str. 5 Komponenta navora:

$$\vec{M} = \vec{r} \times \vec{F} \rightarrow M_L = \varepsilon_{Lmn} r_m F_n$$

$$M_{ij} = \varepsilon_{ijk} M_k$$

$$M_{ij} = ?$$

$$M_{ij} = \varepsilon_{ijk} \varepsilon_{kmn} r_m F_n = \varepsilon_{kij} \varepsilon_{kmn} r_m F_n =$$
$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) r_m F_n = \delta_{im}\delta_{jn} r_m F_n - \delta_{in}\delta_{jm} r_m F_n =$$
$$= r_i F_j - r_j F_i$$

Operator nabla in krivočrtne koordinate

Pri ortog. bazah ne rabiš metričnega tenzorja.

$$\nabla = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i}$$

h_i ... skalni faktorji
$\hat{e}_i$ ... bazni vektor
q_i ... premiki

$$h_i = \left| \frac{\partial \vec{r}}{\partial q_i} \right|$$

$$\hat{e}_i = \frac{1}{h_i} \cdot \frac{\partial \vec{r}}{\partial q_i}$$

Opomba:

$$ds^2 = \sum_i \underline{h^2} \, \underline{dq_i^2} = \underline{g_{ij}} \, \underline{dq_i \, dq_j}$$

- Gradient skalarnega polja:

$$\nabla f = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} f$$

- Divergenca vektorja:

$$\nabla \cdot \vec{v} = \sum_i \hat{e}_i \frac{1}{h_i} \underline{\frac{\partial}{\partial q_i}} \cdot \sum_j v_j \boxed{\hat{e}_j}$$

$$\vec{v} = \sum_j \hat{e}_j v_j$$

V krivočrtnih koordinatah krajevno odvisno in odvod da dodaten člen.

- Rotor vektorja:

$$\nabla \times \vec{v} = \sum_i \hat{e}_i \frac{1}{h_i} \frac{\partial}{\partial q_i} \times \sum_j v_j \hat{e}_j$$

- Laplace skalarja (divergenca gradienta):

$$\Delta f = \nabla^2 f = \sum_i \hat{e}_i \frac{1}{h_i}\frac{\partial}{\partial q_i} \cdot \sum_j \hat{e}_j \frac{1}{h_j}\frac{\partial}{\partial q_j} f$$

- Laplace vektorja:

$$\Delta \vec{v} = \nabla^2 \vec{v} = \sum_i \hat{e}_i \frac{1}{h_i}\frac{\partial}{\partial q_i} \cdot \sum_j \hat{e}_j \frac{1}{h_j}\frac{\partial}{\partial q_j} \otimes \sum_k v_k \hat{e}_k$$

Sedaj poskusimo to v cilindričnih koordinatah:

$x = r\cos\varphi$

$y = r\sin\varphi$

$\vec{r} = x\hat{e}_x + y\hat{e}_y$

$$\hat{e}_r = \left(\left|\frac{\partial \vec{r}}{\partial r}\right|^{-1} \frac{\partial \vec{r}}{\partial r}\right) = \frac{1}{\underset{\substack{\shortparallel\\ 1}}{h_r}} \cdot \left(\hat{e}_x \cos\varphi + \hat{e}_y \sin\varphi\right) = \hat{e}_x \cos\varphi + \hat{e}_y \sin\varphi$$

Don't think about it, je samo norma napisana na kompliciran način!

$$\hat{e}_\varphi = \frac{1}{\underset{\substack{\shortparallel\\ r}}{h_\varphi}}\left(-r\hat{e}_x \sin\varphi + r\hat{e}_y \cos\varphi\right) = -\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi$$

Izračunamo odvode:

$$\frac{\partial \hat{e}_r}{\partial \varphi} = \left(-\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi\right) = \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\varphi}{\partial r} = 0$$

$$\frac{\partial \hat{e}_r}{\partial r} = 0$$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = \left(-\hat{e}_x \cos\varphi - \hat{e}_y \sin\varphi\right) = -\hat{e}_r$$

Sedaj pa rotor in divergenca:

$$\nabla \cdot \vec{v} = \cancel{\left(\sum_i e\right.}$$

Ta je odvisen od φ zato nov člen

$$= \left(\hat{e}_r \cdot \frac{\partial}{\partial r} + \hat{e}_\varphi \frac{1}{r} \cdot \frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z}\right) \cdot \left(v_v \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z\right) =$$

$$= \frac{\partial v_r}{\partial r} + \boxed{\frac{v_r}{r}} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z}$$

dodaten člen!

$$\nabla \times \vec{v} = \left(\hat{e}_r \cdot \frac{\partial}{\partial r} + \hat{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{e}_z \frac{\partial}{\partial z}\right) \times \left(v_r \hat{e}_r + v_\varphi \hat{e}_\varphi + v_z \hat{e}_z\right) =$$

$$= \ldots = \frac{\partial v_\varphi}{\partial r}\hat{e}_z + \frac{\partial v_z}{\partial r}(-\hat{e}_\varphi) + \frac{1}{r}\left(\frac{\partial v_r}{\partial \varphi}(-\hat{e}_z) + v_\varphi \cdot (\hat{e}_z) + \frac{v_z}{\partial \varphi}\hat{e}_r\right)$$

$$+ \frac{\partial v_r}{\partial z}\hat{e}_\varphi + \frac{\partial v_\varphi}{\partial z}(-\hat{e}_r) =$$

$$= \hat{e}_r\left(-\frac{\partial v_\varphi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \varphi}\right) + \hat{e}_\varphi\left(-\frac{\partial v_z}{\partial r} + \frac{\partial v_\varphi}{\partial z}\right) +$$

$$+ \hat{e}_z\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \varphi} + v_\varphi\right)\right)$$

Deformacijski tenzor v sfernih koordinatah

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

je že normiran

norma $=1$

$$\hat{e}_r = \frac{1}{h_r}\frac{\partial \vec{r}}{\partial r} =$$

$$= \left(\hat{e}_x \sin\theta\cos\varphi + \hat{e}_y \sin\theta\sin\varphi + \hat{e}_z \cos\theta\right)\frac{1}{h_r}\cdot 1$$

$$\hat{e}_\varphi = \frac{1}{h_\varphi}\frac{\partial \vec{r}}{\partial \varphi}\left(-\hat{e}_x r\sin\theta\sin\varphi + \hat{e}_y \sin\theta\cos\varphi\, r + 0\cdot\hat{e}_z\right)$$

$h_\varphi = r\sin\theta$ -> da je normiran

$$\hat{e}_\theta = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta} =$$

$$= \frac{1}{h_\theta}\left(\hat{e}_x r\cos\theta\cos\varphi + \hat{e}_y r\cos\theta\sin\varphi - \hat{e}_z \sin\theta\right)$$

