Ponovitev

$$H\Psi(x) = i\hbar \frac{\partial \Psi(x,t)}{\partial t}$$

OSE: $H\Psi_n(x) = E_n \Psi_n(x)$

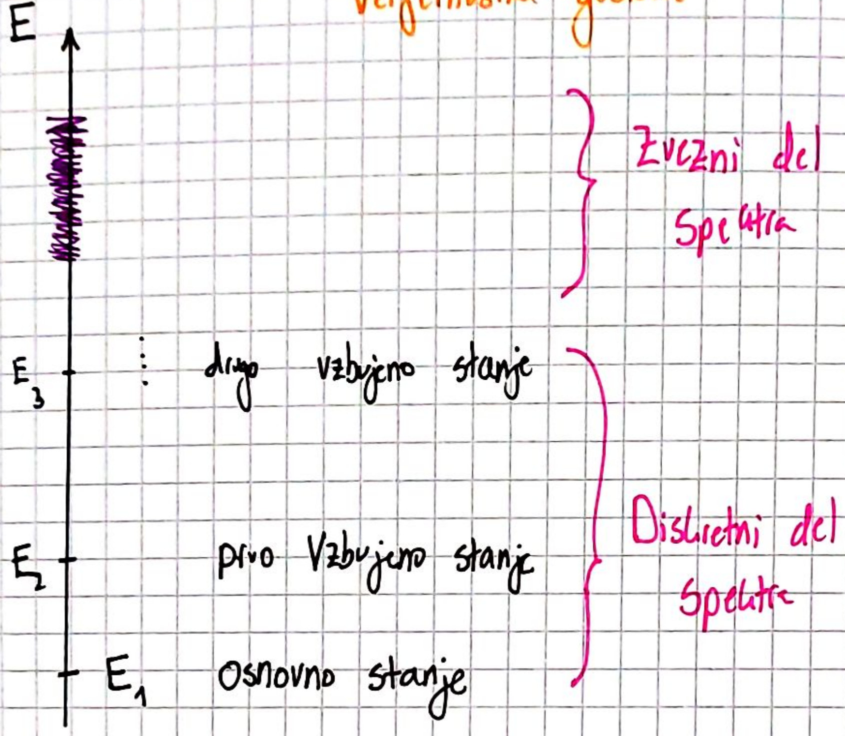
Eni E_n pripada le ena Ψ_n (E_n ni degenerirano)

Eni E_n pripada n lin. neod. Ψ_n (E_n je n-krat degenerirano)

$$\hat{H} = H = \frac{p^2}{2m} + V(x) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

$$\hat{p} = p = -i\hbar \frac{\partial}{\partial x}$$

$|\Psi(x)|^2$ — Verjetnostna gostota



Ni nujno, da ima vsak sistem oboje, lahko tudi samo enega.

Rešitve za delec v konstantnem potencialu $V(x) = V_0 = \text{konst.}$

1.
$$\Psi(x) = Ae^{ikx} + Be^{-ikx} \; ; \; k = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \qquad E > V_0$$

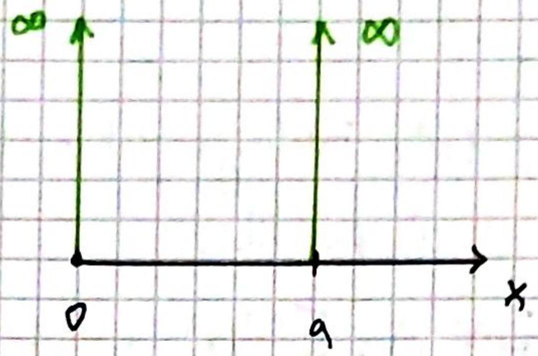
Imamo dvojno degeneracijo (za isti E ravni val v - in v +)

drugа param.: $\Psi(x) = C\sin kx + D\cos kx = F\cos[k(x-x_0)]$

2.
$$\Psi(x) = Ae^{-\varkappa x} + Be^{\varkappa x} \; ; \; \varkappa = \sqrt{\frac{2m(V_0-E)}{\hbar^2}} \qquad E < V_0$$

druga param.: $\Psi(x) = C\,\mathrm{ch}(\varkappa x) + D\,\mathrm{sh}(\varkappa x)$

∞ potencialna jama



Robni pogoji: $\psi(0) = \psi(a) = 0$

Rešitve: $\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x \quad ; \; n = 1, 2, 3, \ldots$

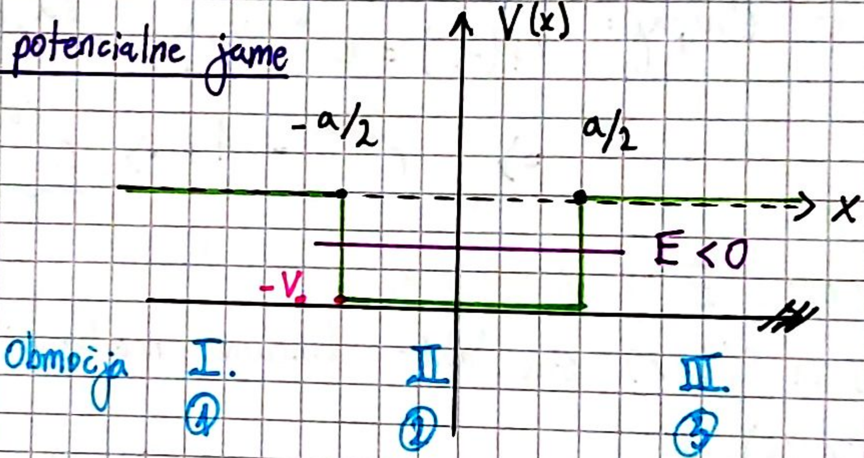
Normalizacija $\quad \int |\psi_n(x)|^2 dx = 1$

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

$$\psi_n(x) \rightarrow e^{i\alpha} \psi_n(x) \; ; \; |e^{i\alpha}| = 1$$

V QM so ψ nao nedoločene do faznega faktorja natančno

Poišči vezana stanja končne potencialne jame



$$\psi_1(x) = Ae^{-\varkappa x} + Be^{\varkappa x};$$

$$\varkappa = \sqrt{\frac{2m(-E)}{\hbar^2}}$$

$$\psi_2(x) = Ce^{ik_2 x} + De^{-ik_2 x}$$

$$k_2 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$\psi_3(x) = Fe^{-\varkappa x} + Ge^{\varkappa x}$$

Robni pogoji:

$\left.\begin{array}{l} \psi(\infty) = 0 \\ \psi(-\infty) = 0 \end{array}\right\}$ Iščemo vezana stanja

$\psi_1(-\frac{a}{2}) = \psi_2(-\frac{a}{2}) \qquad \psi_2(\frac{a}{2}) = \psi_3(\frac{a}{2})$

$\psi_1'(-\frac{a}{2}) = \psi_2'(-\frac{a}{2}) \qquad \psi_2'(\frac{a}{2}) = \psi_3'(\frac{a}{2})$

"Zlepimo" na robih

Da zadostimo v ∞: $B = 0, \; G = 0$

Prej rabimo izrek:

Če je $V(x)$ soda funkcija, lahko najdemo take lastne funkcije, ki so sode ali pa lihe.

Dokaz: $H\psi(x) = E\psi(x)$; $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

$x \to -x$: $H\psi(-x) = E\psi(-x)$ Hamiltonian je invarianten na to zamenjavo.

• 1. primer: E ni degeneriran

Lahko se razlikujeta le za fazni faktor (in ne kršimo degeneracije) $\psi(-x) = e^{i\alpha}\psi(x)$; $|e^{i\alpha}| = 1$

Zopet $x \to -x \Rightarrow \psi(x) = e^{i\alpha}\psi(-x)$

Vstavimo skupaj: $\psi(-x) = e^{i\alpha}\psi(x) = e^{i\alpha}e^{i\alpha}\psi(-x)$

$\Rightarrow e^{2i\alpha} = 1 \Rightarrow e^{i\alpha} = \pm 1 \to \psi(-x) = \pm\psi(x)$

• 2. primer: E je degeneriran

$\psi(x)$ in $\psi(-x)$ sta lahko linearno neodvisni (če sta odvisni je isto kot pri 1. primeru).

$\psi_+(x) = \alpha\psi(x) + \psi(-x) = \psi_+(-x)$

$\psi_-(x) = \alpha\psi(x) - \psi(-x) = -\psi_-(-x)$

Našli bazo, kjer sta bazni funkciji sodi in lihi $\to$ Vedno lahko najdemo tako bazo!

Sedaj lahko rešujemo naprej:

Iskanje sodih:

$A = F$

$\psi_1 = Ae^{\varkappa x}$

$\psi_3 = Ae^{-\varkappa x}$

$\psi_2 = B\cos(kx)$

Robni pogoji: $Ae^{-\varkappa a/2} = B\cos(k\frac{a}{2})$

$-\varkappa Ae^{-\varkappa\frac{a}{2}} = kB\sin(k\frac{a}{2})$

Iskanje lihih:

$\psi_1 = Ae^{\varkappa x}$

$\psi_3 = -Ae^{-\varkappa x}$

$\psi_2 = B\sin(kx)$

Robni pogoji: $-Ae^{-\varkappa a/2} = B\sin(k\frac{a}{2})$

$\varkappa Ae^{-\varkappa\frac{a}{2}} = kB\cos(k\frac{a}{2})$

S tem, smo razdelili 4x4 problem na dva 2x2.

Robni pogoji pri $a/2$:

Sode:

$$A e^{-\varkappa \frac{a}{2}} = B\cos\left(k \frac{a}{2}\right)$$
$$-\varkappa A e^{-\varkappa \frac{a}{2}} = -Bk\sin\left(k\frac{a}{2}\right)$$

Lihe:

$$-A e^{-\varkappa a/2} = B\sin\left(k\, a/2\right)$$
$$\varkappa A e^{-\varkappa a/2} = kB\cos\left(k\, a/2\right)$$

$$\varkappa = k\,\mathrm{tg}\left(k\frac{a}{2}\right) \quad \xleftarrow{\text{Sicer}} \; \xrightarrow{\text{transcendentni}} \quad -\varkappa = k\,\mathrm{ctg}\left(k\frac{a}{2}\right)$$

$$u = ka$$

$$\mathrm{tg}\left(\frac{u}{2}\right) = \frac{\varkappa a}{u} \qquad -\mathrm{ctg}\left(\frac{u}{2}\right) = \frac{\varkappa a}{u}$$

$$E = -V_0 + \frac{\hbar^2 k^2}{2m} = -V_0 + \frac{\hbar^2 u^2}{2ma^2}$$

Potrebujemo še povezavo med k in $\varkappa$:

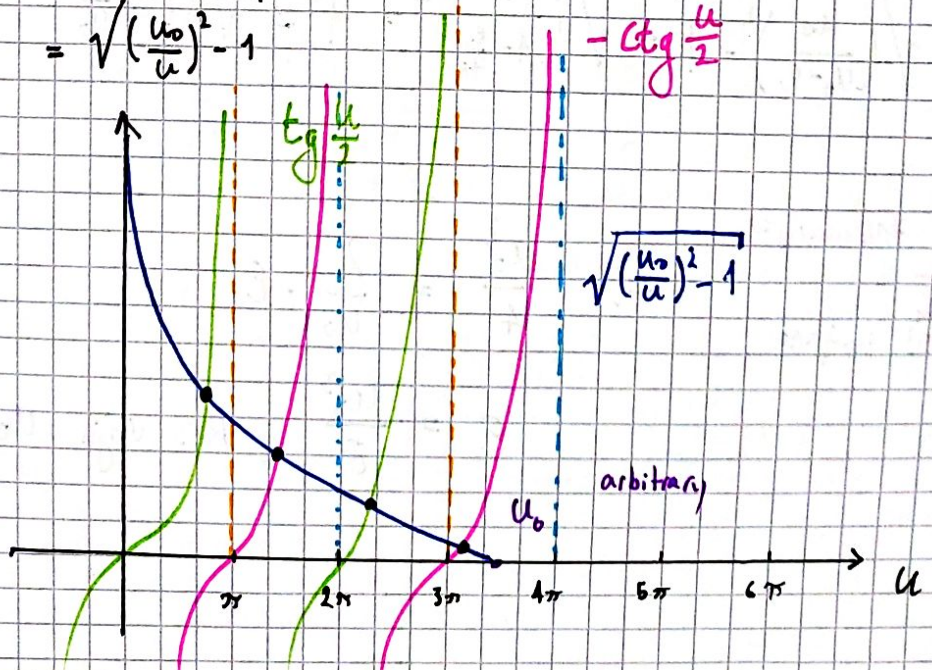
$$\varkappa^2 + k^2 = \frac{2mV_0}{\hbar^2} \quad / \cdot a^2$$

$$\varkappa^2 a^2 + u^2 = \frac{2ma^2V_0}{\hbar^2} \equiv u_0^2$$

Opisuje našo potencialno jamo

$$\mathrm{tg}\left(\frac{u}{2}\right) = \frac{\varkappa a}{u} = \frac{\sqrt{u_0^2 - u^2}}{u} = \sqrt{\left(\frac{u_0}{u}\right)^2 - 1}$$

$$-\mathrm{ctg}\left(\frac{u}{2}\right) = \frac{\varkappa a}{u} = \sqrt{\left(\frac{u_0}{u}\right)^2 - 1}$$



V primeru ∞ pot. jami:

$$u_0^2 = \frac{2mV_0a^2}{\hbar^2} \qquad \left.\begin{matrix} a \text{ končen} \\ V_0 \to \infty \end{matrix}\right\} u_0 \to \infty$$

$$\Rightarrow \quad u = n\pi$$

$$\Rightarrow \quad E_n = -V_0 + \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

Zanimiv limitni primer izračuna od zadnjič

$$a \to 0 \qquad V_0 \to \infty$$

$$aV_0 = \lambda = \text{konst.}$$

$$V(x) = \underbrace{-\lambda\delta}(x)$$

↳ Navzdol obrnjena delta funkcija

$$u_0 = \sqrt{\frac{2mV_0a^2}{\hbar}} \xrightarrow[\substack{\text{ko} \\ V_0 \to \infty \\ a \to 0}]{} 0$$

Vezano stanje bo eno samo ampak ali res obstaja?

$$\operatorname{tg}\frac{u}{2} = \sqrt{\left(\frac{u_0}{u}\right)^2 - 1}$$

Če je u_0 majhen bo tudi u majhen.

$u = u_0 - \varepsilon$; $\varepsilon \ll u_0$ } Predpostavimo, da bo tako veljalo. Zato naredimo razvoj

$$\frac{u_0-\varepsilon}{2} = \frac{u}{2} = \sqrt{\left(\frac{u_0}{u_0-\varepsilon}\right)^2 - 1} = \sqrt{\left(\frac{1}{1-\frac{\varepsilon}{u_0}}\right)^2 - 1} \simeq \sqrt{1 + 2\frac{\varepsilon}{u_0} - 1} \simeq \sqrt{\frac{2\varepsilon}{u_0}}$$

↑

ε je tu gotovo zanemarljiv proti $\sqrt{\frac{\varepsilon}{u_0}}$ majhno

$$\Rightarrow \quad \frac{u_0^2}{4} = \frac{2}{u_0} \cdot \varepsilon$$

$$\varepsilon = \frac{u_0^3}{8} \text{ ; Res velja } u_0 \gg \varepsilon!$$

$$u = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}\, a$$

$$\left(\frac{u}{a}\right)^2 \hbar^2 = 2m(E+V_0) \quad \Rightarrow \quad E = -V_0 + \frac{\hbar^2}{2m}\left(\frac{u}{a}\right)^2$$

In vstavimo $u = u_0 - \varepsilon = u_0 - \frac{u_0^3}{8}$

Dobimo:

$$E_0 = -V_0 + \frac{\hbar^2}{2m}\left(\frac{u_0 - \frac{u_0^3}{8}}{a}\right)^2 =$$

Temu členu ne zaupamo nujno ker je višjega reda kot razvoj

$$= -V_0 + \frac{\hbar^2}{2m}\frac{u_0^2}{a^2}\left(1 - \frac{u_0^2}{4} + \frac{u_0^4}{64}\right) =$$

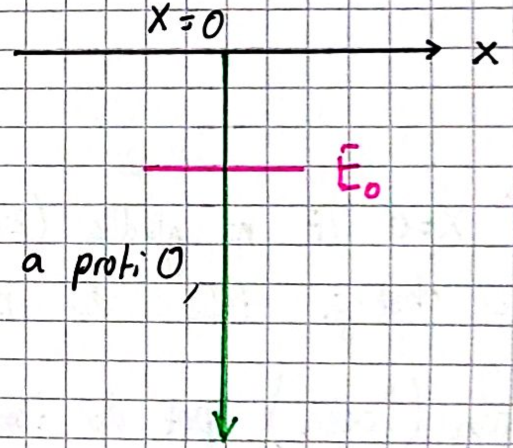
$$= -V_0 + \frac{\cancel{\hbar^2}}{\cancel{2m}}\frac{\cancel{2m}\, V_0 \cancel{a^2}}{\cancel{\hbar^2}\,\cancel{a^2}}\left(1 - \frac{u_0^2}{4} + \cancel{\frac{u_0^4}{64}}\right) =$$

Torej rata ta res prvi netrivialni popravek

$$= -\frac{V_0 u_0^2}{4} =$$

Vezano!

$$= -\frac{2m V_0^2 a^2}{4\hbar^2} = -\frac{m\lambda^2}{2\hbar^2}$$



Prepričali smo se tudi, da kljub temu da limitiramo a proti 0, imamo še vedno osnovno vezano stanje.

Kako zgleda VF v tej limiti?

Sredinski del v jami gre v tej limiti iz kosinusa proti eni točki, torej bo VF le še iz eksponentnih repov $\psi_0(x) = A e^{-\varkappa_0 |x|}$

$$\varkappa_0 = \sqrt{\frac{-2mE_0}{\hbar^2}} = \sqrt{\frac{2m^2\lambda^2}{2\hbar^2 \cdot \hbar^2}} = \frac{m\lambda}{\hbar^2}$$

A pa dobimo iz normalizacije $\quad 1 = \int_{-\infty}^{\infty} \psi \cdot \psi^* dx$

Ker je soda lahko samo od 0 do ∞.

$$2\int_0^\infty A e^{-\varkappa_0 x} A^* e^{-\varkappa_0 x} = 1 \quad \Rightarrow \quad \frac{1}{2} = AA^* \int_0^\infty e^{-2\varkappa_0 x} dx$$

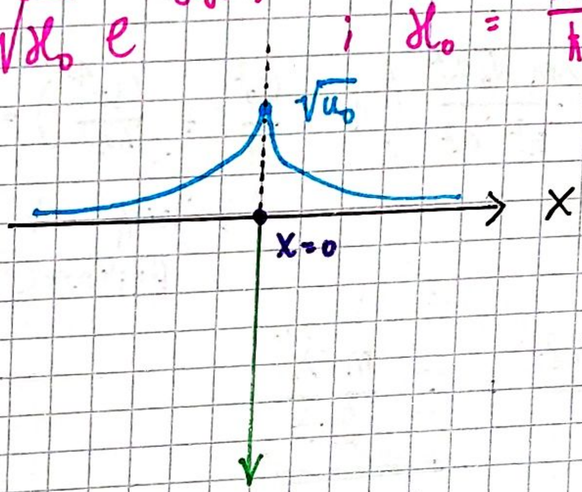
$$AA^*\left(\frac{-1}{2\varkappa_0}\right)e^{-2\varkappa_0 x}\Big|_0^\infty = \frac{1}{2}$$

$$\Rightarrow |A|^2 = AA^* = \varkappa_0 \Rightarrow |A| = \sqrt{\varkappa_0}$$ Ne zvemo pa amplitude

Ampak VF je neodvisna do faznega faktorja $A = \sqrt{\varkappa_0}\, e^{i\alpha}$; $\alpha \in \mathbb{R}$

Zato lahko vzamemo kar $A = \sqrt{\varkappa_0}$ ker ne sme biti odvisno od faze.

Torej: $\psi(x) = \sqrt{\varkappa_0}\, e^{-\varkappa_0 |x|}$; $\varkappa_0 = \frac{m\lambda}{\hbar^2}$

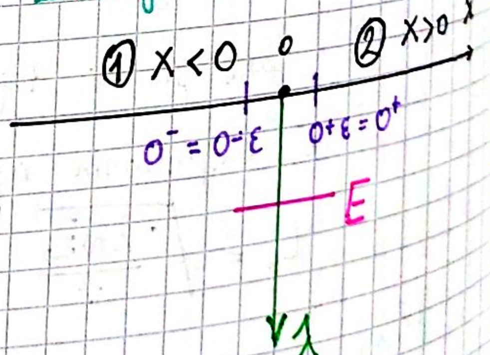


V x=0 VF ni gladka (zvezno odvedljiva) a v naravi tak potencial zares ne obstaja tako da ne pride do take nezveznosti.

Druga (krajša) pot do tega rezultata:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \lambda\,\delta(x)\,\psi = E\psi$$

Zanimajo nas $E<0$ (vezana stanja)



Spet lot jamo ločimo v območja. Na območjih ① in ② je potencial konstanten.

$$\psi_1 = Ae^{\varkappa x} + \cancel{Be^{-\varkappa x}} = Ae^{\varkappa x}$$

$$\psi_2 = \cancel{Ce^{\varkappa x}} + De^{-\varkappa x} = De^{-\varkappa x}$$

Da bo vezano $\psi \to 0$ ko $|x| \to \infty \Rightarrow B = 0 = C$

Tu robni pogoj o ~~[illegible]~~ lepljenju odvoda ne velja.

Integriramo SE:

$$-\int_{0^-}^{0^+} \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}dx - \lambda\int_{0^-}^{0^+}\delta(x)\psi dx = E\int_{0^-}^{0^+}\psi dx$$

$$-\frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x}\Big|_{0^-}^{0^+} - \lambda\psi(0) = \underline{\underline{0}}$$

Pričakujemo 0 ker je ψ končna in jo integriramo na infinitezimalnem intervalu

Izpeljali smo robni pogoj za razliko levega in desnega odvoda. Še vedno pa velja da mora biti VF zvezna: $\psi(0^-) = \psi(0^+)$.

