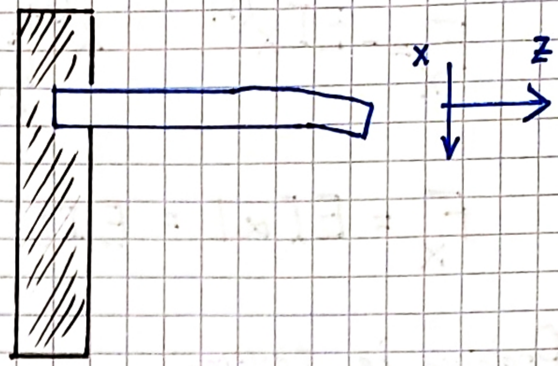
Naloga na str. 71

Poves pod lastno težo?

$$\vec{M} \approx EI(-\ddot{y}, \ddot{x}, 0)$$

$$F_x = -EIx^{(3)} + F_z\dot{x}$$

$$F_y = -EIy^{(3)} + F_z\dot{y}$$



$$EIx^{(4)} - F_z\ddot{x} - \dot{F}_z\dot{x} - K_x = 0$$

$$EIy^{(4)} - F_z\ddot{y} - \dot{F}_z\dot{y} - K_y = 0$$

Pri nas:

zunanja sila

$$EIx^{(4)} - \cancel{F_z}\ddot{x} - \cancel{\dot{F}_z}\dot{x} - K_x = 0 \quad ; \quad K_x = \frac{mg}{L}$$

ni sile v smeri z

$$\Rightarrow \quad x^{(4)} = \alpha \quad ; \quad \alpha = \frac{mg}{LEI}$$

To lahko zelo hitro rešimo: $x = \frac{\alpha}{24}z^4 + Az^3 + Bz^2 + \underbrace{\cancel{Cz}}_{0\ \boxed{RP2}} + \underbrace{\cancel{D}}_{0\ \boxed{RP1}}$

Robni pogoji:

1. $x(z=0) = 0$
2. $\dot{x}(z=0) = 0$ } ne bi veljalo če bi bilo vrtljivo vpeto
3. $M(z=L) = 0 \Rightarrow \ddot{x}(z=L) = 0$ $\ddot{y}(z=L) = 0$
4. $F_x(z=L) = 0 \Rightarrow x^{(3)}(z=L) = 0$

   $F_y(z=L) = 0 \Rightarrow y^{(3)}(z=L) = 0$

$$\dot{x} = \frac{\alpha}{6}z^3 + 3Az^2 + 2Bz$$

$$\ddot{x} = \frac{\alpha}{2}z^2 + 6Az + 2B \quad \Rightarrow 0 = \frac{\alpha}{2}L^2 + 6AL + 2B \Rightarrow B = -\frac{\alpha}{4}L^2 + \frac{\alpha L^2}{2}$$

$$\Rightarrow \underline{\underline{B = +\frac{\alpha}{4}L^2}}$$

$$x^{(3)} = \alpha z + 6A \quad \Rightarrow \alpha L = -6A$$

$$\Rightarrow \underline{\underline{A = -\frac{\alpha L}{6}}}$$

$$\Rightarrow x = \frac{\alpha}{24}z^4 - \frac{\alpha L}{6}z^3 + \frac{\alpha}{3}L^2z^2$$

$$F_x = -EI x^{(3)} \qquad x^{(3)}\Big|_{z=0} = -\frac{\alpha L^2}{6} z + \alpha L^2$$

$$F_x = EI\alpha L = EIL\frac{mg}{ELI} = mg$$ } Sila na desni presek (v smeri naraščujočega z)

Poglejmo še navore:

0 se dogaja v ravnini

$$\vec M(z=0) = EI\left(\cancel{-\ddot y(z=0)}, \ddot x(z=0), 0\right)$$

$$\vec M = EI\frac{\alpha}{2}L^2 \hat e_y = \frac{EI}{2}\frac{mg}{LEI}L^2\hat e_y$$

$$\Rightarrow \quad \vec M(z=0) = \frac{L}{2} mg\, \hat e_y$$

Naloga na strani. [Eulerjeva nestabilnost]

Startamo iz:

$$EIx^{(4)} - F_z\ddot x - \cancel{\dot F_z \dot x} - K_x = 0$$

ker se F_z ne spreminja vzdolž z

$= -F_z$

$$v = \ddot x$$

$$EI\ddot v + |F_z| v = 0 \;/ : EI$$

$$\ddot v + \overbrace{\frac{|F_z|}{EI}}^{k^2} v = 0$$

Vzamemo rešitev/nastavek:

$$v = A'\sin(kz) + B'\cos(kz) = \ddot x$$

$$\Downarrow$$

$$x = D\sin(kz) + C\cos(kz) + A + Bz$$

Robni pogoji:

i) $x(0) = 0$

ii) $x(L) = 0$

iii) $\ddot x(0) = 0$ (ni navorov)

iv) $\ddot x(L) = 0$ (ni navorov) $\Rightarrow$

i) $A + C = 0$

iii) $Ck^2 = 0 \Rightarrow \boxed{C = 0}$ } $\boxed{A = 0}$

$$x = D\sin(kz) + Bz$$

(ii) $\Rightarrow D\sin(kL) + BL = 0$

(iv) $-Dk^2\sin(kL) = 0$

$$\sin(kL) = 0$$

$$k = \frac{n\pi}{L}$$

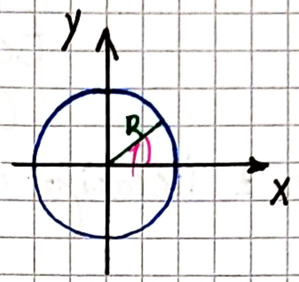
Samo $n=1$ je fizikalen mode. Ostalo bi bila ostala vpetja na polovici recimo ipd.:

$$x = D\sin\left(\frac{z\pi}{L}\right)$$

$$\frac{\pi}{L} = \sqrt{\frac{|F_z|}{EI}}$$

$$|F_z| = \frac{EI\pi^2}{L^2}$$

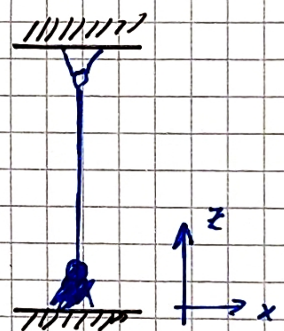
Izračunajmo še moment I:



$$I = \int x^2 dS$$

$$I = 4\int_0^R \int_0^{\frac{\pi}{2}} r^2\cos^2\varphi \, r\,dr\,d\varphi = \pi\frac{R^4}{4}$$

naloga isto kot prej le Asimetrično vpetje



Vzamemo lahko rešitev od prej. Le robni pogoji se spremenijo.