$h_\theta = r$

z upoštevano normo
$$\begin{cases}\hat{e}_\varphi = (-\hat{e}_x \sin\varphi + \hat{e}_y \cos\varphi) \\ \hat{e}_\theta = \hat{e}_x \cos\theta\cos\varphi + \hat{e}_y \cos\theta\sin\varphi - \hat{e}_z \sin\theta\end{cases}$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_x \cos\theta\cos\varphi + \hat{e}_y \cos\theta\sin\varphi - \hat{e}_z \sin\theta = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_r}{\partial \varphi} = \hat{e}_x \sin\theta\sin\varphi + \hat{e}_y \sin\theta\cos\varphi + 0 = \hat{e}_\varphi \sin\theta$$

$$\frac{\partial \hat{e}_\varphi}{\partial \theta} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_x \sin\theta\cos\varphi - \hat{e}_y \sin\theta\sin\varphi - \hat{e}_z \cos\theta = -\hat{e}_r$$

$$\frac{\partial e_\theta}{\partial \varphi} = -\hat{e} \cos\theta\sin\varphi + \hat{e}_y \cos\theta\cos\varphi = \hat{e}_\varphi \cos\theta$$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_x \cos\varphi - \hat{e}_y \sin\varphi = -\sin\theta\,\hat{e}_r - \cos\theta\,\hat{e}_\theta$$

Nadaljevanje:

$$\left.\begin{aligned}\vec{u} &= \sum_k u_k \hat{e}_k \\ \nabla &= \sum_k \hat{e}_k \frac{1}{h_k}\frac{\partial}{\partial q_k}\end{aligned}\right\}$$ To je ponavljanje od prej

$$\frac{1}{h_i}\frac{\partial \hat{e}_j}{\partial q_i} = \sum_k \Gamma_{i,j}^{k} \hat{e}_k$$

Kristoffelov simbol!

$$\nabla\vec{u} = \sum_l \hat{e}_l \frac{1}{h_l}\frac{\partial}{\partial q_l} \otimes \sum_k u_k \hat{e}_k$$

$$= \sum_{l,k} \frac{1}{h_l}\left(\frac{\partial u_k}{\partial q_l}\right)\hat{e}_l \otimes \hat{e}_k + \frac{u_k}{h_l}\hat{e}_l \otimes \frac{\partial \hat{e}_k}{\partial q_l} =$$

$$= \sum_{l,k}\left[\frac{1}{h_l}\frac{\partial u_k}{\partial q_l}\hat{e}_l \otimes \hat{e}_k + u_k \sum_m \Gamma_{l,k}^{m}\hat{e}_l \otimes \hat{e}_m\right]$$

Napišimo sedaj po komponentah:

$$(\nabla u)_{ij}=\sum_{l,k}\left[\frac{1}{h_l}\frac{\partial u_k}{\partial q_l}\boxed{\hat{e}_i\cdot\hat{e}_l}\otimes\boxed{\hat{e}_k\cdot\hat{e}_j}+\sum_m \Gamma^m_{l,k}\boxed{\hat{e}_i\cdot\hat{e}_l}\otimes\boxed{\hat{e}_m\hat{e}_j}u_k\right]=$$

Iz desne množimo z $\cdot\hat{e}_j$ iz leve pa $\hat{e}_i\cdot$

$$=\sum_{l,k}\left[\frac{1}{h_l}\frac{\partial u_k}{\partial q_l}\delta_{il}\delta_{kj}+u_k\sum_m\Gamma^m_{l,k}\,\delta_{il}\delta_{mj}\right]=$$

$$=\frac{1}{h_i}\frac{\partial u_j}{\partial q_i}+\sum_k u_k\Gamma^k_{i,k}$$

Sedaj simetriziramo tenzor, ker je to linearni del deformacijskega tenzorja:

$$\underline{\underline{u^{(1)}_{ij}}}=\frac{1}{2}\left((\nabla\vec{u})_{ij}+(\nabla\vec{u})_{ji}\right)=$$

$$\underline{\underline{=\frac{1}{2}\left(\frac{1}{h_i}\frac{\partial u_j}{\partial q_i}+\frac{1}{h_j}\frac{\partial u_i}{\partial q_j}\right)+\frac{1}{2}\sum_k\left(\Gamma^j_{ik}+\Gamma^i_{jk}\right)}}$$

Naredimo to zdaj konkretno v sferičnih koordinatah:

$$\nabla u=\left(\hat{e}_r\frac{1}{1}\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\otimes\left(u_r\hat{e}_r+u_\theta\hat{e}_\theta+\hat{e}_\varphi u_\varphi\right)=$$

$$=\left(\underline{\frac{\partial u_r}{\partial r}\hat{e}_r\otimes\hat{e}_r}+\frac{\partial u_\theta}{\partial r}\hat{e}_r\otimes\hat{e}_\theta+\frac{\partial u_\varphi}{\partial r}\hat{e}_r\otimes\hat{e}_\varphi\right)+$$

$$+\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}\hat{e}_\theta\times\hat{e}_r+\underline{u_r\hat{e}_\theta\otimes\hat{e}_\theta}+\underline{\frac{\partial u_\theta}{\partial\theta}\hat{e}_\theta\otimes\hat{e}_\theta}-u_\theta\hat{e}_\theta\otimes\hat{e}_r+\frac{\partial u_\varphi}{\partial\theta}\hat{e}_\theta\otimes\hat{e}_\varphi\right)+$$

$$+\frac{1}{r\sin\theta}\left(\frac{\partial u_r}{\partial\varphi}\hat{e}_\varphi\otimes\hat{e}_r+\underline{u_r\sin\theta\,\hat{e}_\varphi\otimes\hat{e}_\varphi}+\frac{\partial u_\theta}{\partial\varphi}\hat{e}_\varphi\otimes\hat{e}_\theta+\underline{\cos\theta\, u_\varphi\hat{e}_\varphi\otimes\hat{e}_\varphi}+\underline{\frac{\partial u_\varphi}{\partial\varphi}\hat{e}_\varphi\otimes\hat{e}_\varphi}-\right.$$

$$\left.-\sin\theta\, u_\theta\hat{e}_\varphi\otimes\hat{e}_r-u_\varphi\cos\theta\,\hat{e}_\varphi\otimes\hat{e}_\theta\right)$$

Preberemo ven komponente. Ne diag. lahkati še simetriziramo

$$u_{rr} = \frac{\partial u_r}{\partial r}$$

$$u_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)$$

$$u_{\varphi\varphi} = \frac{1}{r}u_r + \frac{1}{r}\frac{1}{tg\theta}u_\theta + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi}$$

$$u_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{1}{r}u_\theta\right)$$

$$u_{r\varphi} = \frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} - \frac{1}{r}u_\varphi\right)$$

$$u_{\theta\varphi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\varphi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{1}{r\,tg\theta}u_\varphi\right)$$

Nal. na str. 14 [Rotacija okoli z osi]

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{x}' = \vec{x}$$

$$\vec{u} = \vec{x}' - x = R\vec{x} - \vec{x} = (R - I)\vec{x} =$$

$$= \begin{pmatrix} \cos\varphi - 1 & -\sin\varphi & 0 \\ \sin\varphi & \cos - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$= \begin{pmatrix} (\cos\varphi - 1)x - \sin\varphi y \\ \sin\varphi x \quad (\cos - 1)y \\ 0 \end{pmatrix}$$

u... deformacijsko polje

∇u... deformacijski tenzor?

