1. [Zvezda z linearno spreminjajočo gostoto]

M

$\rho(r) = \rho_c\left(1 - \frac{r}{R}\right)$

a) $p_c(M, R) = ?$

b) $p_c > \frac{GM^2}{8\pi R^4}$

a) $\frac{dp}{dr} = -\rho(r)\frac{Gm(r)}{r^2}$

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\,dr'$$

$$= \int_0^r 4\pi r'^2 \rho_c\left(1 - \frac{r'}{R}\right)dr' =$$

$$= \frac{4\pi}{3}r^3\rho_c - \frac{4\pi\rho_c}{R}\frac{r^4}{4}$$

$$\Rightarrow m(r) = 4\pi\rho_c\left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

Sedaj pa lahko uporabimo HS ravnovesje:

$$\frac{dp}{dr} = -\frac{G}{r^2}\rho_c\left(1 - \frac{r}{R}\right)4\pi\rho_c\left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

$$\frac{dp}{dr} = -\frac{4\pi G\rho_c^2}{r^2}\left(\frac{r^3}{3} \underbrace{- \frac{r^4}{4R} - \frac{r^4}{3R}}_{-7r^4/12R} + \frac{r^5}{4R^2}\right)$$

$$\frac{dp}{dr} = -4\pi G\rho_c^2\left(\frac{r}{3} - \frac{7r^2}{12R} + \frac{r^3}{4R^2}\right)$$

$$\int_{p_c}^{p(r)} dp = -4\pi G\rho_c^2\int_0^R\left(\frac{r}{3} - \frac{7r^2}{12R} + \frac{r^3}{4R^2}\right)dr = -4\pi G\rho_c^2\left(\frac{r^2}{6} - \frac{7r^3}{36R} + \frac{r^4}{16R^2}\right)$$

$$\Rightarrow p(r) - p_c = -4\pi G\rho_c^2\left(\frac{r^2}{6} - \frac{7r^3}{36R} + \frac{r^4}{16R^2}\right)$$

Primer: $p(r) = p(R) = 0$

$$p_c = 4\pi G\rho_c^2\left(\frac{1}{6} - \frac{7}{36} + \frac{1}{16}\right) = 4\pi G\rho_c^2 R^2\frac{5}{144}$$

$$= \rho_c^2\, 4\pi G R^2\,(\underbrace{0{,}0347}_{=\alpha})$$

b) $P_c > \frac{GM^2}{8\pi R^4}$

$$M = m(R) = 4\pi \rho_c R^3 \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{3}\pi \rho_c R^3$$

$$\rho_c = \frac{3M}{\pi R^3}$$ To lahko vstavimo v zapis za P_c od prej

$$P_c = \frac{9M^2}{(\pi R^3)^2} 4\pi G R^2 \alpha = \frac{9 \cdot 4\pi R^2 \alpha G M^2}{\pi^2 R^6} = 36\alpha \frac{GM^2}{\pi R^4}$$

$$36\alpha \frac{GM^2}{\pi R^4} > \frac{GM^2}{8\pi R^4}$$

$36\alpha > \frac{1}{8}$ ✓ Smo preverili da neenakost velja

2. [Računanje Hala]

ρ_c, P_c $$P_c < (4\pi)^{1/3} \cdot 0{,}347\, G M^{2/3} \rho_c^{4/3}$$

Ocenimo:

$P_c' > P_c$ $$\frac{dP}{dm} = \frac{Gm}{4\pi r^4}$$

Neka "največja" možna gostota, ki bi jo lahko imela ← pri $\rho = \rho_c = \text{konst.}$

$$P_c' = \int_0^M \frac{Gm\,dm}{4\pi r^4} = \int_0^R \frac{G\rho_c \frac{4}{3}\pi r^3 4\pi r^2 \rho_c\, dr}{4\pi r^4} = \frac{G\rho_c^2 4\pi}{3}\int_0^R r\,dr$$

$$\Rightarrow P_c' = \frac{4\pi G \rho_c^2 R^2}{6}$$

Povežemo še: $R^2 = \left(\frac{3M}{\rho 4\pi}\right)^{2/3}$

$$\Rightarrow P_c' = \frac{4\pi G \rho_c^2}{6}\left(\frac{3M}{4\pi\rho}\right)^{2/3} = (4\pi)^{1/3} G M^{2/3} \rho_c^{4/3} \underbrace{3^{2/3}\frac{1}{6}}_{0{,}347}$$