1.) $-\frac{\hbar^2}{2m}(-\varkappa D - \varkappa A) - \lambda\psi(0) = 0$

2.) $\psi_1(0) = \psi_2(0) \Rightarrow \underline{\underline{A = D}}$

Torej iz 1.) =>

$$\frac{\hbar^2}{2m}\varkappa 2A = \lambda A \rightarrow \lambda = \varkappa\frac{\hbar^2}{m} \Rightarrow \varkappa_0 = \frac{\lambda m}{\hbar^2}$$

Tako kot prej!

In reproducirali smo, da je samo eno vezano stanje.

Izračunajmo še energijo vezanega stanja:

$$\sqrt{-\frac{2mE}{\hbar^2}} = \varkappa_0 = \frac{m\lambda}{\hbar^2}$$

$$-\frac{2mE}{\hbar^2} = \frac{m^2\lambda^2}{\hbar^4} \Rightarrow E_0 = -\frac{m\lambda^2}{2\hbar^2}$$

Kot prej!

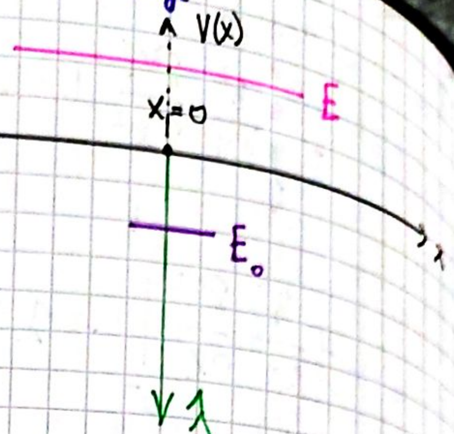
Valovna funkcija se skriva v nastavku. Ker sta $A = D$ lahko:

$$\psi(x) = A e^{-\varkappa_0|x|}$$

Kjer bi A določili identično kot prej iz normalizacije.

Sipalna stanja ($E > 0$) navzdol obrnjene δ funkcije

Kolikšna je verjetnost za prepustnost, odbojnost...



$$V(x) = -\lambda \delta(x)$$

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

Prepustnost in odbojnost sta definirana preko verjetnostnega toka:

$$j(x) = \frac{\hbar}{2mi}\left(\psi^*(x)\frac{\partial \psi}{\partial x}(x) - \psi(x)\frac{\partial \psi}{\partial x}(x)\right)$$

$$z - z^* = 2i\,\mathrm{Im}(z)$$
$$z + z^* = 2\,\mathrm{Re}(z)$$

Izračunajmo pro tok na območju kjer je potencial konstanten:

$$j(x) = \frac{\hbar}{m}\,\mathrm{Im}\left(\psi^*(x)\,\psi'(x)\right) = \frac{\hbar}{2m}\,\mathrm{Im}\left((A^* e^{ikx} + B^* e^{-ikx})(Aik e^{ikx} - Bik e^{ikx})\right)$$

$$= \frac{\hbar}{m}\,\mathrm{Im}\left(AA^* ik + B^*A\, ik\, e^{2ikx} - A^*B\, ik\, e^{-2ikx} + BB^* ik\right) =$$

Konjugirano

$$v \equiv \frac{\hbar}{m}k$$

$$= \frac{\hbar}{m}k\left(|A|^2 - |B|^2 + \cancel{\mathrm{Im}(-\,||\,-)}\right)$$

0

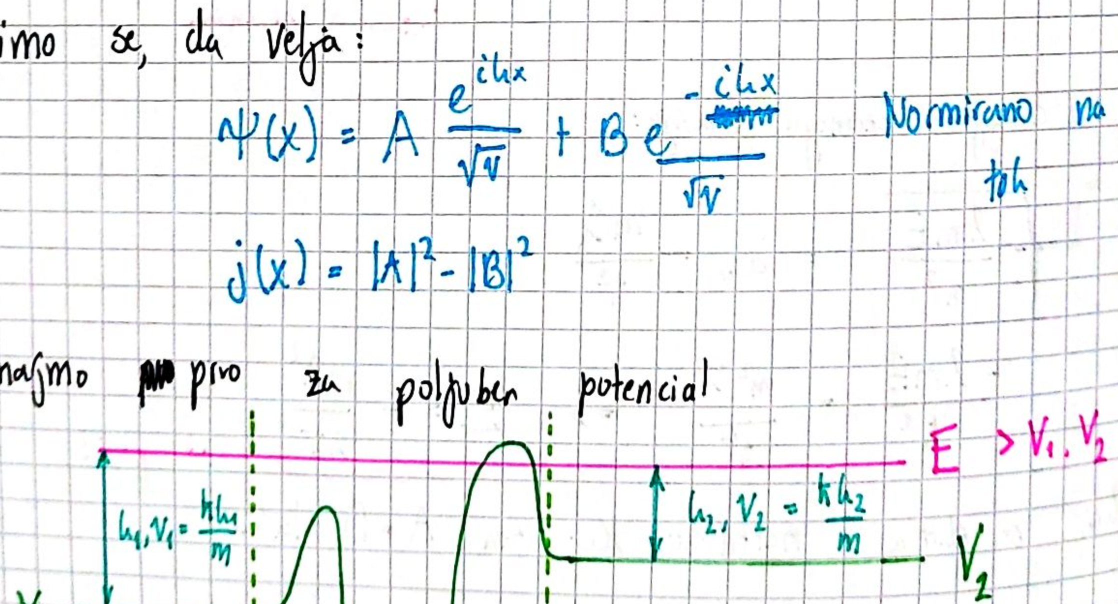
Dogovorimo se, da velja:

$$\psi(x) = A\frac{e^{ikx}}{\sqrt{v}} + B\frac{e^{-ikx}}{\sqrt{v}}$$

Normirano na verjetnostni tok

$$j(x) = |A|^2 - |B|^2$$

Izračunajmo pro za poljuben potencial



$$-\frac{\hbar^2}{2m}\psi'' + V_3\psi = E\psi$$

$$\psi_1 = A_1 \frac{e^{ik_1x}}{\sqrt{N_1}} + B_1 \frac{e^{-ik_1x}}{\sqrt{N_1}}$$

$$\psi_2 = B_2 \frac{e^{ik_2x}}{\sqrt{N_2}} + A_2 \frac{e^{-ik_2x}}{\sqrt{N_2}}$$

$$\psi_3 = \alpha\, f(x) + \beta\, g(x)$$

Linearna DE 2. reda (kar je SE) ima rešitev, ki je linearna kombinacija linearno neodvisnih rešitev. Zato lahko tudi ψ_3 zapišemo kot lin. kom. (ne vemo pa kaki sta rešitvi).

Dodati moramo še robne pogoje, da lahko rešimo ta problem:

$$\psi_1(x_1) = \psi_3(x_1) \qquad \psi_2(x_2) = \psi_3(x_2)$$

$$\psi_1'(x_1) = \psi_3'(x_1) \qquad \psi_2'(x_2) = \psi_3'(x_2)$$

6 neznank in 4 pogoji. V resnici poznamo A_1, A_2 ker sta to vhodna parametra delcev, ki jih pri eksperimentu pošljemo na potencial, da se sipljejo, če bi rešili sistem bi dobili B_1, B_2, α, β.

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = S \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} ; \quad S \in Mat(2\times 2, \mathbb{C})$$

Sipalna matrika

$$\underline{\underline{B = SA}}$$

Število delcev se ohranja (procesov kjer delci izginjajo/nastajajo ne znamo še opisati => QFT jih zna).

$$\underbrace{|A_1|^2 + |A_2|^2}_{\text{Toliko kot notr pošljemo}} = \underbrace{|B_1|^2 + |B_2|^2}_{\text{Toliko kot leti ven}}$$

Hermitska adjungacija

dagger $\rightarrow$ $B^\dagger = [B_1^*, B_2^*] = B^h$

Zapišimo to z vektorjema:

$$|\vec{A}|^2 = |\vec{B}|^2$$

oz.

$$A^\dagger A = B^\dagger B$$

Preverimo:

$$R = \frac{-\kappa_0}{ik+\kappa_0}\left(\frac{-\kappa_0}{ik+\kappa_0}\right)^* = \frac{-\kappa_0}{ik+\kappa_0}\,\frac{-\kappa_0}{-ik+\kappa_0} = \frac{\kappa_0^2}{-i^2k^2+\kappa_0^2} =$$

$$= \frac{\kappa_0^2}{k^2+\kappa_0^2}$$

$$T = \frac{ik}{ik+\kappa_0}\,\frac{-ik}{-ik+\kappa_0} = \frac{k^2}{k^2+\kappa_0^2}$$

Očitno je, da velja $R+T=1$.

Poglejmo še pogoj unitarnosti

$$S^\dagger \cdot S = \begin{pmatrix} r^* & t^* \\ t'^* & r'^* \end{pmatrix}\begin{pmatrix} r & t' \\ t & r' \end{pmatrix} = \begin{pmatrix} \boxed{r^*r+t^*t} & r^*t'+t^*r' \\ t'^*r+r'^*t & t'^*t'+r'^*r' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Torej je to eden od štirih pogojev za unitarnost (za ohranjanje št. delcev).

Izrazimo oboje z energijami:

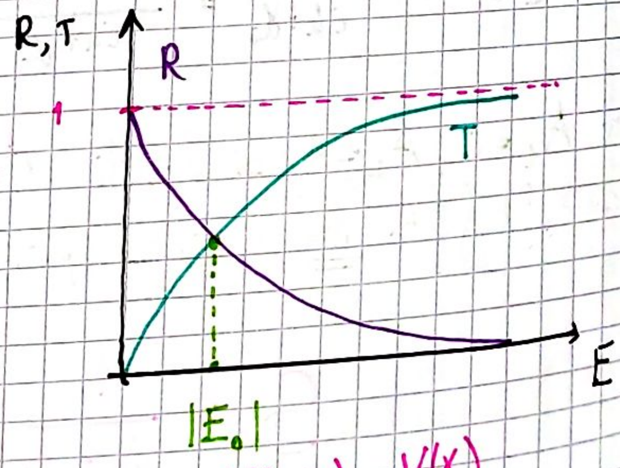
$$k = \sqrt{\frac{2mE}{\hbar^2}} \rightarrow E = \frac{\hbar^2k^2}{2m}$$

Spomnimo se edinega vezanega stanja:

$$E_0 = -\frac{m\lambda^2}{2\hbar^2} = -\frac{\hbar^2\kappa_0^2}{2m}$$

$$R = \frac{\frac{\hbar^2}{2m}\kappa_0^2}{E+\frac{\hbar^2}{2m}\kappa_0} = \frac{|E_0|}{1+|E_0|}$$

$$T = \frac{E}{E+\frac{\hbar^2}{2m}\kappa_0^2} = \frac{E}{E+|E_0|}$$



Izkoristimo dejstvo, da je potencial soda funkcija $V(-x)=V(x)$

$$H\psi(x) = E\psi(x)$$

$$x \rightarrow -x$$

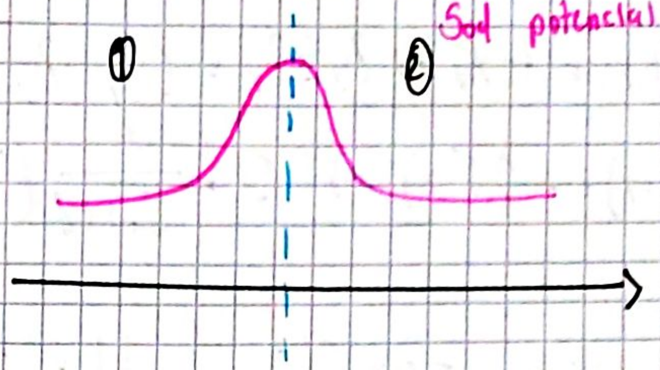
$$H\psi(-x) = E\psi(-x)$$

Če je $\psi(x)$ sipalno stanje, je $\psi(-x)$ tudi sipalno stanje.