→

Sedaj pa rabimo še odvode:

$$u_{ij} = \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)\frac{1}{2}$$

$$u_{xx}^{(1)} = \frac{\partial u_x}{\partial x} = \cos\varphi - 1$$

$$u_{yy}^{(1)} = \cos\varphi - 1$$

} Res nič v prvem redu!

$$u_{xy}^{(1)} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = 0$$

$$u_{xy}^{(2)} = \frac{1}{2}\left(\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y}\right) = \frac{1}{2}\left(-(\cos\varphi - 1)\sin\varphi + \sin\varphi \cdot (\cos\varphi - 1)\right) = 0$$

$$u_{xx}^{(2)} = \frac{1}{2}\left(\frac{\partial u_x}{\partial x}\cdot\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\left((\cos\varphi - 1)^2 + \sin^2\varphi\right)$$

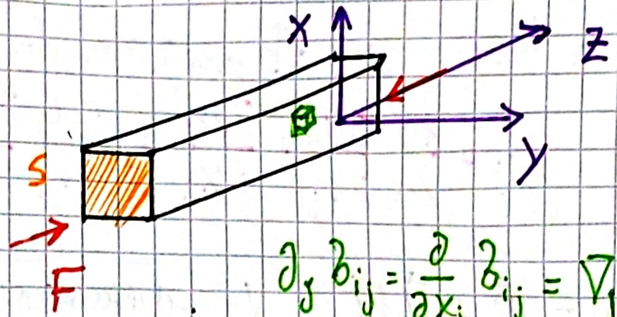
$$u_{yy}^{(2)} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\left((\cos\varphi - 1)^2 + \sin^2\varphi\right)$$

Poglejmo za X:

$$u_{xx} = u_{xx}^{(1)} + u_{xx}^{(2)} = \cos\varphi - 1 + \frac{1}{2}\left(\cos^2\varphi + \sin^2\varphi - 2\cos\varphi + 1\right) =$$
$$= \cos\varphi - 1 + 1 - \cos\varphi = \underline{\underline{0}}$$

Res tudi v višjem redu!

naloga na strani 27.



$\Delta V = ?$

$$f_{el.} = 0 = f^{el.}$$

$$\partial_j \sigma_{ij} = \frac{\partial}{\partial x_j} \sigma_{ij} = \nabla_j \sigma_{ij}$$

napetostni tenzor

1.) $\boxed{\nabla_j \sigma_{ij} = 0}$ bulk pogoj (v vsaki točki)

Če predpostavimo da je napetostni tenzor konst. avtomatsko zadostimo "bulk" pogoju (odvod konstante). Zdaj bojo pa robni pogoji povedali če to smemo.

2.) R. P.

$$\sigma_{zz} = -\frac{F}{S} \qquad \sigma_{xz} = 0 \qquad \sigma_{yz} = 0$$

↑

sila v z smeri, ki deluje na ploskev z normalo z

Še ostali dve ploskvi:

$$\sigma_{xx} = 0 \qquad \sigma_{xy} = 0$$

$$\sigma_{yx} = 0 \qquad \sigma_{yy} = 0$$

$$\sigma_{zx} = 0 \qquad \sigma_{zy} = 0$$

Rabimo Hookov zakon da izračunamo ΔV. Mi smo rekli, da je σ tak povsod kot je na robu.

Izpeljava ~~Lagrangeovega~~ Eulerjevega def. tenzorja.

$$u_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \rightarrow$$ Lagrangeov def. tenzor

$\vec{r}'(\vec{r}) = \vec{r} + \vec{u}(\vec{r})$ v Lagrangeovi sliki, a ... premik izrazimo z starimi

$\vec{r}(\vec{r}') = \vec{r}' - \vec{u}(\vec{r}')$ v Eulerjevi. ... premik izrazimo z novimi

$$\hookrightarrow d\vec{r} = d\vec{r}' - d\vec{u}$$

$$dx_i = dx_i' - \frac{\partial u_i}{\partial x_j'} dx_j'$$

Def. tenzor pove kvadrat razlike premika med dvema bližnjima točkama.

Vstavimo izraz za $d\vec{r}$:

$$d\vec{r}'^2 - d\vec{r}^2 = d\vec{r}'^2 - (d\vec{r}' - d\vec{u})^2 =$$

$$= dx_i'^2 - \left(dx_i' - \frac{\partial u_i}{\partial x_j'} dx_j'\right)^2 =$$

kvadriramo $$= dx_i'^2 - \left(dx_i'^2 - 2dx_i' \frac{\partial u_i}{\partial x_j'} dx_j' + \frac{\partial u_i}{\partial x_j'}\frac{\partial u_i}{\partial x_k'} dx_j' dx_k'\right) =$$

Prepišemo indekse $$= 2\frac{\partial u_i}{\partial x_j'} dx_i' dx_j' - \frac{\partial u_k}{\partial x_i'}\frac{\partial u_k}{\partial x_j'} dx_i' dx_j' =$$

Simetriziramo $$= \left(\frac{\partial u_i}{\partial x_j'} + \frac{\partial u_j}{\partial x_i'}\right) dx_i' dx_j' - \frac{\partial u_k}{\partial x_i'}\frac{\partial u_k}{\partial x_j'} dx_i' dx_j' = 2u_{ij}^{(E)} dx_i' dx_j'$$

odstranimo črtice in izpostavimo

$$\Rightarrow u_{ij}^{(E)} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \ominus \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]$$

$\uparrow$ To je razlika!

Nadaljevanje naloge iz str. 27.

$\partial_j \sigma_{ij} = 0 \quad \sigma_{ij} = \begin{cases} \sigma_{zz} = -\frac{F}{S} \\ \text{sicer } 0 \end{cases}$

Hookov zakon:

$$u_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{kk}\delta_{ij}$$

Zanima nas relativna sprememba prostornine: $\frac{dV}{V} = tr\, \sigma_{ij}$

$$u_{xx} = \frac{+\lambda \frac{F}{S}}{2\mu(2\mu+3\lambda)} = u_{yy}$$

$$u_{zz} = -\frac{F}{2\mu S} + \frac{\lambda F/S}{2\mu(2\mu+3\lambda)}$$

Torej sled:

$$tr = u_{xx} + u_{yy} + u_{zz} =$$

$$= \frac{\lambda F/S}{\mu(2\mu+3\lambda)} + \frac{\lambda F/S}{2\mu(2\mu+3\lambda)} - \frac{F}{2\mu S} =$$

$$= \frac{\lambda F/S}{\mu}\left(\frac{1}{(2\mu+3\lambda)} + \frac{1}{2(2\mu+3\lambda)} - \frac{1}{2\lambda}\right) =$$

$$= \frac{\lambda F/S}{\mu}\left(\frac{2\lambda+\lambda-2\mu-3\lambda}{(2\mu+3\lambda)\,2\lambda}\right) = \frac{F}{2S\mu}\left(\frac{-2\mu}{2\mu+3\lambda}\right) =$$

$$= -\frac{F}{S}\,\frac{1}{2\mu+3\lambda}$$

Poissonovo razmerje
(če stisnemo v z smeri,
koliko se raztezu v x smeri)

$$\sigma = -\frac{u_{xx}}{u_{zz}} = -\frac{u_{yy}}{u_{zz}}$$

$$\sigma = \frac{-F/S\,\lambda}{2\mu(2\mu+3\lambda)}\,\frac{2\mu(2\mu+3\lambda)}{F/S(\lambda-2\mu-3\lambda)} \rightarrow \underline{\underline{\sigma = \frac{\lambda}{2(\lambda+\mu)}}}$$