Robni pogoji so tako:

$$x = A + Bz + C\cos(kz) + D\sin(kz)$$

① $x(0) = 0$	① $A + C = 0 \Rightarrow A = -C$
② $x(L) = 0$	③ $0 = kD - B \Rightarrow B = -kD$
③ $\dot{x}(0) = 0$	② $-\frac{B}{k}\sin(kL) + C\cos(kL) + BL - C = 0$
④ $\ddot{x}(L) = 0$	④ $\ddot{x} = -k^2 D\sin(kL) - k^2 C\cos(kL) = 0$

Torej:

$$\text{②}, \text{④} \Rightarrow \det\begin{vmatrix} -k^2 D\sin(kL) & -k^2 C\cos(kL) \\ D(\sin(kL) - kL) & + C(\cos(kL) - 1) \end{vmatrix} = 0$$

$$\det = -k^2 \sin(kL)[\cos(kL) - 1] + k^2 \cos(kL)[\sin(kL) - kL] = 0$$

$$\sin(kL)[\cos(kL) - 1] = \cos(kL)[\sin(kL) - kL]$$

$$\mathrm{tg}(kL)[\cos(kL) - 1] = \sin(kL) - kL$$

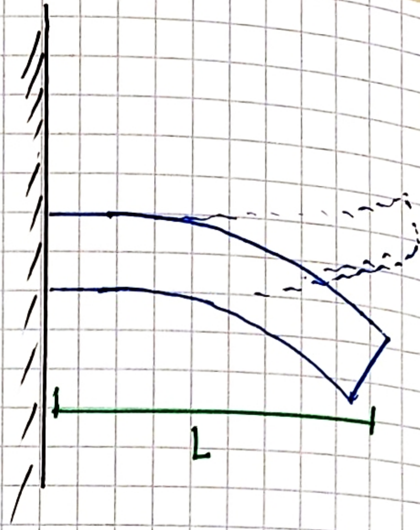
$$\sin(kL) - \mathrm{tg}(kL) = \sin(kL) - kL$$

$$\mathrm{tg}(kL) = kL$$

To rešimo z iteracijo arctan na kalkulatorju. $kL \approx 4{,}49$.

Naloga na str. 76

$$EI x^{(4)} - F_z \ddot{x} - \dot{F}_z \dot{x} - K_x = -\rho \frac{d^2 x}{dt^2}$$



$$EI x^{(4)} = -\rho \frac{\partial^2 x}{\partial t^2}$$

Rešujemo:

$$x^{(4)} + \frac{\rho}{EI} \frac{\partial^2 x}{\partial t^2} = 0$$

Vzemimo časovni nastavek:

$$u(z,t) = V(z) e^{-i\omega t}$$

$$\Rightarrow \quad V^{(4)} e^{-i\omega t} - \frac{\rho}{EI} V \omega^2 e^{-i\omega t} = 0$$

$$V^{(4)} - \frac{\rho}{EI} V = V^{(4)} - \alpha^4 V = 0$$

Rešitev so eksponenti oz.:

$$V = A\,\mathrm{sh}\,\alpha z + B\,\mathrm{ch}\,\alpha z + C \sin \alpha z + D \cos \alpha z$$

Robni pogoji:

i) $V(0) = 0 \quad \Rightarrow \quad B + D = 0 \quad \boxed{D = -B}$

ii) $\dot{V}(0) = 0 \quad \Rightarrow \quad A\alpha + C\alpha = 0 \quad \boxed{C = -A}$

iii) $\ddot{V}(L) = 0 \Rightarrow A\alpha^2 \mathrm{sh}\,\alpha L + B\alpha^2 \mathrm{ch}\,\alpha L - C\alpha^2 \sin\alpha L - D\alpha^2 \cos\alpha L = 0$

iv) $V^{(3)}(L) = 0 \Rightarrow A\alpha^3 \mathrm{ch}\,\alpha L + B\alpha^3 \mathrm{sh}\,\alpha L - C\alpha^3 \cos\alpha L + D\alpha^3 \sin\alpha L = 0$

$$A(\mathrm{sh}\,\alpha L + \sin\alpha L) + B(\mathrm{ch}\,\alpha L + \cos\alpha L) = 0$$

$$A(\mathrm{ch}\,\alpha L + \cos\alpha L) + B(\mathrm{sh}\,\alpha L - \sin\alpha L) = 0$$

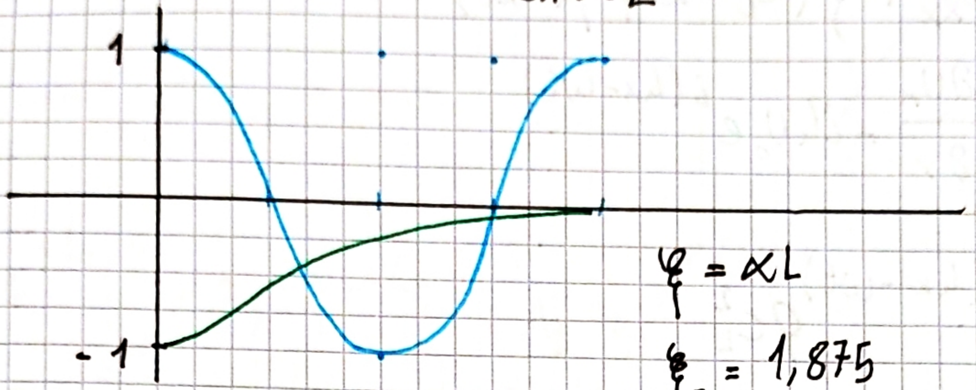
Zahtevamo det = 0, da ima sistem rešitve:

$$\mathrm{sh}^2\alpha L - \sin^2\alpha L - \mathrm{ch}^2\alpha L - \cos^2\alpha L - 2\,\mathrm{ch}\,\alpha L \cos\alpha L = 0$$

Uporabimo: $\sin^2 + \cos^2 = 1$ $\quad sh^2 - ch^2 = 1$

$$-2 - 2\,ch\,\alpha L \cos\alpha L = 0$$

$$\Rightarrow \quad \cos\alpha L = \frac{-1}{ch\,\alpha L}$$



$\xi = \alpha L$

$\xi_0 = 1{,}875$

$\xi_1 = 4{,}694$

$\xi_2 = 7{,}855$

$\xi_n = \frac{(2n+1)}{2}\pi$

Poglejmo sedaj še frekvence:

$$\alpha^4 = \frac{\varrho}{EI}\omega^2 \qquad \omega = \pm\sqrt{\left(\frac{\xi}{L}\right)^4 \frac{EI}{\varrho}}$$
$$= \pm\left(\frac{\xi}{L}\right)^2 \sqrt{\frac{EI}{S}}$$

Naloga na str. 82

$$Z = \frac{p}{v} \quad \dots \text{impedanca}$$

Koliko tlaka rabimo, da dodamo" hitrost v?

Imejmo tlačni val v x smeri:

$$\vec{u} = \vec{u}_0 e^{i(kx - \omega t)}$$

Mi imamo za longitudinalni val odmik v x in div v x:

$$Z = \frac{\sigma_{xx}}{v_x}$$

$$\cancel{\sigma_{xx} = 2\rho},$$

$$\sigma_{ij} = 2\rho c_T^2 u_{ij} + \rho(c_L^2 - 2c_T^2) u_{kk}\delta_{ij}$$

edini neničelen

$$\sigma_{xx} = 2\rho c_T^2 u_{xx} + \rho(c_L^2 - 2c_T^2)(u_{xx}) =$$

$$U_{xx} = \frac{\partial U_x}{\partial x} = ikU_0 e^{i(kx-\omega t)}$$

$$= ikU_0 e^{i(kx-\omega t)} \rho c_L^2$$

$$v_x = \frac{\partial U_x}{\partial t} = -i\omega U_0 e^{i(kx-\omega t)}$$

Dobimo vmesen rezultat:

$$Z = \frac{k\rho c_L^2}{\omega}$$

Tu upoštevamo še:

$$\omega^2 = k^2 c_L^2$$

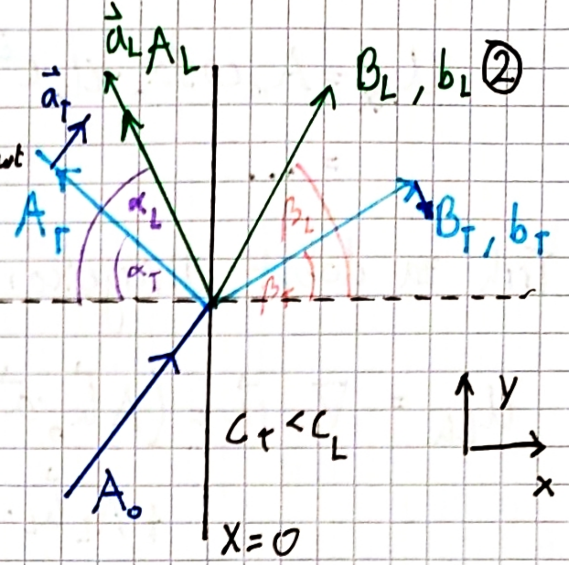
In dobimo:

$$Z = \rho c_L \; ; \; c_L^2 = \frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}$$

$$Z = \frac{\sqrt{\rho}\sqrt{E(1-\sigma)}}{\sqrt{(1+\sigma)(1-2\sigma)}}$$

Za transverzalno (vzameš σ_{xy}) pride pa $Z = \rho c_L$.