$\psi(x)$:

$$\psi_1(x) = A_1 \frac{e^{ikx}}{\sqrt{v}} + B_1 \frac{e^{-ikx}}{\sqrt{v}}$$

$$\psi_2(x) = A_1 \frac{e^{-ikx}}{\sqrt{v}} + B_2 \frac{e^{ikx}}{\sqrt{v}}$$



$x \to -x \Rightarrow \psi(-x)$:

$$\psi_1 = A_2 \frac{e^{ikx}}{\sqrt{v}} + B_2 \frac{e^{-ikx}}{\sqrt{v}}$$

$$\psi_2 = A_1 \frac{e^{-ikx}}{\sqrt{v}} + B_2 \frac{e^{ikx}}{\sqrt{v}}$$

Obrne strani in vlogi A in B.

$$\begin{pmatrix} A_2 \\ A_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x \,;\, \sigma_x^2 = I} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\begin{pmatrix} B_{\cancel{1}2} \\ B_{\cancel{2}1} \end{pmatrix} = S \begin{pmatrix} A_2 \\ A_1 \end{pmatrix}$$

$$\Rightarrow \sigma_x \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = S \sigma_x \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \to \sigma_x B = S \sigma_x A$$

$$B = \sigma_x^{-1} S \sigma_x A$$

$$\Rightarrow \quad B = \underbrace{\sigma_x S \sigma_x}_{= S} A$$ (hkrati pa velja $B = SA$)

$$\sigma_x S \sigma_x = S$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t' & r \\ r' & t \end{pmatrix} = \begin{pmatrix} r' & t \\ t' & r \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

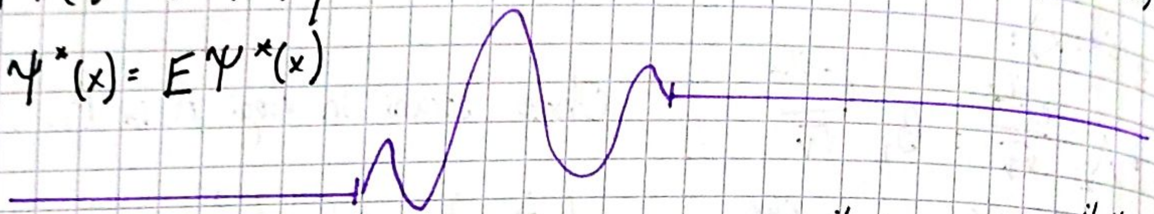
Sodost potenciala je tista, ki nam da, da je $r = r'$ in $t = t'$!

Za $V(x) = V(-x) \to \sigma_x S \sigma_x = S$; $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Za $V(x) = -\lambda\delta(x) \to S = \frac{1}{\varkappa_0 + i\varkappa}\begin{pmatrix} -\varkappa_0 & i\varkappa \\ i\varkappa & -\varkappa_0 \end{pmatrix}$; $\varkappa_0 = \frac{m\lambda}{\hbar^2}$

Torej: $H\psi(x) = E\psi(x) \,/^*$ $\qquad H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) = H^*$

$H\psi^*(x) = E\psi^*(x)$



$$A_1^* \frac{e^{-ik_1x}}{\sqrt{v_1}} + B_1 \frac{e^{ik_1x}}{\sqrt{v_1}} \qquad A_2^* \frac{e^{ik_2x}}{\sqrt{v_2}} + B_2^* \frac{e^{-ik_2x}}{\sqrt{v_2}}$$

$$\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix} = S \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix}$$

$$S^\dagger \,/\, A^* = S B^*$$

$$S^\dagger A^* = S^\dagger S B$$

$$\Rightarrow S^\dagger A^* = I B^* = B^* \,/^* \quad \Rightarrow B = (S^\dagger A^*)^* = S^T A \text{ in } B = SA$$

Iz tega sledi, da je $\underline{\underline{S = S^T}} \Rightarrow t = t'$ če je hamiltonian realen $H = H^*$

Primer ko $H \neq H^*$ (delec v magnetnem polju):

$$H = \frac{(\vec{p} - e\vec{A})^2}{2m} + V(\vec{r}) =$$

$$= \frac{(-i\hbar\nabla - e\vec{A})^2}{2m} + V(\vec{r}) \neq H^*$$

Spet gledamo nekaj za δ potencial

$$V(x) = -\lambda\delta(x) \qquad S = \frac{1}{\varkappa_0 + ik}\begin{pmatrix} -\varkappa_0 & ik \\ ik & -\varkappa_0 \end{pmatrix}$$

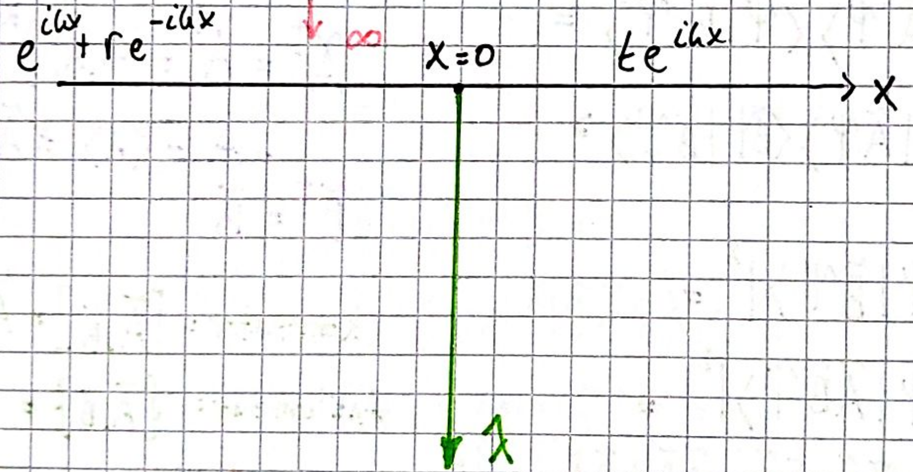
Pol imamo pri $\varkappa_0 + ik = 0 \Rightarrow k = i\varkappa_0$

$$E = \frac{\hbar^2 k^2}{2m} = -\frac{\hbar^2 \varkappa_0^2}{2m} = E_0$$ } Pol imamo pri ~~vsr~~ energiji vezanega stanja

$\Rightarrow$ Sipalna matrika nosi informacije tudi o vezanih stanjih

Iz sipalne matrike lahko rekonstruiramo tudi VF tega vezanega stanja:

Pri $k = i\varkappa_0$:
$$S = \underbrace{\frac{1}{\varkappa_0 + ik}}_{\infty}\begin{pmatrix} -\varkappa_0 & -\varkappa_0 \\ -\varkappa_0 & -\varkappa_0 \end{pmatrix}$$



V točki, kjer je pol:

$k \to i\varkappa_0$

Zanemarljiva amplituda

$$\cancel{e^{-\varkappa_0 x}} + \infty(-\varkappa_0)e^{\varkappa_0 x} \qquad \infty(-\varkappa_0)e^{-\varkappa_0 x}$$ } To lahko interpretiramo kot VF vezanega stanja

$\downarrow$ normalizacija $\downarrow$

$$A e^{-\varkappa_0 |x|} \to \sqrt{\varkappa_0}\, e^{-\varkappa_0 |x|}$$

To pa je ravno VF vezanega stanja, kot smo jo izračunali prej

Heisenbergovo načelo nedoločenosti

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

[Poišči vse valovne funkcije $\Psi(x)$, za katere velja $\sigma_x \sigma_p = \frac{\hbar}{2}$]

Najprej dokažimo načelo in tekom dokaza ~~bomo~~ bomo identificirali take VF:

1) Dokaz Heisenbergovega načela nedoločenosti

$\sigma A \cdot \sigma B$ A in B opazljivki; $\underbrace{A^\dagger = A, B^\dagger = B}_{\text{Hermitski operatorji}}$

(lastnost sebi-adjungirana)

$$\sigma A = \sqrt{\langle (A - \langle A \rangle)^2 \rangle}$$

Da poenostavimo notacijo vpeljemo:

$\tilde{A} = A - \langle A \rangle$

$\tilde{B} = B - \langle B \rangle$ Realno število

$\tilde{A}^\dagger = A^\dagger + (-\langle A \rangle)^* = A - \langle A \rangle = \tilde{A}$

$\tilde{B}^\dagger = \tilde{B}$

$$\sigma^2 A \, \sigma^2 B = \langle (A - \langle A \rangle)^2 \rangle \langle (B - \langle B \rangle)^2 \rangle =$$

$$= \langle \tilde{A}^2 \rangle \langle \tilde{B}^2 \rangle =$$

V splošnem $\langle \Psi | C \Psi \rangle = \langle C^\dagger \Psi | \Psi \rangle$

$$= \langle \Psi | \tilde{A}^2 \Psi \rangle \langle \Psi | \tilde{B}^2 \Psi \rangle =$$

$$= \langle \tilde{A} \Psi | \tilde{A} \Psi \rangle \langle \tilde{B} \Psi | \tilde{B} \Psi \rangle \geq$$

Cauchy-Schwartzeva neenakost

$$\geq |\langle \tilde{A} \Psi | \tilde{B} \Psi \rangle|^2 =$$

$$= |\langle \Psi | \tilde{A} \tilde{B} \Psi \rangle|^2 =$$

$$= \left| \langle \Psi | \frac{\tilde{A}\tilde{B} - \tilde{B}\tilde{A} + \tilde{A}\tilde{B} + \tilde{B}\tilde{A}}{2} \Psi \rangle \right|^2 =$$

$$= \left| \frac{1}{2} \langle \Psi | ([\tilde{A}, \tilde{B}] + \{\tilde{A}, \tilde{B}\}) \Psi \rangle \right|^2 =$$

Komutator: $[\tilde{A}, \tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A}$

Antikomutator: $\{\tilde{A}, \tilde{B}\} = \tilde{A}\tilde{B} + \tilde{B}\tilde{A}$

$[\tilde{A}, \tilde{B}]^\dagger = (\tilde{A}\tilde{B} - \tilde{B}\tilde{A})^\dagger =$ Antihermitski

$= \tilde{B}^\dagger \tilde{A}^\dagger - \tilde{A}^\dagger \tilde{B}^\dagger =$

$= \tilde{B}\tilde{A} - \tilde{A}\tilde{B} = \ominus [\tilde{A}, \tilde{B}]$

$\{\tilde{A}, \tilde{B}\}^\dagger = \tilde{B}^\dagger \tilde{A}^\dagger + \tilde{A}^\dagger \tilde{B}^\dagger = \tilde{B}\tilde{A} + \tilde{A}\tilde{B} = \{\tilde{A}, \tilde{B}\}$

Antikomutator dveh hermitskih operatorjev je hermitski.