Še Youngov modul: $\underset{\sigma_{zz}}{\frac{F}{S}} = E \frac{dl}{l} \; u_{zz}$ Youngov modul:

$$E = \frac{\sigma_{zz}}{u_{zz}} = \frac{-F/S}{\frac{F/S(-2\lambda-2\mu)}{2\mu(2\mu+3\lambda)}} = \frac{-2\mu(2\mu+3\lambda)}{-2(\lambda+\mu)} \rightarrow E = \frac{(2\mu+3\lambda)\mu}{\lambda+\mu}$$

Izraženo z E in σ je sprememba volumna:

$$\frac{\Delta V}{V} = \frac{\sigma_{zz}(1-2\sigma)}{E} = -\frac{F(1-2\sigma)}{SE}$$

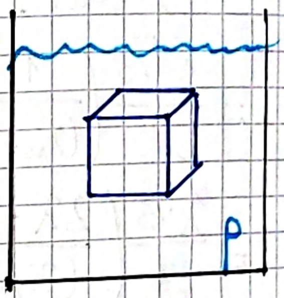
Da je to positive: $1-2\sigma > 0$

$$-2\sigma > -1$$

$$\sigma < \frac{1}{2}$$

Naloga na str. 29

Kocko izotropno obremenimo z tlakom (jo potopimo pod vodo)



$$\frac{\Delta V}{V} = u_{kk} \qquad \sigma_{ij} = -p\,\delta_{ij}$$

$$u_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\sigma_{kk}\delta_{ij}$$

$$\mathrm{tr}\, u_{ij} = \frac{1}{2\mu}\sigma_{kk} - \frac{3\lambda\sigma_{kk}}{2\mu(2\mu+3\lambda)} =$$

$$= \left(\frac{1}{2\mu} - \frac{3\lambda}{2\mu(2\mu+3\lambda)}\right)\sigma_{kk} = \frac{1}{\cancel{2\mu}}\left(\frac{\cancel{2}\mu}{2\mu+3\lambda}\right)\sigma_{kk}$$

Sedaj vstavimo σ_{kk}:

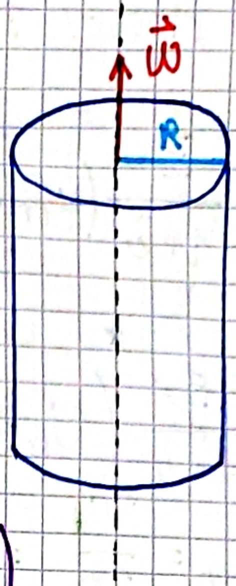
$$\frac{\Delta V}{V} = -\chi \cdot p$$

$$\mathrm{tr}\, u_{ij} = \frac{-3p}{2\mu+3\lambda} = \frac{1-2\sigma}{E}(-3p)$$

Stisljivost $\Rightarrow 3\frac{1-2\sigma}{E} > 0$

Naloga na str. 35

$$\rho\ddot{u} = \vec{f} + \frac{E}{2(1+\sigma)}\left(\nabla^2\vec{u} + \frac{1}{1-2\sigma}\nabla\nabla\cdot\vec{u}\right)$$



$\vec{f} = \rho\omega^2\vec{r}$ } Centripetalna sila

Rešujemo:

$$0 = \rho\omega^2\vec{r} + \frac{E}{2(1+\sigma)}\left(\nabla^2\vec{u} + \frac{1}{1-2\sigma}\nabla\nabla\cdot\vec{u}\right)$$

$\vec{u}(r) = u(r)\hat{e}_r$

$$\nabla(\nabla\cdot\vec{u}) = \nabla^2\vec{u} + \underline{\nabla\times(\nabla\times\vec{u})}$$

Sumimo, da je to 0.
(če preverimo je res)

Izrazimo enačbo z gradientom divergence:

$$0 = \rho\omega^2\vec{r} + \frac{E}{2(1+\sigma)}\left(1 + \frac{1}{1-2\sigma}\right)\nabla\nabla\cdot\vec{u}$$

Ta enačba vsebuje 3 komponente, imamo pa samo r smer:

$$\nabla\nabla\cdot\vec{u} = -\underbrace{\frac{\rho\omega^2(1+\sigma)(1-2\sigma)}{E(1-\sigma)}}_{\alpha}\vec{r} = -\alpha\vec{r}$$

Vzamemo samo r komponento:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru)\right] = -\alpha r \Big/ \int dr$$

$$\frac{\partial(ru)}{\partial r} = -\frac{\alpha r^3}{2} + Ar \Big/ \int dr$$ } divergenca (for later)

$$ru = -\frac{\alpha r^4}{8} + A\frac{r^2}{2} + B$$

$$\Rightarrow u = -\frac{\alpha}{8}r^3 + \frac{Ar}{2} + \frac{B}{r}$$

$B = 0$
↑
ni divergence v r=0

A pa določimo iz tega, da je rob neobremenjen. Tam ni sil!

$$\sigma_{rr}\big|_{r=R} = 0$$

Potrebujemo Hookov zakon: $\sigma_{ij} = \frac{E}{1+\sigma}\left(u_{ij} + \frac{\sigma}{1-2\sigma} u_{kk}\delta_{ij}\right)$

$$\sigma_{rr} = \frac{E}{1+\sigma}\left(u_{rr} + \frac{\sigma}{1-2\sigma}\nabla\cdot u\right)$$

$$\downarrow$$

$$0 = \frac{\cancel{E}}{\cancel{1+\sigma}}\left(u_{rr} + \frac{\sigma}{1-2\sigma}\left(-\frac{\alpha r^2}{2} + A\right)\right)$$

$$u_{rr} = \frac{\partial u}{\partial r} = -\frac{3\alpha}{8}r^2 + \frac{A}{2}$$

$$0 = -\frac{3\alpha}{8}r^2 + \frac{A}{2} + \frac{\sigma}{1-2\sigma}\left(A - \frac{\alpha}{2}r^2\right)$$

$$A\left(\frac{1}{2} + \frac{\sigma}{1-2\sigma}\right) = \alpha r^2\left(\frac{3}{8} + \frac{\sigma}{2(1-2\sigma)}\right)$$

$$A\,\frac{1-2\sigma+2\sigma}{2(1-2\sigma)} = \alpha r^2\,\frac{3-6\sigma+4\sigma}{8(1-2\sigma)}$$

$$\Rightarrow A = \alpha R^2\,\frac{3-2\sigma}{4}$$

Rešitev je tako torej:

$$u = -\frac{\alpha}{8}r^3 + \frac{3-2\sigma}{8}\alpha R^2 r$$

Vstavi α

$$\hookrightarrow u = \frac{\rho\omega^2(1+\sigma)(1-2\sigma)}{8E(1-\sigma)}\left[(3-2\sigma)R^2 r - r^3\right]$$