nalaga na str. 83



$$\vec{U}_1 = \left(A_0 \vec{a}_0 e^{i\vec{k}_0 \cdot \vec{r}} + A_T \vec{a}_t e^{i\vec{k}_T \cdot \vec{r}} + A_L \vec{a}_L e^{i\vec{k}_2 \cdot \vec{r}}\right) e^{-i\omega t}$$

$$\vec{U}_2 = \left(B_T \vec{b}_T e^{i\vec{q}_T \cdot \vec{r}} + B_L \vec{b}_L e^{i\vec{b}_L \cdot \vec{r}}\right) e^{-i\omega t}$$

To enačimo na meji $x = 0$:

$$A_0 \vec{a}_0 e^{ik_{0y} y} + A_y \vec{a}_T e^{ik_{Ty} y} + A_L \vec{a}_L e^{ik_{Ly} y} =$$

$$= B_T \vec{b}_T e^{iq_{Ty} y} + B_L \vec{b}_L e^{iq_{Ly} y}$$

$$\Rightarrow k_{0y} = k_{Ty} = k_{Ly} = q_{Ty} = q_{Ly}$$

Upoštevajmo disperzijsko relacijo!

Tako dobimo odbojni zakon:

$$\frac{\sin\alpha_0}{c_{1L}} = \frac{\sin\alpha_T}{c_{1T}} = \frac{\sin\alpha_L}{c_{1L}} = \frac{\sin\beta_T}{c_{2T}} = \frac{\sin\beta_L}{c_{2L}}$$

Poglejmo sedaj amplitude odbitega valovanja če drugega sredstva sploh ni.

$$\sigma_{ix} = 0 \;!$$ ni sile na meji prehoda

Zapišemo Hookov zakon:

$$\sigma_{xx} = 2\rho c_T^2 u_{xx} + \rho(c_L^2 - 2c_T^2) u_{kk}$$

$$\sigma_{yx} = 2\rho c_T^2 u_{yx}$$

$$u_{xx} = \frac{\partial u_x}{\partial x} \qquad u_x = A_0 \cos\alpha\, e^{i\vec{k}_0 \cdot \vec{r}} - A_L \cos\alpha_L e^{i\vec{k}_L \cdot \vec{r}} +$$

$$+ A_T \sin\alpha_T e^{i\vec{k}_T \cdot \vec{r}}$$

$$u_y = A_0 \sin\alpha_0 e^{i\vec{k}_0 \cdot \vec{r}} + A_L \sin\alpha\, e^{i\vec{k}_L \cdot \vec{r}} + A_T \cos\alpha_T e^{i\vec{k}_T \cdot \vec{r}}$$

$$u_{xx} = A_0 \cos^2\alpha_0 \cdot ik_0 e^{i\vec{k}_0\cdot\vec{r}} + A_L \cos^2 ik_0 e^{i\vec{k}_L\cdot\vec{r}} - A_T \sin\alpha_T \cos\alpha_T ik e^{i\vec{k}_T\cdot\vec{r}}$$

$$= \dots$$

Trik za dobit divergenco:

$$u_{kk} = \nabla\cdot\vec{u} = \Big(A\vec{k}_0\cdot\vec{a}_0 e^{i\vec{k}_0\cdot\vec{r}} + \overbrace{A_T \vec{k}_T\cdot\vec{a}_T}^{0} e^{i\vec{k}_T\cdot\vec{r}} + A_L\vec{k}_L\cdot\vec{a}_L e^{i\vec{k}_L\cdot\vec{r}}\Big) e^{-i\omega t}$$

Naloga na str. 25

$$\Gamma = \oint \vec{v}\cdot d\vec{l} = \int (\nabla \times \vec{v})\cdot d\vec{S} = \int \vec{\omega}\cdot d\vec{S}$$

$$\downarrow$$

$$\Gamma = 2\pi r v \Rightarrow v(r) = \frac{\Gamma}{2\pi r}$$

$$\operatorname{tg}\varphi = \frac{l}{\rho}$$

$$\downarrow$$

$$-\frac{d\varphi}{\cos^2\varphi} = \frac{dl}{\rho}$$

$$\frac{r}{\rho} = \frac{1}{\cos\varphi}$$

Biot-Savart:

$$\nabla \times \vec{H} = \vec{j}$$

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \frac{\vec{j}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Analogija:

$$\nabla \times \vec{v} = \vec{\omega}$$

$$\vec{v}(\vec{r}) = \frac{1}{4\pi}\int d^3r' \frac{\vec{\omega}(\vec{r}) \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} =$$

Samo konstanta

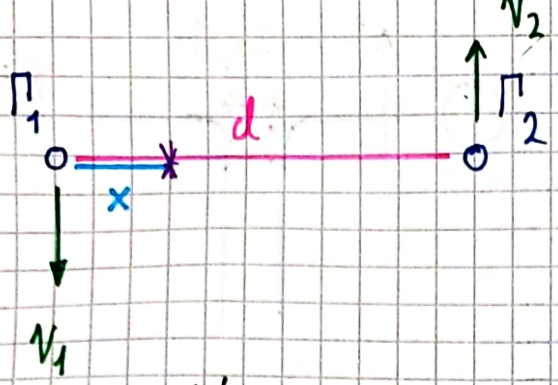
$$= \frac{1}{4\pi}\int \frac{\boxed{\Gamma(\vec{r}')}\, d\vec{l}' \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} =$$

$$= \frac{\Gamma}{4\pi}\int \frac{d\vec{l}'\cos\varphi}{|\vec{r}-\vec{r}'|^2} = \cancel{\frac{\Gamma}{4\pi}\int\left(-\left(\frac{r}{\rho}\right)^2 \cdot \frac{dl}{\rho}\right)} =$$

$$= \frac{\Gamma}{4\pi}\int \frac{-\rho\, d\varphi \cos\varphi}{\cos^2\varphi\, r^2} =$$

$$= \frac{\Gamma}{4\pi}\int_{-\pi/2}^{\pi/2} \frac{\cos\varphi}{\rho}\, d\varphi = \frac{\Gamma}{4\pi\rho}\sin\varphi \Big|_{-\pi/2}^{\pi/2} =$$

$$= \frac{\Gamma}{2\pi\rho}$$



$$\Gamma_1, \Gamma_2 > 0$$

$$v_1 = \frac{\Gamma_2}{2\pi d}$$

$$v_2 = \frac{\Gamma_1}{2\pi d}$$

$$v_1 = \omega x \qquad \Gamma_2 = 2\pi d \omega x$$

$$v_2 = \omega(d-x) \qquad \Gamma_1 = 2\pi d \omega - 2\pi \omega x$$

$$\cancel{\Gamma_1 = 2\pi d \cdot \frac{\Gamma_2}{2\pi d x} - 2\pi \frac{\Gamma_2}{2\pi d x} x}$$

$$\cancel{\Gamma_1 = \frac{\Gamma_2}{x} - \frac{\Gamma_2}{d} x}$$

$$\Gamma_1 = 2\pi d^2 \omega - 2\pi d x \omega$$

$$\Gamma_2 = 2\pi d \omega x$$

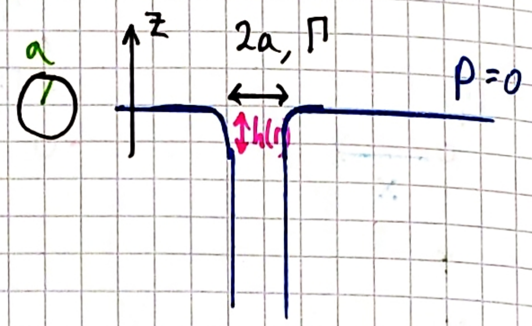
:

$$\Downarrow$$

$$\frac{\Gamma_1}{\Gamma_2} = \frac{d-x}{x} \qquad \frac{\Gamma_1}{\Gamma_2} x = d - x \Rightarrow x = \frac{d}{\frac{\Gamma_1}{\Gamma_2} + 1}$$

$$\omega = \frac{\Gamma_2}{2\pi d} \cdot \frac{\frac{\Gamma_1}{\Gamma_2} + 1}{d} \Rightarrow \omega = \frac{\Gamma_1 + \Gamma_2}{2\pi d^2}$$