3. [Povprečne mase]

$$\frac{1}{\mu_{ion}} = \sum_i \frac{X_i}{A_i}$$

$$\frac{1}{\mu_e} = \sum_i \frac{Z_i X_i}{A_i}$$

$$\frac{1}{\mu} = \frac{1}{\mu_{ion}} + \frac{1}{\mu_e} = \frac{X}{1} + \frac{1X}{1} + \frac{Y}{4} + \frac{2Y}{4} + \underbrace{\sum_i \frac{X_i}{A_i} + \sum_i \frac{Z_i X_i}{A_i}}$$

$$\frac{1}{\mu} = 2X + \frac{3Y}{4} + \frac{1}{2}Z \qquad \Leftarrow \qquad \sum_i \frac{X_i(1+Z_i)}{A_i} = \frac{1}{2}\sum_i X_i = \frac{1}{2}Z$$

Za popolnonoma ioniziran plin.

Za popolnoma neioniziran plin: $\frac{1}{\mu} = \sum_i \frac{(1+Z_i)X_i}{A_i}$

$$\Rightarrow \quad \frac{1}{\mu} = X + \frac{Y}{4} + \underbrace{\left\langle \frac{1}{A_i} \right\rangle}_{1/15{,}5 \text{ za Sonce}} Z$$

4. [Izpelji W_{tot} iz W_g]

$\beta = \frac{P_{gas}}{P}$ uniformen po radiju

$\beta(r) = \beta$

$$W_n = \int (w_{n,gas} + w_{n,rad})\, dm$$

gostota energije na enoto mase

Iz $-3\int_0^M \frac{P}{\rho}\, dm = W_g$ izpelji

$W_{tot} = \frac{\beta}{2} W_g = -\frac{\beta}{2-\beta} W_n$ za idealni plin.

$P_{gas} = \beta P$

$P_{rad} = (1-\beta)P$

$$w_{gas} = \frac{3}{2}\frac{P_{gas}}{\rho} = \frac{3}{2}\frac{\beta P}{\rho}$$

$$w_{rad} = 3\frac{P_{rad}}{\rho} = 3\frac{(1-\beta)P}{\rho}$$

To seštejemo:

$$w_{gas} + w_{rad} = \frac{3}{2}\beta\frac{P}{\rho} + \frac{3P}{\rho} - \frac{3\beta P}{\rho} =$$

$$= \frac{P}{\rho}\left(3 - \frac{3}{2}\beta\right) = \frac{3}{2}\frac{P}{\rho}(2-\beta)$$

Lahko sedaj izrazimo P/ρ z specifičnimi energijami:

$$\frac{P}{\rho} = \frac{2}{3}\frac{1}{(2-\beta)}\left(w_{gas} + w_{rad}\right)$$

Tako je:

$$-3\int_0^M \frac{P}{\rho}\,dm = -\cancel{3}\int_0^M \frac{2}{\cancel{3}}\frac{1}{(2-\beta)}\left(w_{gas} + w_{rad}\right)dm = W_g$$

$$\Rightarrow \quad -\frac{2}{2-\beta}W_n = W_g$$

$$W_{tot} = W_n + W_g = \frac{2-\beta-2}{2-\beta} = -\underline{\underline{\frac{\beta}{2-\beta}W_n}} = -\frac{2-\beta}{2}W_g + W_g = \underline{\underline{\frac{\beta}{2}W_g}}$$

$\beta \to 0 \Rightarrow W_{tot} \to 0$

$\beta \to 1 \Rightarrow W_{tot} = \frac{1}{2}W_g$ kar je pa virialni teorem za idealen plin.

5. [Hitrost krčenja zvezde, ki nima več jedrskih reakcij]

Brez jedrskih reakcij ampak ohranja izsev L.

