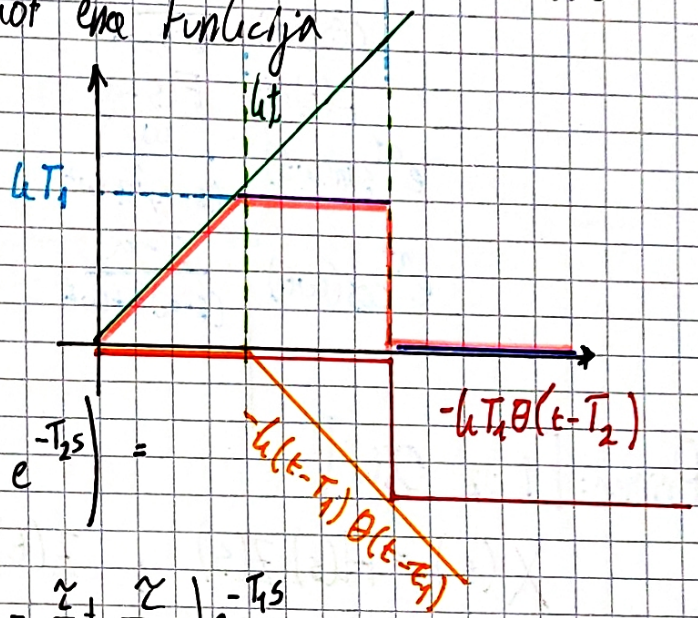
Rešimo odsekoma:

$$X_I(t) = k\left(t - \tau + \tau e^{-t/\tau}\right); \; 0 \le t \le T_1$$

⋮ zlepimo

$$\mathcal{L}\left(f(t-T_1)\theta(t-T_1)\right) = F(s)e^{-T_1 s}$$

Ali pa bolj elegantno zapišemo to kot ena funkcija



$$z(t) = kt - k(t-T_1)\theta(t-T_1) - kT_1\theta(t-T_2)$$

$$Z(s) = \frac{k}{s^2} - \frac{k}{s^2}e^{-T_1 s} - \frac{kT_1}{s}e^{-T_2 s}$$

$$X(s) = \frac{1}{1+\tau s}\left(\frac{k}{s^2} - \frac{k}{s^2}e^{-T_1 s} - \frac{kT_1}{s}e^{-T_2 s}\right) =$$

$$= k\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s+1/\tau}\right) - k\left(\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau}{s+1/\tau}\right)e^{-T_1 s} -$$

$$- kT_1\left(\frac{1}{s} - \frac{1}{1/\tau + s}\right)$$

$$\frac{1}{1+\tau s}\cdot\frac{1}{s} = \frac{A}{1+\tau s} + \frac{B}{s} \rightarrow s(A+\tau B) + B = 1$$

$$B = 1$$

$$= \frac{1}{s} - \frac{1}{s+1/\tau} \qquad A = -\tau$$

Pretvorimo nazaj

$$X(t) = k\left(t - \tau + \tau e^{-t/\tau}\right) - k\left((t-T_1) - \tau + \tau e^{-\frac{(t-T_1)}{\tau}}\right)\theta(t-T_1) -$$

$$- kT_1\left(1 - e^{-\frac{t-T_2}{\tau}}\right)\theta(t-T_2)$$

Npr. za $t > T_2$:

$$X(t) = k\tau e^{-t/\tau} - k\tau e^{-\frac{t-T_1}{\tau}} + kT_1 e^{-\frac{t-T_2}{\tau}}$$

Senzorji 2. reda:

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = \omega_0^2 z + \cancel{2\xi\omega_0}^{\,0} ?$$

$$H_{II}(s) = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}$$

$f(t)$	$F(s)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$f(t)e^{at}$	$F(s-a)$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$

Primer: [1 oz $\Theta(t)$]

$$X(s) = H(s)Z(s) \quad ; \quad Z(t) = 1 \Rightarrow Z(s) = \frac{1}{s}$$

$$X(s) = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2} \cdot \frac{1}{s} = \left(\frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_0 s + \omega_0^2}\right) =$$

$$\omega_0^2 = s^2(A+B) + s(C + 2\xi\omega_0 A) + A\omega_0^2$$

$$\omega_0^2 = A\omega_0^2 \Rightarrow A = 1$$

$$C = -2\xi\omega_0$$

$$B = -1$$

$$= \frac{1}{s} - \underbrace{\frac{s + 2\xi\omega_0}{s^2 + 2\xi\omega_0 s + \omega_0^2}}_{\text{Dopolnemo do popolnega kvadrata}} = (*)$$

$$s^2 + 2\xi\omega_0 s + \omega_0^2 =$$
$$= (s + \xi\omega_0)^2 - \xi^2\omega_0^2 + \omega_0^2 =$$
$$= (s + \xi\omega_0)^2 + \omega_0^2(1 - \xi^2)$$

$\xi\ \xi\ \xi$
$\xi\ \xi$

$$\frac{s+2\xi\omega_0}{\underbrace{(s-\xi\omega_0)^2}_{(s-a)^2}+\underbrace{\omega_0^2(1-\xi^2)}_{\omega^2}} = \underbrace{\frac{s+\xi\omega_0}{\dots}}_{e^{-\xi\omega_0 t}\cos(\omega_0\sqrt{1-\xi^2}\,t)} + \frac{\xi\omega_0}{\dots} \qquad (**)$$