Naloga na str. 26 [Idealni vrtinec]



$$v(r) = \begin{cases} \dfrac{\Gamma}{2\pi r} \ ; \ r > a \\ \dfrac{\Gamma r}{2\pi a^2} \ ; \ r \leq a \end{cases}$$

$\vec{v} = v(r)\hat{e}_\varphi$

Vrtinčnost samo v točki a I think.

$$\omega a = \frac{\Gamma}{2\pi a}$$

$$\omega = \frac{\Gamma}{2\pi a^2}$$

$$v = \frac{\Gamma r}{2\pi a^2}$$

i) $r > a: \nabla \times \vec{v} = 0$

Splošen Bernoulli:

$$\Rightarrow 0 = \rho g h(r) + \frac{1}{2}\rho v^2 = g h \rho + \frac{1}{2}\rho \frac{\Gamma^2}{2^2\pi^2 r^2}$$

$$\rightarrow h = -\frac{\Gamma^2}{8\pi^2 g r^2}$$

ii) $r < a: \nabla \times \vec{v} = 2\vec{\omega}$

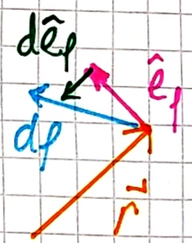
Ne moremo uporabit Bernoullija/(za tokovnice je pa vse itak konstantno in nam nič ne da)

Rešujemo Eulerjevo:

Stac. pogoji.

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + \rho\vec{g}$$

mora bit



$$(\vec{v}\cdot\nabla)\vec{v} = (v(r)\hat{e}_\varphi \cdot \nabla)\, v(r)\hat{e}_\varphi = v(r)\frac{1}{r}\frac{\partial(v(r)\hat{e}_\varphi)}{d\varphi} = \frac{1}{r}v^2\frac{d\hat{e}_\varphi}{d\varphi} = -\frac{v^2(r)}{r}\hat{e}_r$$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_r$$

$$-\rho\omega^2 r\hat{e}_r = -\nabla p - \rho g\hat{e}_z$$

$$z: 0 = -\frac{\partial p}{\partial z} - \rho g \Rightarrow p = -\rho g z + C(r)$$

$$\rho g h = C(r)$$

$$p = \rho g(h(r) - z)$$

To enačbo za z smer bi lahko reproducirali tudi z uporabo Bernoullijeve en. za vrtinčnico.:

$$\frac{1}{2}\rho v^2 + \rho g h + 0 = \frac{1}{2}\rho v^2 + \rho g z + p$$

$$p = \rho g(h-z)$$

Poglejmo še za r smer:

$$r: -\rho\omega^2 r = -\frac{\partial p}{\partial r} = -\rho g \frac{\partial h}{\partial r} \Rightarrow \frac{\partial h}{\partial r} = \frac{\omega^2}{g} r$$

$$h = \int \frac{\omega^2}{g} r \, dr = \frac{\omega^2}{2g} r^2 + \hat{C} = \bigstar$$

Določimo še konstanto tako da zlepimo višine:

$$\frac{\omega^2}{2g} a^2 + \tilde{C} = -\frac{\Gamma^2}{8\pi^2 g a^2}$$

$$\frac{\Gamma^2 a^2}{4\pi^2 a^2 2g 8} + \hat{C} = -\frac{\Gamma^2}{8\pi^2 g a^2}$$

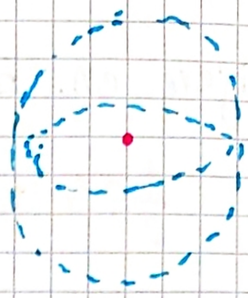
$$\vdots$$

$$\bigstar = -\frac{\Gamma^2}{8\pi^2 g a^2}\left(2 - \frac{r^2}{a^2}\right)$$

Naloga na str. 31 [točkast izvir in točkast dipol] 3D

Izvir:

$$\phi(r) = -\frac{a}{4\pi r}$$

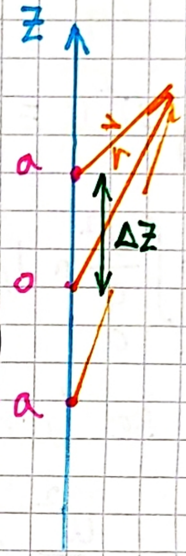


$$\vec{v}(r) = \nabla\phi = \frac{a}{4\pi r^2}\frac{\vec{r}}{r}$$

Preverimo, če je a izdatnost z Gauss izrekom:

$$\oint \vec{v}\cdot d\vec{S} = \frac{a}{4\pi r^2} 4\pi r^2 = \underline{\underline{a}}$$

Točkast dipol:



$$\phi(\vec{r}) = -\frac{a}{4\pi|\vec{r}-\Delta z\hat{e}_z|} + \frac{a}{4\pi|\vec{r}+\Delta z\hat{e}_z|} =$$

$$= -\frac{a}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+(z-\Delta z)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+\Delta z)^2}}\right) =$$

$$= -\frac{a}{4\pi}\left(\frac{1}{\sqrt{x^2+y^2+z^2+\Delta z^2-2z\cdot\Delta z}} - \frac{1}{\sqrt{x^2+y^2+z^2+\Delta z^2+2z\Delta z}}\right) =$$

$\Delta z \ll |\vec{r}| = r$

$\Delta z^2 \approx 0$

$$= -\frac{a}{4\pi}\left(\frac{1}{r\sqrt{1-\frac{2z\Delta z}{r}}} - \frac{1}{r\sqrt{1+\frac{2z\Delta z}{r^2}}}\right) \approx$$

$2\Delta z\cdot a = p_z$

$$\approx -\frac{a}{4\pi r}\left(1+\frac{z\Delta z}{r^2} - 1 + \frac{z\Delta z}{r^2}\right) \approx -\frac{az\cdot 2\Delta z}{4\pi r^3} =$$

$$= -\frac{p_z z}{4\pi r^3} \Rightarrow \phi(\vec{r}) = -\frac{\vec{p}\cdot\vec{r}}{4\pi r^3}$$

V splošnem pa:

$$\nabla^2\phi = 0$$

$$\phi(r,\theta,\varphi) = -\frac{1}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm}r^l + B_{lm}r^{-(l+1)}\right]\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\varphi)$$

naloga na str. 32 [podobno a še za vrtinec in 2D]

$$\nabla^2 \phi = 0 \; ; \; \phi(r, \varphi)$$

V cilindričnih koordinatah

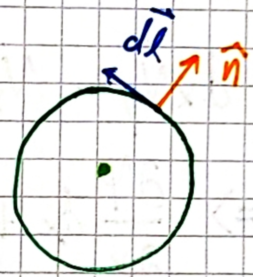
$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 \phi}{\partial \varphi^2}\right)$$