$|x + iy|^2 = |x|^2 + |iy|^2$

$$= \left| \frac{1}{2} \left(\underbrace{\langle \Psi | [\tilde{A}, \tilde{B}] \Psi \rangle}_{\in i\mathbb{R}} + \underbrace{\langle \Psi | \{\tilde{A}, \tilde{B}\} \Psi \rangle}_{\in \mathbb{R}} \right) \right|^2 =$$

$$= \left| \frac{1}{2} \langle \Psi | [\tilde{A}, \tilde{B}] \Psi \rangle \right|^2 + \left(\frac{1}{2} \langle \Psi | \{\tilde{A}, \tilde{B}\} \Psi \rangle \right)^2 \geq$$

$$\geq \left|\frac{1}{2}\langle\psi|[\tilde{A},\tilde{B}]\psi\rangle\right|^2 =$$

$$= \left|\frac{1}{2}\langle\psi|[A,B]\psi\rangle\right|^2$$

Naj bo $C^\dagger = -C$ (antihermitski)

$$\langle C\rangle = \langle\psi|C\psi\rangle = \langle C^\dagger\psi|\psi\rangle = \langle -C\psi|\psi\rangle =$$
$$= -\langle C\psi|\psi\rangle = -\langle\psi|C\psi\rangle^* = -\langle C\rangle^*$$

$$\Rightarrow \langle C\rangle \in i\cdot\mathbb{R} \quad (\mathrm{Re} = 0)$$

$$[\tilde{A},\tilde{B}] = [A-\langle A\rangle, B-\langle B\rangle] =$$
$$= (A-\langle A\rangle)(B-\langle B\rangle) - (B-\langle B\rangle)(A-\langle A\rangle) =$$
$$= AB - \cancel{A\langle B\rangle} - \cancel{\langle A\rangle B} + \cancel{\langle A\rangle\langle B\rangle} -$$
$$- BA + \cancel{B\langle A\rangle} + \cancel{\langle B\rangle A} - \cancel{\langle B\rangle\langle A\rangle} = [A,B]$$

$$\underline{\underline{\sigma A\cdot\sigma B \geq \frac{1}{2}|\langle[A,B]\rangle|}}$$

Kaj smo torej dokazali?

$$\Downarrow$$

$$\sigma A\,\sigma B \geq \left|\frac{1}{2}\langle\psi|[A,B]\psi\rangle\right|$$

Za $A = x$ in $B = p$: $\quad [x,p] = i\hbar$

$$\Rightarrow \quad \sigma_x\sigma_p \geq \left|\frac{i\hbar}{2}\right| = \frac{\hbar}{2} \quad \checkmark$$

Kdaj bojo neenakosti postale enakosti? $\left(\sigma A\sigma B = \left|\frac{1}{2}\langle\psi|[A,B]\psi\rangle\right|\right)$

1. $$\langle\tilde{A}\psi|\tilde{A}\psi\rangle\langle\tilde{B}\psi|\tilde{B}\psi\rangle = |\langle\tilde{A}\psi|\tilde{B}\psi\rangle|^2 \Rightarrow \underbrace{|\tilde{B}\psi\rangle = \lambda|\tilde{A}\psi\rangle}_{\text{Vektorja sta vzporedna}}; \quad \lambda\in\mathbb{C}$$

2. $$\langle\psi|\{\tilde{A},\tilde{B}\}|\psi\rangle = 0$$

$$\langle\psi|\{\tilde{A},\tilde{B}\}\psi\rangle = \langle\psi|(\tilde{A}\tilde{B}+\tilde{B}\tilde{A})\psi\rangle =$$
$$= \langle\psi|\tilde{A}\tilde{B}\psi\rangle + \langle\psi|\tilde{B}\tilde{A}\psi\rangle =$$
$$= \langle\tilde{A}\psi|\tilde{B}\psi\rangle + \langle\tilde{B}\psi|\tilde{A}\psi\rangle =$$
$$= \langle\tilde{A}\psi|\lambda\tilde{A}\psi\rangle + \langle\lambda\tilde{A}\psi|\tilde{A}\psi\rangle =$$
$$= \lambda\langle\tilde{A}\psi|\tilde{A}\psi\rangle + \lambda^*\langle\tilde{A}\psi|\tilde{A}\psi\rangle =$$
$$= (\lambda+\lambda^*)\langle\tilde{A}\psi|\tilde{A}\psi\rangle = 0$$

$$\Rightarrow (\lambda+\lambda^*) \Rightarrow \lambda \in \mu i; \quad \mu\in\mathbb{R}$$

ali

$$\langle\tilde{A}\psi|\tilde{A}\psi\rangle = 0 = \sigma^2 A$$ (v tem primeru $\sigma B \to \infty$)

Torej smo dobili:

$$|\tilde{B}\psi\rangle = \lambda|\tilde{A}\psi\rangle$$
$$|\tilde{B}\psi\rangle = i\mu|\tilde{A}\psi\rangle$$

Od tu naprej pa se omejimo na $A = x,\ B = p$

$$(p - \langle p\rangle)|\psi\rangle = i\mu(x - \langle x\rangle)|\psi\rangle$$

$$\left(-i\hbar\frac{d}{dx} - \langle p\rangle\right)\psi(x) = i\mu(x - \langle x\rangle)\psi(x)$$

$$-i\hbar\frac{d\psi(x)}{dx} - \langle p\rangle\psi(x) = i\mu x\psi(x) - i\mu\langle x\rangle\psi(x)$$

$$\psi(x)\left(-\langle p\rangle - i\mu x + i\mu\langle x\rangle\right) = i\hbar\frac{d\psi}{dx}$$

$$\int(-\langle p\rangle - i\mu x + i\mu\langle x\rangle)dx = i\hbar\int\frac{d\psi}{\psi}$$

$$-\langle p\rangle x - \frac{i\mu}{2}x^2 + \frac{i\mu}{1}\langle x\rangle x = i\hbar\ln\psi + C$$

$$x\left(i\mu\langle x\rangle - \langle p\rangle - \frac{i\mu}{2}x\right) = i\hbar\ln\psi + C'$$

$$x\left(\frac{\mu}{\hbar}\langle x\rangle - \frac{1}{i\hbar}\langle p\rangle\right) - \frac{1}{2\hbar}\mu x^2 = \ln\psi + C'$$

Dopolnimo do kvadrata, dopolnitev v konstanto

$$-\frac{\mu}{2\hbar}(x - \langle x\rangle)^2 + \frac{i\langle p\rangle}{\hbar}x = \ln\psi + C'' \quad /\exp$$

$$C\exp\left(-\frac{\mu}{2\hbar}(x - \langle x\rangle)^2 + \frac{i\langle p\rangle}{\hbar}x\right) = \psi$$

$$\Rightarrow \psi = C\cdot \underbrace{e^{\frac{i\langle p\rangle x}{\hbar}}}_{\text{ravni val}}\ \underbrace{e^{-\frac{\mu}{2\hbar}(x - \langle x\rangle)^2}}_{\text{Gauss}}$$

↳ Iz normalizacije

$$1 = \int_{-\infty}^{\infty} \left(C^* e^{-\frac{i\langle p\rangle x}{\hbar}} e^{-\frac{\mu}{2\hbar}(x-\langle x\rangle)^2}\right)\left(C e^{\frac{i\langle p\rangle x}{\hbar}} e^{-\frac{\mu}{2\hbar}(x-\langle x\rangle)^2}\right) dx$$

Normalizacija:
$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

$$|C|^2 \int_{-\infty}^{\infty} e^{-\frac{\mu}{\hbar}(x-\langle x\rangle)^2} dx = 1$$

Verjetnostna gostota je res Gaussova, uporabimo standardno oznako:

$$e^{-\frac{(x-\langle x\rangle)^2}{2\sigma^2}} \quad |C|^2$$

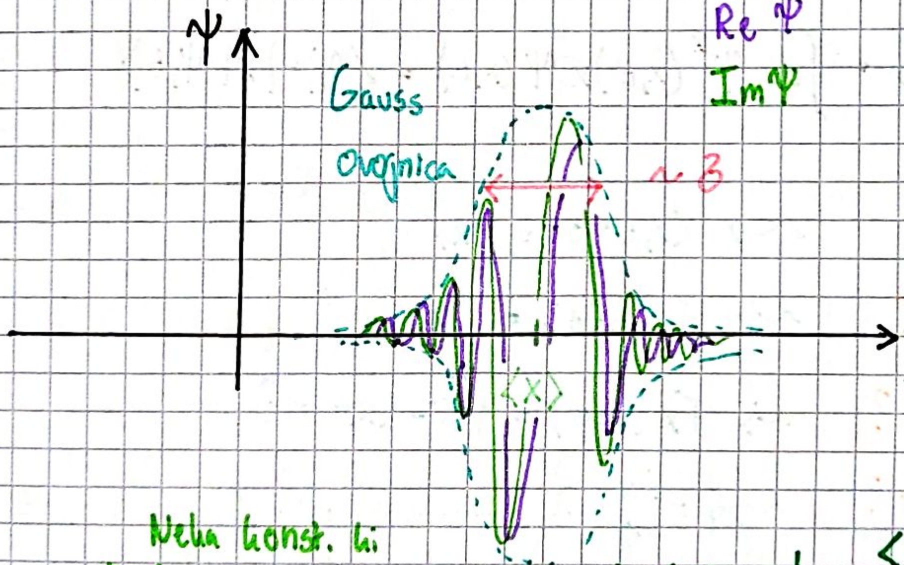
Za Gaussa vemo, da je normalizacija $\frac{1}{\sqrt{2\pi\sigma^2}} \Rightarrow C = \sqrt[4]{\frac{1}{2\pi\sigma^2}}$

Tako imamo:

$$\psi(x) = \sqrt[4]{\frac{1}{2\pi\sigma^2}}\, e^{\frac{i\langle p\rangle x}{\hbar}}\, e^{-\frac{(x-\langle x\rangle)^2}{4\sigma^2}}$$

Gaussovi valovni paketi

↳ Korenimo ker je Gauss $|\psi|^2$ zato je $(e^{\dots})^{\frac{1}{2}}$



Neka konst. ki ne sme vplivati recimo 0

$\langle p\rangle$; $k = \frac{\langle p\rangle}{\hbar}$

$$H = \frac{p^2}{2m} + V(x) = \frac{p^2}{2m} \quad \to \quad H\psi = E\psi$$

$$\Rightarrow \quad \psi_k(x) = e^{ikx}\frac{1}{\sqrt{2\pi}}$$

$$E_k = \frac{\hbar^2 k^2}{2m}$$

$$\Psi(x, t=0) = \frac{1}{\sqrt[4]{2\pi \delta^2}} \, e^{\frac{i\langle p \rangle}{\hbar} x} \, e^{-\frac{(x-\langle x \rangle)^2}{4\delta^2}}$$

k ... so zvezno razporejeni $\Rightarrow$ razvoj je integral

$$\Psi(x, t=0) = \int dk \, \underbrace{\frac{e^{ikx}}{\sqrt{2\pi}}}_{} \, c(k)$$