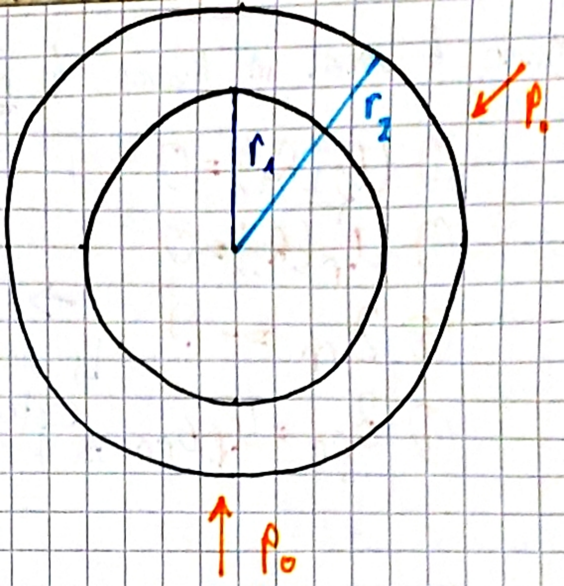
Koliko je upravičen privzetek, da v z smeri ni raztezkov?

$$\sigma_{\phi\phi} = \frac{E}{1+\sigma}\left(u_{\phi\phi} + \frac{\sigma}{1-2\sigma}\nabla\cdot\vec{u}\right) = \qquad u_{\phi\phi} = \cancel{\frac{\partial u_\phi}{\partial\phi}} + \frac{u_r}{r}$$

$$= \frac{E}{1+\sigma}\left(\frac{u}{r} + \frac{\sigma}{1-2\sigma}\left(-\frac{\alpha r^2}{2} + A\right)\right)$$

$$\sigma_{zz} = \frac{E}{1+\sigma}\left(u_{zz} + \frac{\sigma}{1-2\sigma}\nabla\cdot\vec{u}\right) = \frac{E}{1+\sigma}\,\frac{\sigma}{1-2\sigma}\nabla\cdot\vec{u} = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}\left(-\frac{\alpha r^2}{2} + A\right)$$

Naloga na str. 38



$$U = \overbrace{u_r(r)\,\hat{e}_r}^{\vec{u}_1} + \overbrace{e_{\varphi}(r)\,\hat{e}_\varphi}^{\vec{u}_2}$$

$\vec{F} = 0$ … ker je to volumska gostota sile, to ne ve nič o površini. **!**
To bomo uporabili kot robni pogoj

Torej rešujemo:

$$0 = \frac{E}{2(1+\nu)}\left(\nabla^2\vec{u} + \frac{1}{1-2\nu}\nabla\nabla\cdot\vec{u}\right)$$

$$\nabla^2\vec{u} = -\nabla\times\nabla\times\vec{u} + \nabla\nabla\cdot\vec{u}$$

$$\nabla\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \cancel{\frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi}} + \cancel{\frac{\partial v_z}{\partial z}} = 0$$

$$0 = \frac{E}{2(1+\nu)}\left(\left(\frac{2-2\nu}{1-2\nu}\right)\nabla\nabla\cdot\vec{u} - \nabla\times\nabla\times\vec{u}\right)$$

od prej vemo u_1 rot$=0$ u_2 pa zdaj div$=0$

$$0 = \frac{E}{2(1+\nu)}\left[\underbrace{\frac{2-2\nu}{1-2\nu}\nabla\nabla\cdot\vec{u}_1}_{\text{kaže v smeri } \hat{e}_r} - \underbrace{\nabla\times\nabla\times\vec{u}_2}_{\text{kaže v smeri } \hat{e}_\varphi}\right]$$

← Napiši dvojni rotor in poglej da res samo $\hat{e}_\varphi$ smer preživi

$$\hat{e}_r : 0 = \frac{E}{2(1+\nu)}\frac{2-2\nu}{1-2\nu}\nabla\nabla\cdot\vec{u}_1 \qquad \hat{e}_\varphi : 0 = \frac{E}{2(1+\nu)}\nabla\times\nabla\times\vec{u}_2$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) = 0 \;\Big/ \int dr \qquad -\frac{\partial}{\partial r}\left(\frac{1}{r}\left(\frac{\partial(r u_\varphi)}{\partial r}\right)\right)$$

$$\frac{\partial}{\partial r}(r u_r) = A r \;\Big/ \int dr \qquad \Downarrow$$

$$u_\varphi = \frac{1}{2}Cr + \frac{D}{r}$$

$$U_r = \frac{Ar}{2} + \frac{B}{r}$$

Pomembno je:

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\varphi + \frac{1}{r}\left(\frac{\partial(r v_\varphi)}{\partial r} - \frac{\partial v_r}{\partial\varphi}\right)\hat{e}_z$$

Imamo 4 konst. in rabimo 4 RP:

① $u_\phi(R_1) = 0$

② $u_r(R_1) = 0$

③ $\sigma_{rr} = -p$

④ $\sigma_{\phi r} = k_t \sigma_{rr}$

→ bi nam dalo sistem 4 enačb za 4 neznanke

① $0 = \frac{1}{2} C R_1 + \frac{D}{R_1}$

② $0 = \frac{1}{2} A R_1 + \frac{B}{R_1}$

③ $-p = \frac{E}{1+\sigma}\left[u_{rr} + \frac{\sigma}{1-2\sigma} \nabla \cdot \vec{u}_1\right]$

④ $\frac{E}{1+\sigma} \boxed{u_{\phi r}} = k_t \frac{E}{1+\sigma}\left[u_{rr} + \frac{\sigma}{1-2\sigma} \nabla \cdot \vec{u}_1\right]$

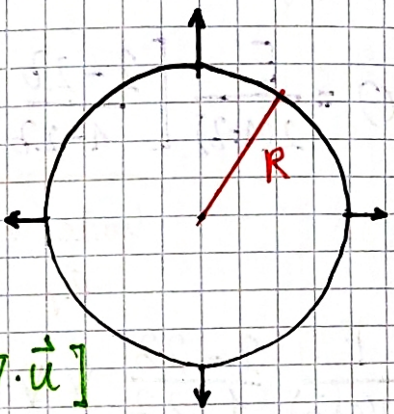
$$\downarrow$$

$$u_{\phi r} = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} + \frac{\partial u_r}{r \partial \phi}\right)$$

Da dobimo to kar naloga zahteva:

$$\Delta \phi = \frac{u_\phi(R_2)}{R_2}$$

Naloga na strani 40.