Izvir:

hitrostno polje nič (nezanimivo)

Iščemo $\phi = \phi(r) = \begin{cases} \phi = \text{konst.} \\ \phi = \ln r \end{cases}$

$$\nabla^2(\ln r) = \frac{1}{r}\frac{\partial}{\partial r}\Big(\underbrace{r \frac{1}{r}}_{1}\Big) = 0$$



$$\phi = \frac{Q}{2\pi}\ln r$$

$$\Rightarrow \quad \vec{v} = \frac{Q}{2\pi r}\frac{\vec{r}}{r} = \frac{Q}{2\pi r}\hat{e}_r$$

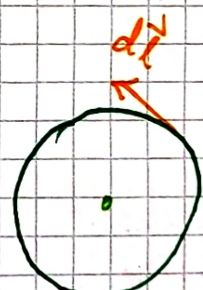
$$\oint \vec{v}\cdot\hat{n}\, dl = 2\pi \frac{Q}{2\pi} = \underline{\underline{Q}}$$

Vrtinec:

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\varphi \frac{\partial}{r\,\partial\varphi}$$

$$\phi \propto \varphi$$

$$\phi = \frac{\Gamma}{2\pi}\varphi$$



$$\vec{v} = \nabla\phi = \hat{e}_\varphi \frac{\partial \phi}{r\,\partial\varphi} = \frac{\Gamma}{2\pi r}\hat{e}_\varphi$$

$$\oint \vec{v}\cdot d\vec{l} = 2\pi r\, v = \frac{\Gamma}{2\pi r} 2\pi r = \underline{\underline{\Gamma}}$$

Kompleksni potencial v 2D (?)

$$\left.\begin{array}{l}\nabla\cdot\vec{v}=0\\ \vec{v}=\nabla\phi\end{array}\right\}\quad \nabla^2\phi=0$$

Uvedemo tokovno funkcijo Ψ

$$v_x=\frac{\partial\Psi}{\partial y}\qquad v_y=-\frac{\partial\Psi}{\partial x}$$

Tokovnica:

$$\frac{dx}{dy}=\frac{v_x}{v_y}$$

$$dx\,v_y=v_x\,dy\ \rightarrow\ v_y\,dx-v_x\,dy=0$$

$$\Rightarrow\ \frac{\partial\Psi}{\partial x}dx+\frac{\partial\Psi}{\partial y}dy=0\ \Rightarrow\ \underline{\underline{d\Psi=0}}$$

Ψ je konst. na tokovnici

$$\boxed{\begin{array}{l}\dfrac{\partial\phi}{\partial x}=\dfrac{\partial\Psi}{\partial y}\\[2ex] \dfrac{\partial\phi}{\partial y}=-\dfrac{\partial\Psi}{\partial x}\end{array}}$$

Cauchy-Riemannov sistem

$$\Downarrow$$

kompleksna analitična funkcija

$$W(z)=\phi(z)+i\,\Psi(z)\ ;\quad z=x+iy$$

Odvodi v kompleksnem so enaki v vse smeri neodvisni od smeri odvajanja. To je def. analitične kompleksne funkcije.

$$\frac{dW}{dz}=\frac{\partial\phi}{\partial x}+i\frac{\partial\Psi}{\partial x}=v_x-i\,v_y$$

$$\frac{dW}{dz}=\frac{\partial\phi}{i\partial y}+i\frac{\partial\Psi}{i\partial y}=-i\,v_y+v_x$$

$$\mathrm{Re}\left(\frac{dW}{dz}\right)=v_x$$

$$\mathrm{Im}\left(\frac{dW}{dz}\right)=-v_y$$

Pretok skozi krivuljo:

$$\underline{\underline{Q}} = \int_A^B \vec{v} \cdot \vec{n}\, dl =$$

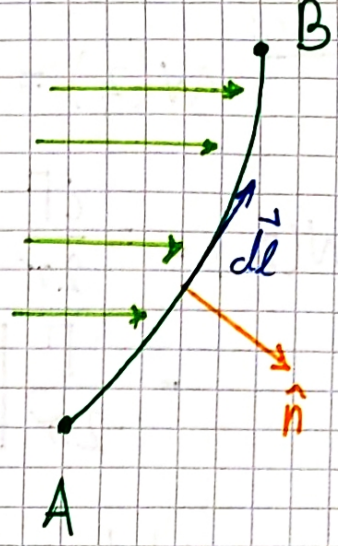
$$= \int \left(\frac{\partial \psi}{\partial y} n_x - \frac{\partial \psi}{\partial x} n_y \right) dl =$$

$$= \int \left(\frac{\partial \psi}{\partial y} \frac{dy}{dl} + \frac{\partial \psi}{\partial x} \frac{dx}{dl} \right) dl =$$

$$= \int \left(\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx \right) = \int d\psi$$

$$= \underline{\underline{\psi(B) - \psi(A)}}$$

Pretok podan z Im delom.



$$dl = (dx, dy)$$

$$\hat{n} = \frac{(dy, -dx)}{dl}$$

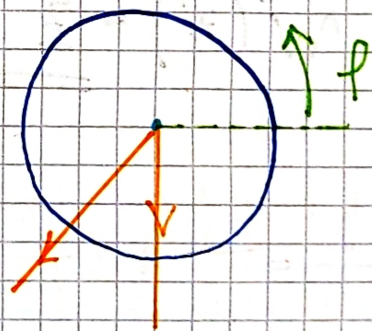
Uganimo potenciala za izvir in Vrtinec:

Izvir:

$$W(z) = \frac{Q}{2\pi} \ln z$$

$$W(z) = \frac{Q}{2\pi} \ln\left(re^{i\varphi}\right) = \frac{Q}{2\pi} \left[\ln r + \ln e^{i\varphi} \right] =$$

$$= \frac{Q}{2\pi} \left[\ln r + i\varphi \right]$$



Izračunajmo pretok s tokovno funkcijo

$$Q = \psi(2\pi) - \psi(0)$$

$$= \underline{\underline{Q}}$$

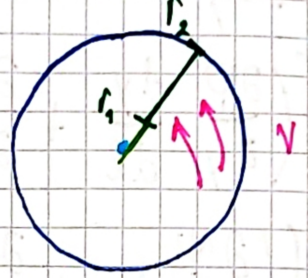
Vrtinec:

$$w(z) = -\frac{i\Gamma}{2\pi}\ln z$$

To je ubistvu rotacija za $90°$ z množenjem s kompleksnim številom i.

$$w(z) = -\frac{i\Gamma}{2\pi}\left[\ln r + i\varphi\right] =$$

$$= \frac{\Gamma}{2\pi}\left[-i\ln r + \varphi\right]$$



Poglejmo pretok med r_1 in r_2:

$$Q = -\frac{\Gamma}{2\pi}\ln\frac{r_1}{r_2} = \frac{\Gamma}{2\pi}\ln\frac{r_2}{r_1}$$