$\hookrightarrow$ Ubistvu Fourierova transformacija

Koeficiente dobimo z inverzno FT:

$$c(k) = \frac{1}{\sqrt{2\pi}} \int \Psi(x, t=0)\, dx \, e^{-ikx} \, dx$$

To bi izvrednotili, ampak bomo šli po drugi po

$$\rightarrow \Psi(x,t) = \int c(k) \, e^{-\frac{iE_k t}{\hbar}} \, \frac{e^{ikx}}{\sqrt{2\pi}} \, dk$$

Uporabimo oznake: $\langle x \rangle = \langle x, t=0 \rangle$

$$\langle x, t \rangle = \int dx \, \Psi^*(x,t)\, x \, \Psi(x,t) = \langle \Psi, t | x | \Psi, t \rangle$$

$\langle x, t \rangle = ?$ $\rightsquigarrow \langle x, t=0 \rangle = \langle x \rangle$

$\langle p, t \rangle = ?$ $\rightsquigarrow \langle p, t=0 \rangle = \langle p \rangle$

$\delta(x)(t) = ?$ $\rightsquigarrow \delta x(t=0) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \delta$

$\delta p(t) = ?$ $\rightsquigarrow \delta_p = \frac{\hbar}{2\delta}$ iz minimalnega produkta nedoločenosti

$\hookrightarrow t=0$ $\qquad \delta x(0)\, \delta p(0) = \frac{\hbar}{2}$

Zanima nas:

$\rightarrow$ Schrödingerjeva slika/reprezentacija

$$\langle A, t \rangle = \langle \Psi, t | A | \Psi, t \rangle = (*)$$

$$|\Psi, t\rangle = \underbrace{e^{-\frac{iHt}{\hbar}}}_{} |\Psi, 0\rangle$$

complex conj.

Operator časovnega razvoja

$$\langle \Psi, t | = \langle \Psi, 0 | \, e^{\frac{iHt}{\hbar}}$$

$$\Rightarrow \langle x \rangle = \langle \psi, t | A \, e^{-\frac{iHt}{\hbar}} | \psi \rangle =$$

$$= \langle \psi, 0 | \underbrace{e^{\frac{iHt}{\hbar}} A \, e^{-\frac{iHt}{\hbar}}}_{\equiv A(t)} | \psi, 0 \rangle = \langle \psi, 0 | A(t) | \psi, 0 \rangle$$

Heisenbergova slika/reprezentacija

Tu sploh ne delamo časovnega razvoja

Poglejmo operatorje v Heisenbergovi sliki:

$$(AB)(t) = e^{i\frac{Ht}{\hbar}} AB \, e^{-i\frac{Ht}{\hbar}} = e^{i\frac{Ht}{\hbar}} A \underbrace{I}_{e^{-i\frac{Ht}{\hbar}} e^{i\frac{Ht}{\hbar}}} B \, e^{-i\frac{Ht}{\hbar}} = A(t)B(t)$$

$$(\alpha A + \beta B)(t) = e^{\frac{iHt}{\hbar}} (\alpha A + \beta B) e^{-i\frac{Ht}{\hbar}} = \alpha \, e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}} + \beta \, e^{\frac{iHt}{\hbar}} B \, e^{-\frac{iHt}{\hbar}} =$$

$\alpha, \beta \in \mathbb{C}$ konst

$$= \alpha A(t) + \beta B(t)$$

Pazi na vrstni red, ker operatorji lahko ne komutirajo

$$\frac{d}{dt} A(t) = \frac{d}{dt}\left(e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}} \right) = \frac{iH}{\hbar} e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}} + e^{\frac{iHt}{\hbar}} A \left(-\frac{iH}{\hbar}\right) e^{-\frac{iHt}{\hbar}} =$$

Operator in funkcija operatorja komutirata

$$= e^{i\frac{Ht}{\hbar}} \left(\frac{iH}{\hbar} A - A \frac{iH}{\hbar} \right) e^{-\frac{iHt}{\hbar}} =$$

$$= e^{\frac{iHt}{\hbar}} \frac{i}{\hbar} [H, A] e^{-\frac{iHt}{\hbar}} = \frac{i}{\hbar} [H, A](t)$$

Lahko se vrnemo nazaj na naš problem: $x(t) = ?$ $\quad p(t) = ?$

$$\frac{d}{dt} x(t) = \frac{i}{\hbar} [H, x](t) = \frac{i}{\hbar} \left[\frac{p^2}{2m}, x \right](t) = (**)$$

$$\frac{d}{dt} p(t) = \frac{i}{\hbar} [H, p](t) = \frac{i}{\hbar} \underbrace{\left[\frac{p^2}{2m}, p \right]}_{=0}(t) = 0$$

$[A,B] = -[B,A]$

$[AB, C] = A[B,C] + [A,C]B$

$[x,p] = i\hbar$

$$\left[\frac{p^2}{2m}, x \right] = \frac{1}{2m} [p^2, x] = \frac{1}{2m} \left(p[p,x] + [p,x]p \right) =$$

$$= \frac{1}{2m} \left(p(-i\hbar) + (-i\hbar)p \right) = -\frac{i\hbar}{m} p$$

$$\Rightarrow (**) = \frac{i}{\hbar}\left(-\frac{i\hbar}{m}\right) p(t) \frac{1}{m} p(t)$$

in

$$\frac{d}{dt} p(t) = 0 \quad \Rightarrow \quad p(t) = C = P$$

Moramo zadostiti še začetnemu pogoju: $A(0) = e^{\frac{iH_0}{\hbar}} A e^{-\frac{iH_0}{\hbar}} = A$

In še:

$$\frac{d}{dt} x(t) = \frac{P}{m} \Rightarrow x(t) = \frac{pt}{m} + C = \frac{pt}{m} + X$$

$$X(0) = X$$

Tako imamo:

$$X(t) = \frac{pt}{m} + X$$
$$p(t) = P$$

Torej:

$$\langle p, t \rangle = \langle \psi, 0 | p(t) | \psi, 0 \rangle = \langle \psi, 0 | p | \psi, 0 \rangle = \langle p, 0 \rangle = \langle p \rangle$$

$$\langle x, t \rangle = \langle \psi, 0 | x(t) | \psi, 0 \rangle = \langle \psi, 0 | \frac{pt}{m} + x | \psi, 0 \rangle =$$
$$= \frac{\langle p \rangle t}{m} + \langle \psi, 0 | x | \psi, 0 \rangle = \langle x, 0 \rangle + \frac{\langle p \rangle t}{m}$$
$$= \frac{\langle p \rangle t}{m} + \langle x \rangle$$

Klasično zgleda kot prema enakomerno gibanje!

$$\Rightarrow \quad \langle p, t \rangle = \langle p \rangle \qquad \langle x, t \rangle = \langle x \rangle + \frac{\langle p \rangle t}{m}$$

$$\delta x^2(t) = \langle x^2, t \rangle - \langle x, t \rangle^2 = ? \quad (*)$$

$$\langle x^2, t \rangle = \langle \psi, 0 | x^2(t) | \psi, 0 \rangle = \langle \psi, 0 | (x(t))^2 | \psi, 0 \rangle =$$
$$= \langle \psi, 0 | \left(\frac{pt}{m} + x\right)^2 | \psi, 0 \rangle =$$

vrstni red pazi

$$= \frac{t^2}{m^2} \langle \psi, 0 | p^2 | \psi, 0 \rangle + \langle \psi, 0 | x^2 | \psi, 0 \rangle + \frac{t}{m} \langle \psi, 0 | px + xp | \psi, 0 \rangle =$$

$$= \frac{t^2}{m^2}\langle p^2, 0\rangle + \langle x^2, 0\rangle + \frac{t}{m}\langle px + xp, 0\rangle$$

$$\delta p^2(t) = \langle p^2, t\rangle - \langle p, t\rangle^2 = ? \quad (**)$$

$$\langle p^2, t\rangle = \langle \Psi, 0 | p^2(t) | \Psi, 0\rangle = \langle \Psi, 0 | (p(t))^2 | \Psi, 0\rangle = \langle \Psi, 0 | p^2 | \Psi, 0\rangle =$$

$$= \langle p^2, 0\rangle$$

Torej za nedoločenost p dobimo:

$$\Rightarrow (**) \quad \delta p^2(t) = \langle p^2, t\rangle - \langle p, t\rangle^2 = \langle p^2, 0\rangle - \langle p, 0\rangle^2 = \delta p^2(0)$$

Nedoločenost gibalne količine se z časom ne spreminja.

Za nedoločenost x pa dobimo:

$$\Rightarrow (*) \quad \delta x(t)^2 = \langle x^2, t\rangle - \langle x, t\rangle^2 = \qquad \sigma^2 \quad \frac{t^2}{m^2}\delta p(0)^2$$

$$= \frac{t^2}{m^2}\langle p^2, 0\rangle + \langle x^2, 0\rangle + \frac{t}{m}\langle px + xp, 0\rangle - \frac{\langle p\rangle^2 t^2}{m^2} - \langle x\rangle^2 -$$

$$- 2\frac{\langle p\rangle t}{m}\langle x\rangle =$$

$$= \sigma^2 + \frac{t^2}{m^2}\frac{\hbar^2}{4\sigma^2} - 2\frac{\langle p\rangle t}{m}\langle x\rangle + \frac{t}{m}\langle px + xp, 0\rangle = (***)$$

$$(px)^\dagger = x^\dagger p^\dagger = xp; \quad x^\dagger = x, \quad p^\dagger = p$$

$$\langle px + xp, 0\rangle =$$

$$= \langle xp + (xp)^\dagger, 0\rangle = \langle xp, 0\rangle + \langle (xp)^\dagger, 0\rangle = (***)$$

$$\langle A^\dagger\rangle = \langle \Psi | A^\dagger \Psi\rangle = \langle A\Psi | \Psi\rangle = \langle \Psi | A \Psi\rangle^* = \langle A\rangle^*$$

$$\Rightarrow (***) = 2\,\mathrm{Re}(\langle xp, 0\rangle)$$

$$p\Psi(x,0) = -i\hbar\frac{\partial}{\partial x}\left(\frac{1}{\sqrt[4]{2\pi\sigma^2}}\exp\left(\frac{i\langle p\rangle x}{\hbar} - \frac{(x-\langle x\rangle)^2}{4\sigma^2}\right)\right) =$$

$$= \frac{-i\hbar}{\sqrt[4]{2\pi\sigma^2}}\left(\left(i\frac{\langle p\rangle}{\hbar}\right)e^{i\frac{\langle p\rangle x}{\hbar}} e^{\frac{(x-\langle x\rangle)^2}{4\sigma^2}} + e^{\frac{i\langle p\rangle x}{\hbar}}\left(-2(x-\langle x\rangle)\right)/(4\sigma^2)\, e^{-\frac{(x-\langle x\rangle)^2}{4\sigma^2}}\right.$$

$$= -i\hbar\left(\frac{i\langle p\rangle}{\hbar} + \left(\frac{-2(x-\langle x\rangle)}{4\sigma^2}\right)\right)\Psi(x,0) =$$

$$= \left(\langle p\rangle + \frac{(x-\langle x\rangle)i\hbar}{2\sigma^2}\right)\Psi(x,0)$$

$$\langle xp,0\rangle = \int\Psi^*(x,0)\, x p\, \Psi(x,0)\,dx =$$

$$= \int\Psi^*(x,0)\, x\left(\langle p\rangle + \frac{(x-\langle x\rangle)i\hbar}{2\sigma^2}\right)\Psi(x,0)\,dx$$

$$\mathrm{Re}\left[\langle xp,0\rangle\right] = \mathrm{Re}\int\Psi^*(x,0)\, x\left(\langle p\rangle + \cancel{\frac{(x-\langle x\rangle)i\hbar}{2\sigma^2}}\right)\Psi(x,0)\,dx$$

$$= \mathrm{Re}\left[\langle p\rangle\int\Psi^*(x,0)\, x\,\Psi(x,0)\,dx\right] = \langle p\rangle\langle x\rangle$$

$$\Rightarrow (***) = \sigma^2 + \frac{t^2}{m^2}\frac{\hbar^2}{4\sigma^2} + \frac{2t}{m}\langle x\rangle\langle p\rangle - 2\frac{t}{m}\langle x\rangle\langle p\rangle =$$