$\vec{u} = u(r)\hat{e}_r$

sumimo $\nabla \times \vec{u} = 0$

Rešujemo: $\rho \ddot{\vec{u}} = \vec{f} + \frac{E}{2(1+\sigma)}\left[\nabla^2 \vec{u} + \frac{1}{1-2\sigma} \nabla \nabla \cdot \vec{u}\right]$

$\nabla \nabla \cdot \vec{u} = \nabla^2 \vec{u} + \nabla \times \nabla \times \vec{u}$

$$\rho \ddot{\vec{u}} = \vec{f} + \frac{E}{2(1+\sigma)}\left[\nabla \nabla \cdot \vec{u} + \frac{1}{1-2\sigma} \nabla \nabla \cdot \vec{u}\right]$$

$$= \frac{E}{2(1+\sigma)} \frac{1-2\sigma+1}{1-2\sigma} \nabla \nabla \cdot \vec{u} =$$

$$= \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \nabla \nabla \cdot \vec{u}$$

Uporabimo: $\vec{u} = \vec{u}_0 e^{i\omega t}$ da pridemo do Helmholtzove enačbe:

$$-\omega^2 \rho \vec{u}_0 \cancel{e^{i\omega t}} = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \nabla \nabla \cdot \vec{u}_0 \cancel{e^{i\omega t}}$$

$$\nabla \nabla \cdot \vec{u}_0 = -\overbrace{\frac{\omega^2 \rho (1+\sigma)(1-2\sigma)}{E(1-\sigma)}}^{k^2} \vec{u}_0$$

$$\Rightarrow \nabla \nabla \cdot \vec{u}_0 + k^2 \vec{u}_0 = 0$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_0)\right) + k^2 u_0 = 0$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2}\left(2r u_0 + r^2\frac{\partial u_0}{\partial r}\right)\right) + k^2 u_0 = 0$$

$$\frac{\partial}{\partial r}\left(\frac{2u_0}{r} + \frac{\partial u_0}{\partial r}\right) + k^2 u_0 = 0$$

$$\frac{2}{r}\frac{\partial u_0}{\partial r} - \frac{2u_0}{r^2} + \frac{\partial^2 u_0}{\partial r^2} + k^2 u_0 = 0 \Rightarrow \frac{\partial^2 u_0}{\partial r^2} + \frac{2}{r}\frac{\partial u_0}{\partial r} + u_0\left(k^2 - \frac{2}{r^2}\right) = 0$$

Prepoznamo lahko Besselovo enačbo, ta ima standardno obliko:

$$\frac{\partial^2 z_l}{\partial r^2} + \frac{2\partial z_l}{r\partial r} + \left(1 - \frac{l(l+1)}{r^2}\right) z_l = 0 \qquad Z = \{\underset{\substack{\uparrow \\ \text{Besselova funkcija}}}{j},\ \underset{\substack{\uparrow \\ \text{Neumannova funkcija}}}{n}\}$$

Sumimo $l=1$

Podelimo z k^2:

$$\Rightarrow \frac{\partial^2 u_0}{\partial (kr)^2} + \frac{2}{(kr)}\frac{\partial u_0}{\partial (kr)} + u_0\left(1 - \frac{2}{(kr)^2}\right) = 0$$

Res $l=1$!

Rešitev: $u_0 = A j_1(kr) + B n_1(kr)$

divergira v izhodišču

Potrebujemo še robni pogoj za določitev konstante A:

$$\sigma_{rr} = 0$$

$$\sigma_{rr} = \frac{E}{1+\sigma}\left(u_{rr} + \frac{\sigma \nabla\cdot\vec{u}}{1-2\sigma}\right)\Bigg|_{r=R} = 0$$

Vstavimo našo rešitev u_0:

$$\frac{\partial j_1(kr)}{\partial r} + \frac{\sigma}{1-2\sigma}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 j_1(kr)\right)\Bigg|_{r=R} = 0$$

(comment later)

$$\frac{k\,\partial j_1(kr)}{\partial(kr)} + \frac{z}{1-2z}\frac{k}{(kr)^2}\frac{\partial}{\partial(kr)}\left((kr)^2 j_1(kr)\right)\Big|_{r=R} = 0 \quad (\star)$$

Uporabimo zveze med odvodi:

Zveze med odvodi Besselovih funkcij:

$$-x^{-l} z_{l+1}(x) = \left(x^{-l} z_l(x)\right)'$$

$$x^{l+1} z_{l-1}(x) = \left(x^{l+1} z_l(x)\right)'$$

$$-j_1 = \dot{j}_0 = \left(\frac{\sin x}{x}\right)' =$$

$$= \left(\frac{\sin(kr)}{(kr)}\right)' = \frac{\cos(kr)\,kr - \sin(kr)}{(kr)^2}$$

$$\underbrace{\frac{\partial}{\partial(kr)}\left((kr)^2 j_1(kr)\right) = (kr)^2 j_0(kr)}$$

$$\frac{\partial j_1}{\partial(kr)} = -\frac{\partial}{\partial(kr)}\left(\frac{\cos(kr)}{kr} - \frac{\sin(kr)}{(kr)^2}\right) =$$

$$= -\left[-\frac{\sin(kr)}{kr} - \frac{\cos(kr)}{(kr)^2} - \frac{\cos(kr)}{(kr)^2} + \frac{2\sin(kr)}{(kr)^3}\right]$$

Poglejmo nazaj naš pogoj $(\star)$. Vržemo stran trivialno rešitev $k=0$ (comment later):

Smemo ker cos ni rešitev. Dodili bi nekaj=0 od + sinusov

$$0 = \Big|_{r=R} \frac{\sin(kr)}{kr} + \frac{2\cos(kr)}{(kr)^2} - \frac{2\sin(kr)}{(kr)^3} + \underbrace{\frac{z}{1-2z}\frac{1}{(kr)^2}(kr^2)\frac{\sin(kr)}{(kr)}} \Big/ :\cos(kr) \cdot (kr)^3$$

$$(kr)^2 tg(kr) + 2(kr) - 2\,tg(kr) + \frac{z}{1-2z}(kr)^2 tg(kr) = \Big|_{R=r} 0$$

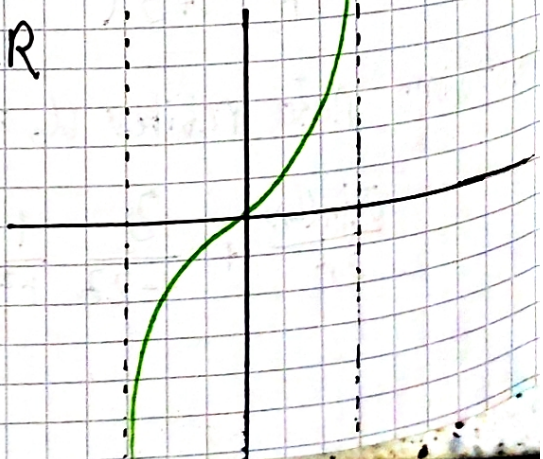
$$tg(kR)\underbrace{\left[(kR)^2 - 2 + \frac{z}{1-2z}(kR)^2\right]} + 2kR = 0$$