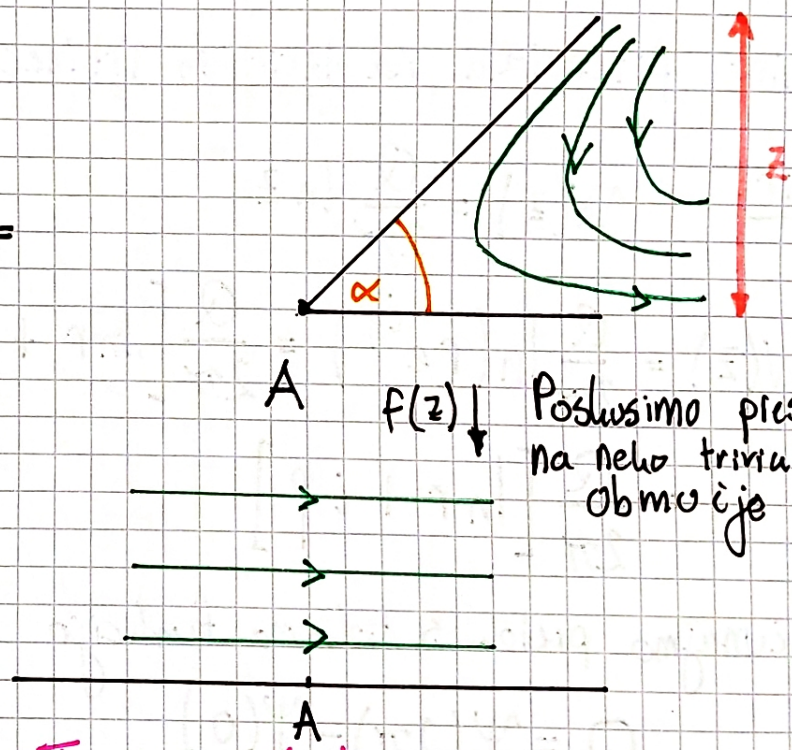
naloga na str. 38

$$f(z) = z^{\pi/\alpha}$$

$$w'(z') = w(z(z')) =$$

$$= w(z'^{\pi/\alpha}) =$$

$$\underline{\underline{w'(z') = z'^{\pi/\alpha}}}$$



Poskusimo preslikati na neko trivialno območje

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} \leftarrow$$

$$= v_0$$

Tu je $w(z) = v_0 \cdot z$
(odvajamo in dobimo $\vec{v}_x$ konst)

Zapišimo v polarni obliki, da preverimo hitrostno polje.

$$z' = re^{i\varphi}$$

$$w'(z') = r^{\pi/\alpha} e^{i\frac{\varphi\pi}{\alpha}} = r^{\pi/\alpha}\Big(\underbrace{\cos\frac{\varphi\pi}{\alpha}}_{\phi} + i\sin\frac{\varphi\pi}{\alpha}\Big)$$

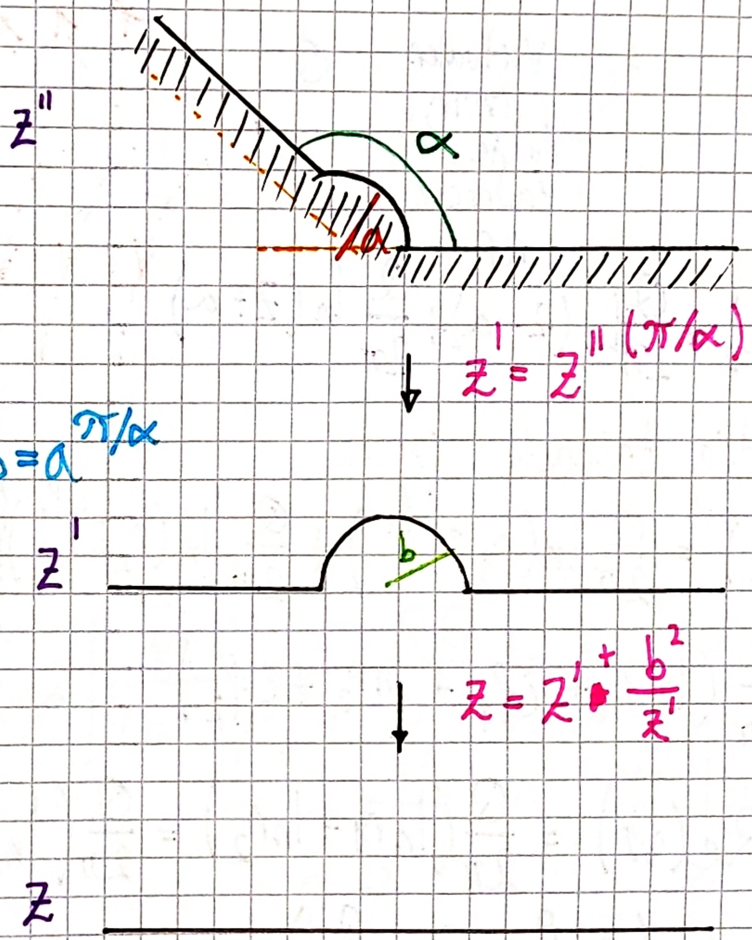
$$\vec{v} = \nabla\phi$$

$$\vec{v}_r \propto \nabla\left(r^{\pi/\alpha}\cos\frac{\varphi\pi}{\alpha}\right) = \frac{\partial}{\partial r}\left(r^{\pi/\alpha}\cos\frac{\varphi\pi}{\alpha}\right) =$$

$$= \frac{\pi}{\alpha} r^{\pi/\alpha - 1} \cos\frac{\varphi\pi}{\alpha}$$

$$v_\varphi \propto \frac{\partial}{r\partial\varphi}\left(r^{\pi/\alpha}\cos\frac{\varphi\pi}{\alpha}\right) = r^{\pi/\alpha - 1}\frac{\pi}{\alpha}\sin\frac{\varphi\pi}{\alpha}$$

naloga na str. 40



$$w''(z'') = w'(z'(z'')) =$$

$$= w(z(z'(z''))) =$$

$$= hz(z'(z'')) =$$

$$= h\left(z''^{\pi/\alpha} + \frac{b^2}{z''^{\pi/\alpha}}\right) =$$

$$\propto z''^{\pi/\alpha} + \frac{a^{2\pi/\alpha}}{z''^{\pi/\alpha}} =$$

$$= (re^{i\varphi})^{\pi/\alpha} + \frac{a^{2\pi/\alpha}}{(re^{i\varphi})^{\pi/\alpha}} =$$

$$= (re^{i\varphi})^{\pi/\alpha} + a^{2\pi/\alpha}(re^{i\varphi})^{-\pi/\alpha} =$$

$$= r^{\pi/\alpha}\left(\cos(\pi\varphi/\alpha) + i\sin(\pi\varphi/\alpha)\right) + a^{2\pi/\alpha} r^{-\pi/\alpha}\left(\cos(\pi\varphi/\alpha) - i\sin(\pi\varphi/\alpha)\right)$$

$$= \cos(\pi\varphi/\alpha)\left(r^{\pi/\alpha} + a^{2\pi/\alpha} r^{-\pi/\alpha}\right) + i\sin\left(\frac{\pi\varphi}{\alpha}\right)\left(r^{\pi/\alpha} - r^{-\pi/\alpha}a^{2\pi/\alpha}\right)$$

Tokovnica: $\sin\left(\frac{\pi\varphi}{\alpha}\right)\left(r^{\pi/\alpha} - r^{-\pi/\alpha}a^{2\pi/\alpha}\right) = \text{konst.}$

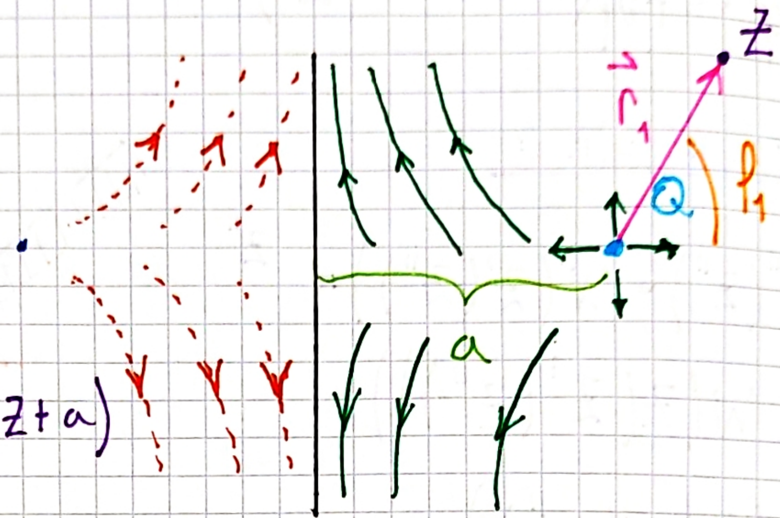
Zanima nas hitrost:

$$v_\varphi = \left(\frac{1}{r}\frac{\partial}{\partial \varphi}\dots\right) = -\frac{1}{r}\sin\left(\frac{\pi\varphi}{\alpha}\right)\left(r^{\pi/\alpha} + a^{2\pi/\alpha} r^{-\pi/\alpha}\right)\frac{\pi}{\alpha}$$

$$v_r = \cos\left(\frac{\pi\varphi}{\alpha}\right)\left(\frac{\pi}{\alpha} r^{\pi/\alpha - 1} - a^{2\pi/\alpha}\frac{\pi}{\alpha} r^{-\pi/\alpha - 1}\right)$$