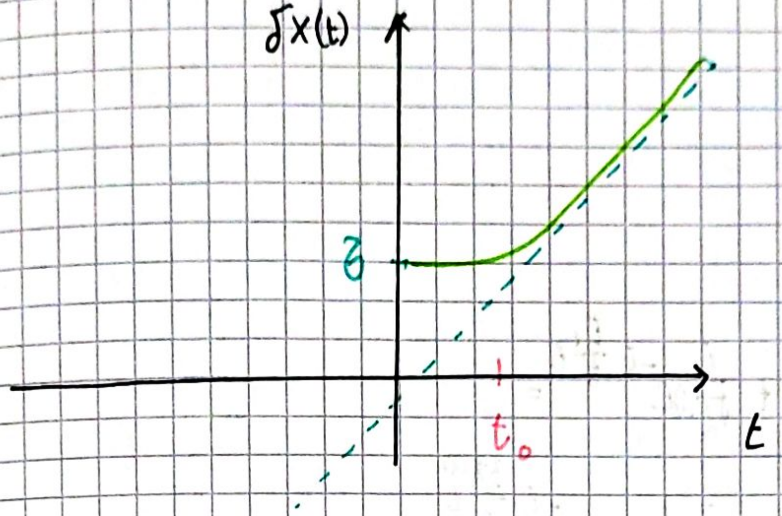
$$\delta x^2(t) = \sigma^2 + \frac{t^2}{m^2}\frac{\hbar^2}{4\sigma^2}$$

$$\delta x(t) = \sigma\sqrt{\left(1 + \frac{t^2\hbar^2}{m^2 4\sigma^4}\right)}$$

Poglejmo še produkt nedoločenosti:

$$\delta x(t)\,\delta_p(t) = \frac{\hbar}{2}\sqrt{1 + \frac{t^2\hbar^2}{4m^2\sigma^4}}$$

Nedoločenost (produkt) je bil minimalen <u>samo</u> na začetku. Ob poznejših časih se ta funkcijska oblika ne ohranja.



$$\frac{t_0^2 \hbar^2}{4m^2\delta^4} = 1 \quad \Rightarrow \quad t_0 = \frac{2m\delta^2}{\hbar}$$

Značilna energija

$$t_0 \cdot \frac{\hbar^2}{2m\delta^2} = \hbar$$

Značilni čas

Harmonski oscilator

$$H = \frac{p^2}{2m} + \frac{kx^2}{2} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) ; \quad \omega = \sqrt{\frac{k}{m}}$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right) ; \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} \quad p_0 = \frac{\hbar}{x_0}$$

Ortogonalni

$$\langle n, m \rangle = \delta_{n,m}$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{p}{p_0}\right)$$

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle ; \quad n = 0, 1, 2, \ldots$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$x = \frac{x_0}{\sqrt{2}}(a + a^\dagger)$$

$$a^\dagger|n\rangle = \sqrt{n-1}\,|n\rangle$$

$$p = \frac{p_0}{\sqrt{2}i}(a - a^\dagger)$$

$$a^\dagger a|n\rangle = n|n\rangle$$

$$[a, a^\dagger] = 1$$

[Linearni harmonski oscilator]

$|\Psi,t\rangle = ?$ $|\Psi,0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$ $E_n = \hbar\omega(n+\frac{1}{2})$

$\langle x,t\rangle = ?$

$\langle p,t\rangle = ?$

$\sigma x(t) = ?$

a) $|\Psi,t\rangle = ?$

$$|\Psi,t\rangle = \frac{1}{\sqrt{2}}|0\rangle e^{-i\frac{E_0}{\hbar}t} + \frac{i}{\sqrt{2}}|1\rangle e^{-i\frac{E_1}{\hbar}t} =$$

$$= \frac{1}{\sqrt{2}}|0\rangle e^{-i\frac{\omega}{2}t} + \frac{i}{\sqrt{2}}|1\rangle e^{-i\frac{3\omega}{2}t}$$

b) $\langle x,t\rangle = ?$ V obeh slikah

c) $\langle p,t\rangle = ?$

Konjugirana

$$\langle x,t\rangle = \langle \frac{x_0}{\sqrt{2}}(a+a^\dagger),t\rangle = \frac{x_0}{\sqrt{2}}\left(\langle a,t\rangle + \langle a^\dagger,t\rangle\right) =$$

$$= \sqrt{2}x_0 \, Re(\langle a,t\rangle)$$

$$\langle p,t\rangle = \frac{p_0}{\sqrt{2}i}\left(\underbrace{\langle a,t\rangle - \langle a^\dagger,t\rangle}_{2iIm \text{ ker Konjugirana}}\right) = \sqrt{2}p_0 \, Im(\langle a,t\rangle)$$

Schrödinger:

aplicirammo $a|1\rangle = \sqrt{1}|0\rangle = |0\rangle$

$$\langle\Psi,t|a|\Psi,t\rangle = \langle a,t\rangle =$$

$$= \left(\frac{1}{\sqrt{2}}e^{i\frac{\omega}{2}t}\langle 0| - \frac{i}{\sqrt{2}}e^{i\frac{3\omega}{2}t}\langle 1|\right) a\left(\frac{1}{\sqrt{2}}|0\rangle e^{-i\frac{\omega}{2}t} + \frac{i}{\sqrt{2}}e^{-i\frac{3\omega}{2}t}|1\rangle\right) =$$

skalarni prod. $a|0\rangle = \sqrt{0}|-1\rangle = 0$ $\langle 0,0\rangle = 1$ $\langle 1,0\rangle = 0$

$$= \left(\frac{1}{\sqrt{2}}e^{i\frac{\omega}{2}t}\langle 0| - \frac{i}{\sqrt{2}}e^{i\frac{3\omega}{2}t}\langle 1|\right)\left(\frac{i}{\sqrt{2}}|0\rangle e^{-i\frac{3\omega}{2}\omega t}\right) =$$

$$= \frac{i}{2}e^{-i\omega t} = \frac{i}{2}\left(\cos(\omega t) - i\sin(\omega t)\right) = \frac{\sin(\omega t)}{2} + i\frac{\cos(\omega t)}{2}$$

Tako je torej:

$$\langle x,t\rangle = \sqrt{2}\,x_0\,Re(\langle a,t\rangle) = \sqrt{2}\,x_0\,\frac{\sin(\omega t)}{2}$$

$$\langle p,t\rangle = \sqrt{2}\,p_0\,Im(\langle a,t\rangle) = \sqrt{2}\,p_0\,\frac{\cos(\omega t)}{2}$$

Ehrenfestov teorem:

i) $\frac{d}{dt}\langle x,t\rangle = \frac{\langle p,t\rangle}{m}$ Analogno klasično: $v = \frac{p}{m}$

ii) $\frac{d}{dt}\langle p,t\rangle = \langle -\frac{d}{dx}V(x)\rangle$ Analogno klasično: $F = -\nabla V$

$$\frac{d}{dt}\langle p,t\rangle = -\frac{1}{2}k\,2\langle x\rangle$$

Preverimo ali res velja i):

$$\frac{d}{dt}\left(\sqrt{2}\,x_0\,\frac{\sin(\omega t)}{2}\right) = \frac{\sqrt{2}\,p_0}{m}\,\frac{\cos(\omega t)}{2}$$

$$\frac{\sqrt{2}}{2}\,x_0\,\omega\cos(\omega t) = \frac{\sqrt{2}}{2}\,\frac{p_0}{m}\cos(\omega t)$$

$$\frac{\sqrt{2}}{2}\sqrt{\frac{\hbar}{m\omega}}\,\omega\cos(\omega t) = \frac{\sqrt{2}}{2}\,\frac{\hbar}{\sqrt{\frac{\hbar}{m\omega}}\,m}\cos(\omega t)$$

$$\frac{\sqrt{2}}{2}\sqrt{\frac{\hbar\omega}{m}}\cos(\omega t) = \frac{\sqrt{2}}{2}\sqrt{\frac{\hbar\omega}{m}}\cos(\omega t) \quad \checkmark$$

Velja!

Heisenberg:

$$\langle x,t\rangle = \sqrt{2}\,x_0\,Re(\langle a(t)\rangle)$$

$$\langle p,t\rangle = \sqrt{2}\,p_0\,Im(\langle a(t)\rangle)$$

$$a(t) = e^{i\frac{H}{\hbar}t}\,a\,e^{-i\frac{H}{\hbar}t}$$

$$\frac{d}{dt}a(t) = \frac{i}{\hbar}[H,a](t)\ ;\ a(0) = a$$

Torej bomo potrebovali: Množenje komutira

$$[H, a] = \left[\hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right), a\right] = \hbar\omega\left[a^\dagger a + \cancel{\tfrac{1}{2}}, a\right] =$$

Opravimo sam s tem komutira

$[AB, C] = A[B, C] + [A, C]B$

$$= \hbar\omega\left(a^\dagger \cancel{[a, a]} + \underbrace{[a^\dagger, a]}_{-1} a\right) =$$

$$= \hbar\omega(-a) = -\hbar\omega a$$

Imamo: Rešitev bo neka eksponentna

$$\frac{d}{dt} a(t) = \frac{i}{\hbar}\left(-\hbar\omega a(t)\right) \cdot$$

$$a(t) = a_0 e^{-i\omega t}$$

Upoštevimo še začetni pogoj: $a(0) = a_0 e^0 = a$

$$\Rightarrow \quad a(t) = a e^{-i\omega t}$$

$$\left.\begin{aligned} \langle x, t\rangle &= \sqrt{2} x_0 \, \mathrm{Re}\left(\langle a \rangle e^{-i\omega t}\right) \\ \langle p, t\rangle &= \sqrt{2} p_0 \, \mathrm{Im}\left(\langle a \rangle e^{-i\omega t}\right) \end{aligned}\right\}$$ Popolnoma še splošno

ob $t=0$

$$\langle a, 0\rangle = \langle\psi|a|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right) a \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) =$$

$$= \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{i}{\sqrt{2}}|0\rangle\right) = \frac{i}{2} = \langle a \rangle$$

To lahko vstavimo in preverimo, če dobimo isto kot prej...