Ker je krožek poglejmo to

$$2\left[-1 + \frac{1-z}{2-4z}(kR)^2\right]$$ → če bi bilo to 0 bi imeli neke čudne nerealne z

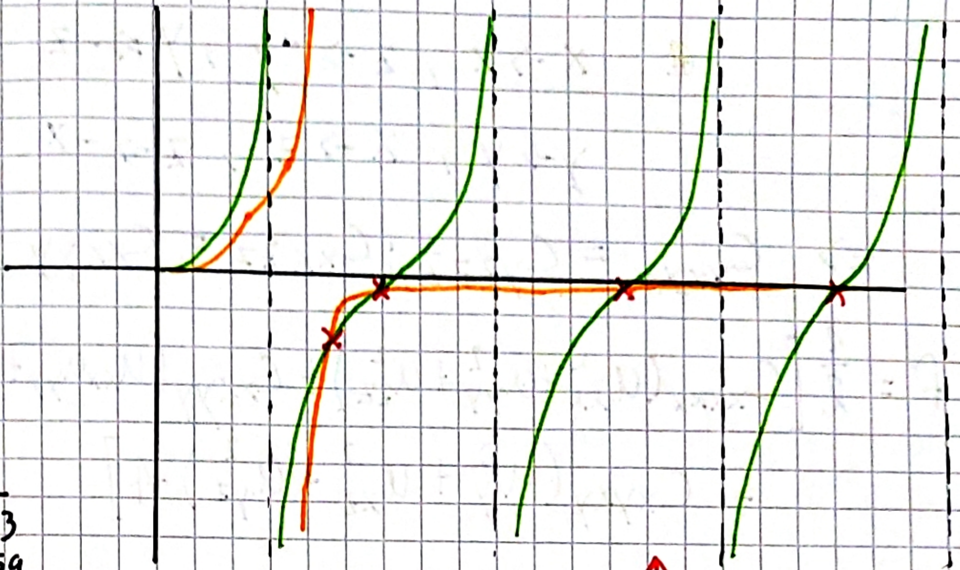
$$\Rightarrow \quad 2\,tg(kR)\left[\frac{1-z}{2(1-2z)}(kR)^2 - 1\right] = -2kR$$

$$\rightarrow \quad tg(kR) = \frac{kR}{1 - \frac{1-z}{2(1-2z)}(kR)^2}$$



$$k^2 = \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)}\rho\omega^2 > 0$$

ξ
↓
$$\frac{X}{1-AX^2}$$



V primeru realnih števil:

$\sigma = 0{,}25$ ... jeklo

$\xi_i = k_i R$

$\xi_i = i\pi$

za $k_i R \gg 1$

i	ξ_i
1	2,563
2	6,059
3	9,280
4	12,489

Za pozne rešitve seka tangens praktično v ničlah tangensa.

Torej:

$$\omega_i = k_i\sqrt{\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)\rho}} = \frac{\xi_i}{R}\sqrt{\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)\rho}}$$

Naloga na strani 20.

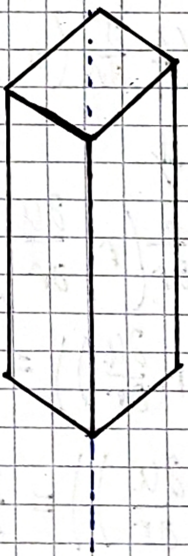
Simetrijska grupa
↓
C_{4V}

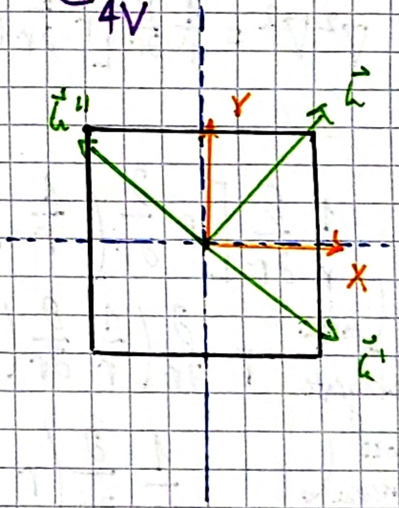
$$f = \frac{1}{2}C_{ijkl}u_{ij}u_{kl}$$

$x \to -x$, $y \to y$, $z \to z$

$x \to x$, $y \to -y$, $z \to z$

$x \to y$, $y \to -x$, $z \to z$





$$\Rightarrow C_{xxxx} = C_{yyyy},\quad C_{xxzz} = C_{yyzz},\quad C_{xxxz} = C_{yyyz}$$

$$f = \frac{1}{2}\Big[C_{xxxx}(u_{xx}^2 + u_{yy}^2) + C_{zzzz}u_{zz}^2 + C_{xxzz}(u_{xx}u_{zz} + u_{xy}u_{zz})\cdot 2 +$$
$$+ C_{xxyy}u_{xx}u_{yy}\cdot 2 + C_{xyxy}u_{xy}^2\cdot 4 + C_{xzxz}(u_{xz}^2 + u_{yz}^2)\cdot 4\Big]$$

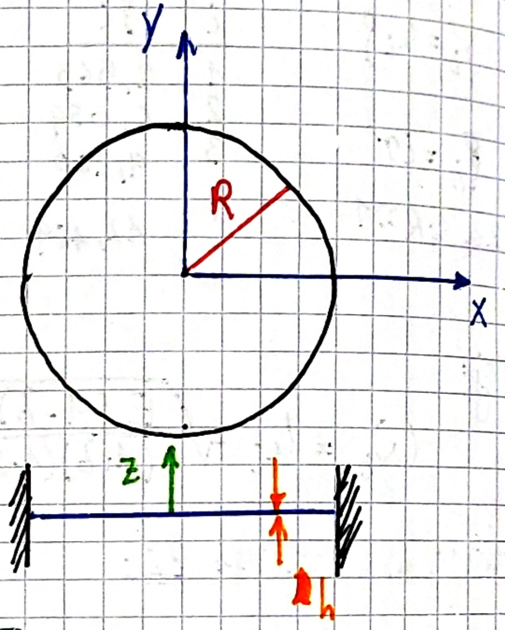
Še za kubično simetrijo velja še dodatno:

- $x \to x\ ,\ z \to y\ ,\ y \Rightarrow -z$

$$y \to y\ ,\ x \to z\ ,\ z \to -x$$

$$\Rightarrow C_{xxxx} = C_{yzzz}\ ;\ C_{xxzz} = C_{xxyy}\ ;\ C_{xyxy} = C_{xzxz}$$

$$f = \frac{1}{2}\Big[C_{xxxx}(u_{xx}^2 + u_{yy}^2 + u_{zz}^2) + C_{xxyy}(u_{xx}u_{yy} + u_{yy}u_{zz} + u_{xx}u_{zz})\cdot 2$$
$$+ C_{xyxy}(u_{xy}^2 + u_{xz}^2 + u_{yz}^2)\cdot 4\Big]$$