Naloga na str. 45

Virtualen izvir, da je zadoščen RP



$$W(z) = \frac{Q}{2\pi}\ln(z-a) + \frac{Q}{2\pi}\ln(z+a)$$

↑ dva točkasta izvira

$$W(z) = \frac{Q}{2\pi}\ln\left(r_1 e^{i\varphi_1}\right) + \frac{Q}{2\pi}\ln\left(r_2 e^{i\varphi_2}\right) \quad \} \, v_x = 0$$

$$= \frac{Q}{2\pi}\left(\ln r_1 + i\varphi_1 + \ln r_2 + i\varphi_2\right)$$

$$\phi = \mathrm{Re}(W) = \frac{Q}{2\pi}\left(\ln r_1 + \ln r_2\right) = \frac{Q}{2\pi}\left(\ln\sqrt{(x-a)^2+y^2} + \ln\sqrt{(x+a)^2+y^2}\right)$$

$$\vec{v} = \nabla\phi = \left(\hat{e}_x\frac{\partial}{\partial x}, \hat{e}_y\frac{\partial}{\partial y}\right) =$$

$$= \frac{Q}{2\pi}\left(\frac{1}{\sqrt{(x-a)^2+y^2}}\,\frac{2(x-a)}{2\sqrt{(x-a)^2+y^2}} + \frac{2(x+a)}{2\left((x+a)^2+y^2\right)},\right.$$

$$\left.\frac{2y}{2\left((x-a)^2+y^2\right)} + \frac{2y}{2\left((x+a)^2+y^2\right)}\right) =$$

$$= \frac{Q}{2\pi}\left(\frac{x-a}{(x-a)^2+y^2} + \frac{x+a}{(x+a)^2+y^2}, \frac{y}{(x-a)^2+y^2} + \frac{y}{(x+a)^2+y^2}\right)$$

Silo izračunamo preko Bernoullija:

$$F_x = -p\int dS$$

V neskončnosti

na steni

$$p + \frac{1}{2}\rho v^2 = \overset{0}{p_0} + \frac{1}{2}\rho \cancel{v_0^2}$$

$$p = -\frac{1}{2}\rho v^2$$

Pri $x=0$:

$$v^2 = \left(\left(\frac{-a}{(-a)^2+y^2} + \frac{a}{a^2+y^2}\right)^2 + \left(\frac{y}{(-a)^2+y^2} + \frac{y}{a^2+y^2}\right)^2\right)\frac{Q^2}{4\pi^2} =$$

$$= \frac{y^2 Q^2}{(a^2+y^2)^2\pi^2}$$

Torej je tlak: $p = -\frac{1}{2}\rho\frac{Q^2 y^2}{\pi^2(a^2+y^2)^2}$

$$\frac{F_x}{l} = -\int_{-\infty}^{\infty} p\,dy = \frac{1}{2}\rho\frac{Q^2}{\pi^2}\,2\int_0^{\infty}\frac{y^2\,dy}{(a^2+y^2)^2} = \frac{\rho Q^2}{\pi^2}\left(-\frac{y}{2(a^2+y^2)} + \frac{1}{2a}\operatorname{arctg}\frac{y}{a}\right)\Bigg|_0^{\infty} =$$

$$= \frac{\rho Q^2}{\pi^2}\left(0 + \frac{1}{2a}\frac{\pi}{2} - 0 - 0\right) = \underline{\underline{\frac{\rho Q^2}{4\pi a}}}$$

naloga na str. 46

$$W(z) = \underbrace{v_0\left(z + \frac{a^2}{z}\right)}_{\text{obtekanje}} +$$

$$+ \underbrace{-i\frac{\Gamma}{2\pi}}_{\text{potencial vrtinca}}\ln z = v_0\left(re^{i\varphi} + \frac{a^2}{r}e^{-i\varphi}\right) - i\frac{\Gamma}{2\pi}\left(\ln r + i\varphi\right) =$$

$$= v_0\left(r\cos\varphi + ir\sin\varphi + \frac{a^2}{r}\cos\varphi - \frac{a^2}{r}i\sin\varphi\right) - i\frac{\Gamma}{2\pi}\ln r + \frac{\Gamma}{2\pi}\varphi$$

$$\vec{v} = \left(v_0\cos\varphi - \frac{a^2}{v_0 r^2}\cos\varphi\right)\hat{e}_r + \frac{1}{r}\left(-v_0 r\sin\varphi - v_0\frac{a^2}{r}\sin\varphi + \frac{\Gamma}{2\pi}\right)\hat{e}_\varphi$$

$$\vec{v}(r=a,\varphi) = \cancel{v_0}\left(\frac{\Gamma}{2\pi a} - 2\sin\varphi\, v_0\right)\hat{e}_\varphi$$

$$\frac{F_x}{l} = -\int_0^{2\pi} p\cos\varphi\, r\, d\varphi$$

$$\frac{F_y}{l} = -\int p\sin\varphi\, r\, d\varphi$$

Tlak iz Bernoullija:

$$p + \frac{1}{2}\rho v^2 = \frac{1}{2}\rho v_0^2 \qquad p = \frac{1}{2}\rho\left(v_0^2 - v^2\right)$$

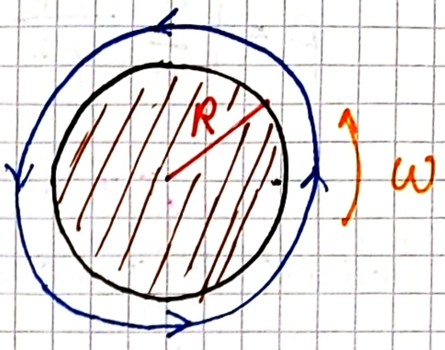
$$p = \frac{1}{2}\rho\left(v_0^2 - \frac{\Gamma^2}{4\pi^2 a^2} - 4v_0^2\sin^2\varphi + \frac{2\Gamma v_0}{\pi a}\sin\varphi\right)$$

$$\frac{F_x}{l} = 0$$ } vsi členi so ničelni pri integriranju

$$\frac{F_y}{l} = -\int_0^{2\pi} p\sin\varphi\, r\, d\varphi = -\frac{1}{2}\rho\frac{2\Gamma v_0}{\pi a}\int_0^{2\pi}\sin^2\varphi\, a\, d\varphi = \underline{\underline{-\Gamma\rho v_0}}$$

Viskozne tekočine

Naloga na str. 51



N.S. $\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + \eta \nabla^2 \vec{v}$