ω

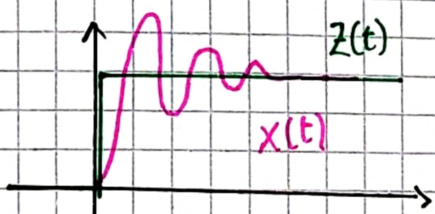
Predfaktor

$$\frac{\xi\,\omega_0\,\frac{\sqrt{1-\xi^2}}{\sqrt{1-\xi^2}}}{\dots} = \frac{\xi}{\sqrt{1-\xi^2}}\,\frac{\omega_0\sqrt{1-\xi^2}}{(s+\xi\omega_0)^2+\omega_0^2(1-\xi^2)}$$

$$(**) \Rightarrow e^{-\xi\omega_0 t}\cos(\omega_0\sqrt{1-\xi^2}\,t) + \mathcal{L}^{-1}\left(\frac{\xi}{\sqrt{1-\xi^2}}\,\frac{\omega_0\sqrt{1-\xi^2}}{(s+\xi\omega_0)^2+\omega_0^2(1-\xi^2)}\right)$$

$$(*) \Rightarrow \quad x(t) = 1-\left[e^{-\xi\omega_0 t}\cos(\omega t) + \frac{\xi}{\sqrt{1-\xi^2}}\,e^{-\xi\omega_0 t}\sin(\omega t)\right]; \quad \omega = \omega_0\sqrt{1-\xi^2}$$

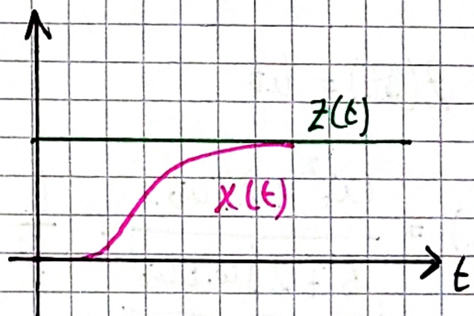
a) $\xi < 1$



Podkritično dušenje

b) $\xi = 1$ Kritično dušenje

$$x(t) = 1 - e^{-\omega_0 t}\left[1+\omega_0 t\right]$$



$$\frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_0\sqrt{1-\xi^2}) \approx \frac{\xi}{(1-\xi^2)^{1/2}}\left(\omega_0\sqrt{1-\xi^2}\,t - \frac{(\omega_0\sqrt{1-\xi^2}\,t)^3}{3!} + \dots\right)$$

Za $t \approx 0$: $x(t) = 1-(1-\omega_0 t)(1+\omega_0 t) = \omega_0^2 t^2$

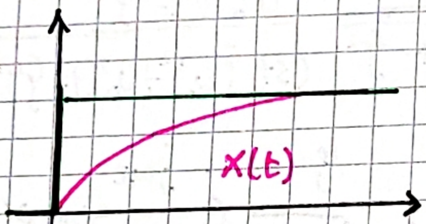
c) $\zeta > 1$ Nadkritično dušenje

$$x(t) = 1 - e^{-\zeta\omega_0 t}\left[\cos\left(i\omega_0\sqrt{\zeta^2-1}\,t\right) + \frac{\zeta}{i\sqrt{\zeta^2-1}}\sin\left(i\omega_0\sqrt{\zeta^2-1}\,t\right)\right] =$$

$$\cos(at) = \frac{e^{iat}+e^{-iat}}{2} \qquad \sin(at) = \frac{e^{iat}-e^{-iat}}{2i}$$

$$\cosh(at) = \frac{e^{-at}+e^{at}}{2} \qquad i\sinh(at) = \sin(iat)$$

$$= 1 - e^{-\zeta\omega_0 t}\left[\cosh\left(\omega_0\sqrt{\zeta^2-1}\,t\right) + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\left(\omega_0\sqrt{\zeta^2-1}\,t\right)\right]$$



DN $z(t) = kt$ 2. red $H(s)$ kot prej

Primer: [Prenosna funkcija podana]

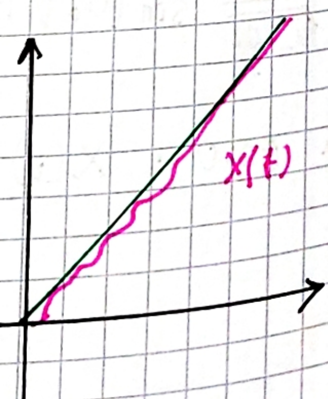
$$H(s) = \frac{\omega_0^2 + 2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$z(t) = kt$$

$$X(s) = \underbrace{\frac{\omega_0^2 + 2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 + \omega_0^2}}_{\left(1 - \frac{s^2}{\dots}\right)} \cdot \frac{k}{s^2} = \frac{k}{s^2} - \frac{k}{(s+\zeta\omega_0)^2 + \omega_0^2(1-\zeta^2)} =$$

$$\left(1 - \frac{s^2}{\dots}\right)\frac{k}{s^2}$$

$$= \frac{k}{s^2} - \frac{k}{\omega_0\sqrt{1-\zeta^2}}\,\frac{\omega_0\sqrt{1-\zeta^2}}{(s+\zeta\omega_0)^2 + \omega_0^2(1-\zeta)}$$



$$x(t) = kt - \frac{k}{\omega_0\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)$$

$$t \Rightarrow \infty \quad x(t) \to kt$$