Naloga na strani

$$D\nabla^2\nabla^2 u - P = 0\ ;\ D = \frac{Eh^3}{12(1-\sigma^2)}$$

$$P = \frac{mg}{S} = \frac{\rho V g}{S} = \frac{\rho S h g}{S} = \rho g h$$

$$\nabla^2 = \nabla\cdot\nabla = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

$$\nabla^2\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)\right] \qquad \alpha$$

$$\Downarrow$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right)\right] = \frac{\rho h g}{Eh^3}12(1-\sigma^2)$$

množi integrate

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) = \frac{1}{2}\alpha r^2 + C_1$$

množi integrate

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{4}\alpha r^2 + \cancel{C_1 \ln r} + C_2$$

ni singularnosti v izhodišču

$$r\frac{\partial u}{\partial r} = \frac{1}{16}\alpha r^4 + \frac{1}{2}C_2 r^2 + \cancel{C_3}$$

$$u = \frac{1}{4\cdot 16}\alpha r^4 + \frac{1}{4}C_2 r^2 + C_4$$

Robni pogoj za Vzidano zadevo je preprost. Vemo, da je na robu:

① $u(R) = 0$ -> miruje na robu

② $\frac{\partial u}{\partial r}(R) = 0$ -> ni vrtenja v normalni smeri

① $\frac{1}{64}\alpha R^4 + \frac{1}{4}C_2R^2 + C_4 = 0$

② $\frac{1}{16}\alpha R^3 + \frac{1}{2}C_2R = 0$

$\Rightarrow C_2 = -\frac{\alpha R^2}{8}$

$C_4 = \frac{1}{64}\alpha R^4$

$\Rightarrow \quad u(r) = \frac{\alpha}{64}\left(r^4 - 2R^2r^2 + R^4\right) = \frac{\alpha}{64}$

nalogа na strani

Naslonjena točkasta sila lastni pores zanemarljiv

$F_0^{\delta(x)\delta(y)}$

$2\pi r$... pride od Jacobi det če greš iz kartezične v cilindrične koordinate

$P = F_0 \dots = F_0 \frac{\delta(r)}{2\pi r}$

$F_0 = \int \delta(r) F_0 \cdot \underbrace{dS}_{2\pi r\,dr} \leftrightarrow \int P\,dS = \int \frac{\delta(r)}{2\pi r} F_0\, 2\pi r\, dr$

$$D\nabla^2\nabla^2 u = P = \frac{\delta(r)}{2\pi r}F_0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)\right]u = \frac{\delta(r)F_0}{2\pi r D} = \alpha\frac{\delta(r)}{r}$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)u = \int\alpha\delta(r)dr = \alpha + \cancel{C_1'}$$

integral je določen, ta konstantu bi pomenila še eno točkasto silo zraven

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)u = \int\frac{\alpha}{r}dr = \alpha\ln r + C_1$$

redefiniramo C_1 $= \alpha\ln\frac{r}{R} + C_1$

$\vdots$

$$r\frac{\partial}{\partial r}u = \alpha\left(\frac{r^2}{2}\ln\frac{r}{R} - \frac{r^2}{4}\right) + \frac{C_1r^2}{2} + C_2$$

$\vdots$

$$u = \alpha\left(\frac{r^2}{4}\ln\left(\frac{r}{R}\right) - \frac{r^2}{8} - \frac{r^2}{8}\right) + \frac{C_1r^2}{4} + C_2\ln\frac{r}{R} + C_3$$

$$= \frac{\alpha r^2}{8}\left(2\ln\frac{r}{R} - 2\right) + \frac{C_1r^2}{4} + C_3$$

Robni pogoji za naslonjeno zadevo:

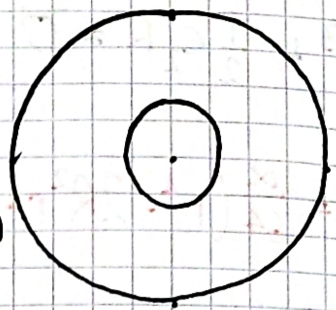
RP1 $u(r=R) = 0$

RP2 $\overbrace{\frac{d^2u}{dr^2} + [illegible]\partial\frac{d\phi}{dl}\frac{du}{dr}}^{\text{Navor je enak } 0} = 0$

naloga na strani

koncentrična sila

$$P = F_0 \frac{\delta(r-r_0)}{2\pi r}$$



Če je pomnožena z heaviside je sila na obroču drugeč je v izhodišča

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial}\right)\right)\right) = \alpha\frac{\delta(r-r_0)}{r}$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right) = \alpha H(r-r_0) + 0$$

∴ ni konstante, ker bi bila dodatna sila

Iz Wiki:

$$\int dx\, f(x) H(x-x_0) = [F(x)-F(x_0)]H(x-x_0) + C$$

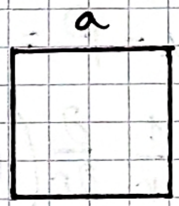
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \int\frac{\alpha}{r}H(r-r_0)dr$$

$$= \alpha\left[\ln\frac{r}{r_0}\right]H(r-r_0) + C_1$$

$$r\frac{\partial u}{\partial r} = \int\left(\alpha r\ln\frac{r}{r_0}H(r-r_0) + C_1 r\right)dr + C_2$$

nalga na strani 52.

Poves zaradi lastne teže, razvoj nastavka; naslonjena plošča.



$$u(x,y) = \sum_{m,n} a_{m,n}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

Štartamo iz funkcionala energije:

$$F = F_0 + \frac{1}{2}K\int\Big[(\Delta_2 u)^2 + 2(1-\sigma)\Big(\Big(\frac{\partial^2 u}{\partial x\partial y}\Big)^2 - \Big(\frac{\partial^2 u}{\partial x^2}\Big)\Big(\frac{\partial^2 u}{\partial y^2}\Big)\Big]dS -$$

$$-\int uP\,dS$$

RP1: $u(\text{rob}) = 0$

RP2: $\frac{\partial^2 u}{\partial n^2} + \sigma\frac{\partial\phi}{\partial l}\frac{\partial u}{\partial n} = 0$

Z izbiro funkcij smo pokrili robne pogoje!

2 vmesna komentarja na prejšnje naloge.

①

- Splošna točkasta sila:

$$f_0(r,l) = F\delta(r-r_0)\delta(l-l_0)$$

Sila porazdeljena po neskončno tankem prstanu:

$$f_1(r) = \int_0^{2\pi r_0} dl_0 \frac{dF}{dl_0} f_0(r,l) = \int_0^{2\pi r_0} dl \frac{F}{2\pi r_0}\delta(r-r_0)\delta(l-l_0) =$$

$$= \frac{F}{2\pi r_0}\delta(r-r_0) = \frac{F}{2\pi r}\delta(r-r_0)$$

② Imeli smo: $\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\right\}u = \alpha\frac{\delta(r)}{r}$

Integral od 0 do $r>0$:

$$r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]u\Big|_0^r = \alpha \boxed{+\beta}$$

! Te konstante ni! Tako smo se zmenili, da ni dodatne točkaste sile.