$\frac{\partial \vec{v}}{\partial t}$ → 0 stacionarno

$\vec{v} = v(r)\cdot\hat{e}_\varphi$

$\nabla\nabla\cdot = \nabla^2 + \nabla\times\nabla\times$, $\nabla\nabla\cdot \to 0$

$$\nabla\times\vec{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \varphi} - \frac{\partial v_\varphi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\varphi + \left(\frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial z}\right)\hat{e}_z = \frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r}\hat{e}_z$$

$$\nabla\times\nabla\times\vec{v} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r}\right)\hat{e}_\varphi$$

$$(\vec{v}\cdot\nabla)\vec{v} = v\frac{1}{r}\frac{\partial}{\partial\varphi}(v\cdot\hat{e}_\varphi) = v\frac{1}{r}(v(-\hat{e}_r)) = -\frac{v^2}{r}\hat{e}_r$$

Ločimo na dve enačbi, za relevantni smeri:

φ smer: $-\eta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r}\right) = 0$

r smer: $\rho\frac{v^2}{r} = \frac{\partial p}{\partial r}$

$$\Rightarrow \frac{1}{r}\frac{\partial(r v_\varphi)}{\partial r} = A$$

$$\frac{\partial(r v_\varphi)}{\partial r} = Ar$$

$$r v_\varphi = \frac{Ar^2}{2} + B \rightarrow v = \frac{Ar}{2} + \frac{B}{r}$$

$A = 0$ ker $v(r\to\infty) = 0$

$v\cdot R = B$

$\omega R^2 = B$

$\Downarrow$

$$v = \frac{\omega R^2}{r}$$

Kot vrtinčna nit

$$p = \int \frac{\rho}{r} \frac{\omega^2 R^4}{r^2} \, dr = -\frac{\rho \omega^2 R^4}{2 r^2} + \underbrace{p_0}_{\text{integracijska konstanta}}$$

S kolikšno močjo ($P = M \cdot \omega$) moramo vrteti gred?

$$\sigma_{ij}^{v} = 2\eta v_{ij}$$

$$\sigma_{r\varphi} = 2\eta v_{r\varphi}$$

$$v_{rr} = \frac{\partial v_r}{\partial r}$$

$$v_{\varphi\varphi} = \frac{\partial v_\varphi}{r \partial \varphi} + \frac{v_r}{r}$$

$$v_{r\varphi} = \frac{1}{2}\left(\frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r} + \frac{\partial v_r}{r \partial \varphi}\right)$$

$$\Rightarrow \sigma_{r\varphi} = 2\eta \frac{1}{2}\left(-\frac{\omega R^2}{r^2} - \frac{\omega R^2}{r^2}\right) = -2\eta \frac{\omega R^2}{r^2}$$

Torej:

$$P = M \cdot \omega$$

$$dM = dF_\varphi \cdot R = \sigma_{r\varphi} \cdot \underbrace{dS}_{R d\varphi \cdot L \,(\text{length})} R = \sigma_{r\varphi} R^2 L \, d\varphi$$

$$\frac{M}{L} = \sigma_{r\varphi} R^2 \int_0^{2\pi} d\varphi = \sigma_{r\varphi} R^2 2\pi = -4\pi \eta \omega R^2$$

$$\frac{P}{L} = -4\pi \eta \omega^2 R^2$$

Sedaj pa zapišimo še disipirano moč (in vidimo, da je enako).

$$P = \int dV \sigma_{ij} v_{ij} = 2\pi \int_R^\infty r\,dr\, L\, 2\eta \underbrace{v_{ij} \cdot v_{ij}}_{\text{Vsota kvadratov}} = 2\pi 2\eta \int_R^\infty r\,dr\, L\left(v_{rr}^2 + v_{\varphi\varphi}^2 + 2 v_{r\varphi}^2\right)$$

Vse z komp. so ničelne

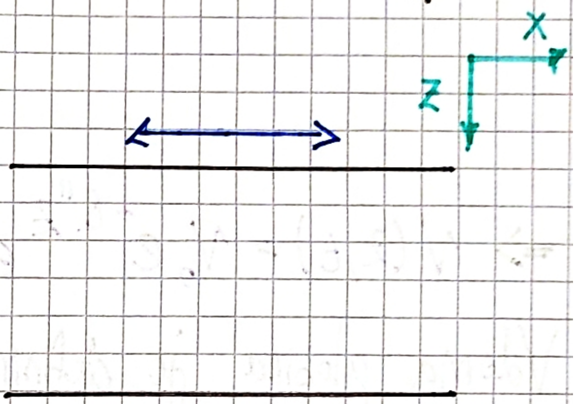
$$= 2\pi \int_R^\infty r\,dr\, 2\eta L \, 2\left(\frac{1}{2}\left(-2\frac{\omega R^2}{r^2}\right)\right)^2 =$$

$$= 8\pi\eta L\omega^2 R^4 \int_R^\infty \frac{1}{r^3}\, dr = 4\pi\eta L R^4 \omega^2 \frac{1}{R^2} =$$

$$= 4\pi\eta L R^2 \omega^2 \Rightarrow \frac{P}{L} = 4\pi\eta\omega^2 R^2$$

disipacija je pozitivno definitna!
Prej smo izračunali moč na gred (gred zgublja moč) in ne moč na tekočino.

Naloga na str. 54



$$\vec{v} = v(z,t)\hat{e}_x$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + \eta\nabla^2\vec{v}$$

Premislimo člene in zapišemo enačbe za x smeri:

$$\rho \dot{v}\hat{e}_x = \eta \frac{\partial^2 v}{\partial z^2}\hat{e}_x \rightarrow \rho\frac{\partial v}{\partial t} = \eta\frac{\partial^2 v}{\partial z^2}$$

in vzamemo nastavek ravnega vala: $v(z,t) = A e^{i(kz-\omega t)}$
ker ni valovna enačba (ampak difuzijska) bomo dobili kompleksni k.

$$\Rightarrow \quad -\rho A i\omega e^{i(kz-\omega t)} = -\eta A k^2 e^{i(kz-\omega t)}$$

$$i\rho\omega = \eta k^2 \;\Big\}\; \text{disperzijska relacija}$$

Pozor!

$$k^2 = \frac{i\rho\omega}{\eta}$$

$$\Rightarrow k = \pm e^{i\pi/4}\sqrt{\frac{\rho\omega}{\eta}} = \pm\left(\sin\left(\frac{\pi}{4}\right) + i\cos\left(\frac{\pi}{4}\right)\right)\sqrt{\frac{\rho\omega}{\eta}} =$$

$$= \pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\sqrt{\frac{\rho\omega}{\eta}} = \pm\frac{\sqrt{2}}{2}(1+i)\sqrt{\frac{\rho\omega}{\eta}} = \pm\frac{1}{\sqrt{2}}(1+i)\sqrt{\frac{\rho\omega}{\eta}} =$$

$$= \pm(k' + ik'')$$

$$V(z,t) = Ae^{i((k'+ik'')z-\omega t)} + Be^{i[-(k'+ik'')z-\omega t]} =$$

$$= Ae^{(ik'-k'')z-i\omega t} + Be^{(-ik'+k'')z-i\omega t} =$$

$$= Ae^{-k''z}e^{i(k'z-\omega t)} + \underbrace{Be^{k''z}e^{-i(k'z+\omega t)}}_{=0}$$

gre proti 0

V neskončnosti ne sme divergirati

(če bi bili plošči neskončno narazen)

(če sta pa bližje je pa vseeno majhen)

$$Ae^{i(\cancel{k'}-\omega t)} = V_0 e^{-i\omega t}$$

$$\Rightarrow \boxed{A = V_0}$$

$$\Rightarrow V(z,t) = V_0 e^{-k''z} e^{i(k'z-\omega t)}$$

Vdorna globina je definirana tu z k'' (iz oblike enačbe):

$$L = \frac{1}{k''} = \sqrt{\frac{\eta}{\rho\omega}}\,\frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{\frac{2\eta}{\rho\omega}}$$

$$d\varphi(z) = k'z = \sqrt{\frac{\rho\omega}{2\eta}}\,z \Big\}\ \text{profil vdorne globine}$$

linearen