$$\langle x, t\rangle = \sqrt{2} x_0 \, \mathrm{Re}\left(\frac{i}{2}(\cos(\omega t) - i\sin(\omega t))\right) = \frac{\sqrt{2}}{2} x_0 \sin(\omega t)$$

$$\langle p, t\rangle = \ldots = \frac{\sqrt{2}}{2} p_0 \cos(\omega t)$$

d) $(\sigma x(t))^2 = \langle x^2, t\rangle - \langle x, t\rangle^2$

$$\langle x^2, t\rangle = \left\langle \frac{x_0^2}{2}(a+a^\dagger)^2, t\right\rangle = \frac{x_0^2}{2}\left\langle (a+a^\dagger)^2, t\right\rangle =$$

$$= \frac{x_0^2}{2}\left\langle (a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}), t\right\rangle =$$

Pazi na vrstni red

Normalni vrstni red:

$$(a^\dagger)^n a^m$$

$$[a, a^\dagger] = 1 = aa^\dagger - a^\dagger a$$

$$= \frac{x_0^2}{2}\left\langle a^2 + 1 + a^\dagger a + a^\dagger a + a^{\dagger 2}, t\right\rangle =$$

complex conjugate

$$= \frac{x_0^2}{2}\left(\langle a^2, t\rangle + \langle 1, t\rangle + 2\langle a^\dagger a, t\rangle + \langle a^{\dagger 2}, t\rangle\right) =$$

$$= \frac{x_0^2}{2}\left(1 + 2\langle a^\dagger a, t\rangle + 2\,\mathrm{Re}\langle a^2, t\rangle\right) = (*)$$ Rešimo za vajo v Heisenbergu

$$a^2(t) = e^{+i\frac{H}{\hbar}t} a^2 e^{-i\frac{H}{\hbar}t} = a(t)^2 \; ; \; a(t) = a e^{-i\omega t}$$

Pomožni račun za $A^\dagger(t)$:

$$A^\dagger(t) = e^{i\frac{H}{\hbar}t} A^\dagger e^{-i\frac{H}{\hbar}t} = \left(e^{i\frac{H}{\hbar}t} A e^{-i\frac{H}{\hbar}t}\right)^\dagger = (A(t))^\dagger$$

Torej:

$$a(t) = a e^{-i\omega t}$$

$$a^\dagger(t) = (a e^{-i\omega t})^\dagger = a^\dagger e^{i\omega t}$$

$$a^2(t) = a(t)^2$$

$$(*) \Rightarrow \; = \frac{x_0^2}{2}\left(2\,\mathrm{Re}\langle a(t)^2\rangle + 2\langle a^\dagger(t) a(t)\rangle + 1\right) =$$

$$= \frac{x_0^2}{2} + x_0^2\left(\mathrm{Re}\langle a^2 e^{-2i\omega t}\rangle + \langle a^\dagger a\rangle\right) =$$

$$= \frac{x_0^2}{2} + x_0^2\left(\mathrm{Re}(\langle a^2\rangle e^{-2i\omega t}) + \langle a^\dagger a\rangle\right)$$

$$\langle a^2\rangle = \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right) a^2 \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) = \underline{\underline{0}}$$

$$\langle a^\dagger a\rangle = \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right) a^\dagger a \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right) =$$

$$= \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{i}{\sqrt{2}}\langle 1|\right)\left(\frac{i}{\sqrt{2}}|1\rangle\right) = \underline{\underline{\frac{1}{2}}}$$

$a^\dagger a|0\rangle = 0|0\rangle$
$a^\dagger a|1\rangle = 1|1\rangle$

$$\Rightarrow \langle x^2, t\rangle = x_0^2\left(\frac{1}{2}+\frac{1}{2}\right) = \underline{x_0^2}$$

In še končni rezultat

$$[\delta x(t)]^2 = x_0^2 - \frac{x_0^2}{2}\sin^2(\omega t)$$

Ponovitev ~~formul~~ formul za LHO:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \qquad a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + i\frac{p}{p_0}\right)$$

$$H|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle \,;\; n = 0,1,\ldots \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - i\frac{p}{p_0}\right)$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad x = \frac{x_0}{\sqrt{2}}(a + a^\dagger)$$

$$a^\dagger|n\rangle = \sqrt{n-1}\,|n\rangle \qquad p = \frac{p_0}{\sqrt{2}}(a - a^\dagger)$$

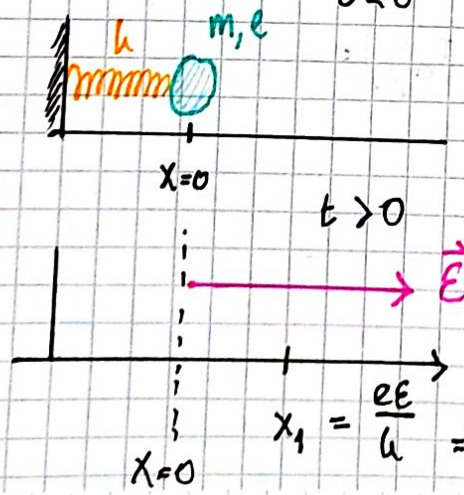
$$[a, a^\dagger] = 1$$

$$\langle x\rangle = \sqrt{2}\,x_0\,\mathrm{Re}\langle a\rangle$$

$$\langle p\rangle = \sqrt{2}\,p_0\,\mathrm{Im}\langle a\rangle$$

[Klasičen primer rešujemo kvantno]

Klasično:



$$x_1 = \frac{e\mathcal{E}}{k} \Rightarrow x(t) = x_1(1-\cos\omega t)$$

Kvantno:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \,;\; t<0$$

$$\tilde{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - e\mathcal{E}x \,;\; t>0$$

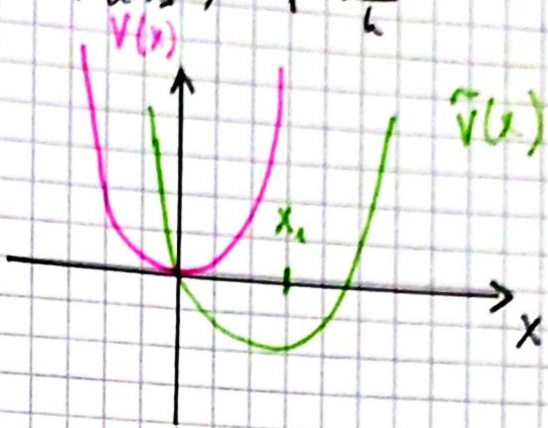
$|\psi, 0\rangle = |0\rangle$ Najbližje temu, da delec miruje (zaradi Heisenberg ne sme mirovati)

Zanima nas $|\psi, t>0\rangle$?

danes od 09.12.2021

$$\frac{1}{2}k\left(x^2 - \frac{2e\varepsilon}{k}x\right) = \frac{1}{2}k\left[\left(x - \frac{e\varepsilon}{k}\right)^2 - \left(\frac{e\varepsilon}{k}\right)^2\right] \quad ; \quad x_1 = \frac{e\varepsilon}{k}$$

$$= \underbrace{\frac{1}{2}k\left[(x-x_1)^2 - x_1^2\right]}_{\tilde{V}(x)}$$



$$x - x_1 = \tilde{x}$$

$$\tilde{p} = -i\hbar\frac{d}{d\tilde{x}} = -i\hbar\frac{d}{dx} = p$$

$$\Rightarrow \tilde{H} = \frac{\tilde{p}^2}{2m} + \frac{1}{2}k\tilde{x}^2 - \frac{1}{2}kx_1^2 = \hbar\omega\left(\tilde{a}^\dagger\tilde{a} + \frac{1}{2}\right) - \frac{1}{2}kx_1^2$$

$$\omega = \tilde{\omega}$$

$$\tilde{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} + i\frac{\hat{p}}{p_0}\right)$$

$$|z,t\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle e^{-i\frac{E_n}{\hbar}t} =$$

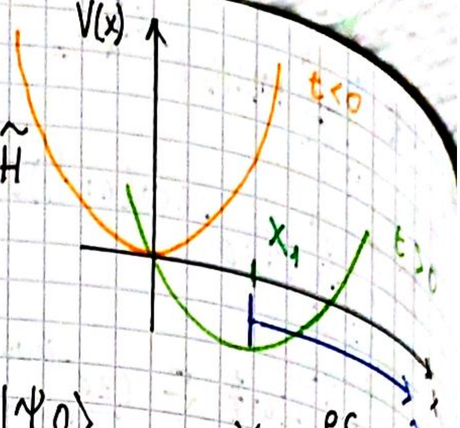
$$= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle e^{-i\omega(n+\frac{1}{2})t} = e^{-\frac{|z|^2}{2}} e^{-i\omega\frac{t}{2}} \sum_{n=0}^{\infty} \frac{(z e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle =$$

$$= e^{-\frac{|z e^{i\omega t}|^2}{2}} e^{-i\omega\frac{t}{2}} \sum_{n=0}^{\infty} \frac{(z e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle = e^{-\frac{i\omega}{2}t} |z e^{-i\omega t}\rangle$$

Nekaj lastnosti koherentnih stanj

$$a|z\rangle = z|z\rangle$$

$$H(t) = \begin{cases} t < 0; & \frac{p^2}{2m} + \frac{1}{2} k x^2 = H \\ t > 0; & \frac{p^2}{2m} + \frac{1}{2} k x^2 - e\mathcal{E}_x = \tilde{H} \end{cases}$$



$$|\psi,0\rangle = |0\rangle \rightarrow a|\psi,0\rangle = 0$$

$$|\psi,t\rangle = ? \rightarrow \tilde{a}|\psi,0\rangle = -\frac{X_1}{\sqrt{2}X_0}|\psi,0\rangle \qquad X_1 = \frac{e\mathcal{E}}{k}$$

Ukvarjali smo se z koherentnimi stanji: (ostanejo koherentna tudi skozi čas)

$$a|z\rangle = z|z\rangle$$

$$|z,t\rangle = e^{-i\frac{\omega}{2}t}|z e^{-i\omega t}\rangle \qquad \langle x\rangle = \sqrt{2}X_0 \,\mathrm{Re}\, z$$

$$\psi_z(x) = \frac{1}{\sqrt[4]{\pi X_0}} e^{-\frac{(x - \sqrt{2}X_0 \,\mathrm{Re}\, z)^2}{2X_0}} e^{\frac{i\sqrt{2}p_0 \,\mathrm{Im}\, z}{\hbar}x}$$

← gaussov valovni paket

Poglejmo si:

$$z(t)$$

$$z(0) = -\frac{X_1}{\sqrt{2}X_0}$$

$$z(t) = -\frac{X_1}{\sqrt{2}X_0} e^{-i\omega t}$$

$\langle x,t\rangle = ?$

$\tilde{x} = x - X_1$

$x = \tilde{x} + X_1$

$$\langle \tilde{x},t\rangle = \sqrt{2}X_0 \,\mathrm{Re}\,(z(t)) =$$

$$= \sqrt{2}X_0 \,\mathrm{Re}\left(-\frac{X_1}{\sqrt{2}X_0} e^{-i\omega t}\right) = -X_0\cos(\omega t)$$

Upoštevamo povezavo med x in $\tilde{x}$:

$$\langle x,t\rangle = -X_1\cos(\omega t) + X_1 = X_1(1-\cos(\omega t))$$

To pa dobimo tudi v klasičnem primeru