M, R

$\frac{dR}{dt} = ?$

$\tau = ?$

τ čas ko se zvezda skrči na polovični radij $R/2$

$$\frac{dW_{tot}}{dt} = -L$$

$$W_{tot} = \frac{1}{2}W_g = \left(\frac{\beta}{2}W_g\right)^{DN}$$

Recimo da velja Virialni teorem oz. pišemo [illegible]

$$W_g = -\alpha\frac{GM^2}{R}$$

Torej:

$$W_{tot} = -\frac{\alpha}{2}\frac{GM^2}{R} \quad ; \quad M = konst.$$

Odvajamo:

$$\frac{dW_{tot}}{dt} = -\frac{\alpha}{2} GM^2 \frac{d}{dt}\left(\frac{1}{R}\right) = -L$$

$$\int_R^{R/2} d\left(-\frac{GM^2\alpha}{2R}\right) = \int_0^{\tau} -L\, dt$$

$$-\frac{\alpha}{2} GM^2\left(\frac{2}{R} - \frac{1}{R}\right) = -L\tau \quad \Rightarrow \quad \frac{\alpha}{2}\frac{GM^2}{R} = L\tau$$

$$\Rightarrow \quad \underline{\tau = \frac{\alpha}{2}\frac{GM^2}{LR}}$$

Za poljuben $R' < R$:

$$\frac{1}{R'} - \frac{1}{R} = \frac{2L}{\alpha GM^2} t$$

$$\frac{R}{R'} - 1 = \frac{2RL}{\alpha GM^2} t \quad \Rightarrow \quad \frac{R}{R'} = \frac{t}{\tau} + 1$$

$$R' = \frac{R}{1 + t/\tau}$$

To še odvajamo in dobimo hitrost:

$$\frac{dR'}{dt} = -\frac{R/\tau}{(1+t/\tau)^2} \qquad \text{če } t \gg \tau \quad \frac{dR'}{dt} \to -\frac{R\tau}{t^2}$$

Za sonce:

$$\tau_\odot = \alpha \cdot 1{,}5 \cdot 10^7 \text{ let}$$

To je ravno Kelvin-Helmholtzova skala oz. termični relaksacijski čas

5. [Rešitev Lane-Emdenove enačbe pri $n=0$]

$n=0$

$\xi_1 = ?$

$M(R) = ?$

$\rho = \rho_c \theta^n$

$r = \alpha \xi$

Lane-Emden:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -1 \quad / \cdot \xi^2$$

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\xi^2 \quad / \cdot \int$$

$$\int \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = -\int \xi^2 d\xi$$

$$\xi^2 \frac{d\theta}{d\xi} = -\frac{\xi^3}{3} + C \quad / : \xi^2$$

$$\Rightarrow \frac{d\theta}{d\xi} = -\frac{\xi}{3} + \frac{C}{\xi^2} \quad / \cdot \int$$

$$\int d\theta = -\int \left(\frac{\xi}{3} + \frac{C}{\xi^2}\right) d\xi \Rightarrow \theta(\xi) = -\frac{1}{6}\xi^2 + C\left(-\frac{1}{\xi}\right) + D$$

Robni pogoji:

$\xi = 0 \quad \theta = 1$

$\frac{d\theta}{d\xi} = 0$

$\Rightarrow$ $C = 0$ (sicer divergira), $D = 1$ $\Rightarrow \underline{\underline{\theta(\xi) = -\frac{1}{6}\xi^2 + 1}}$

$$\theta(\xi_1) = 0 \Rightarrow \frac{1}{6}\xi_1^2 = 1 \Rightarrow \underline{\underline{\xi_1 = \sqrt{6}}}$$

$$M = -4\pi\alpha^3 \rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$$

$$\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} = \left(-\frac{1}{3}\xi\right)_{\xi=\xi_1} = -\frac{\sqrt{6}}{3}$$

$$\Rightarrow M(R) = -4\pi\left(\frac{R}{\xi_1}\right)^3 \rho_c \xi_1^2 \left(-\frac{\sqrt{6}}{3}\right) = \underline{\underline{\frac{4}{3}\pi R^3 \rho_c}}$$

Smiselno, ker smo rekli $\rho = \rho_c$ konst.

6. [Rešitev Lane-Emdenove enačbe za $n=1$]

$n=1$

$\xi_1 = ?$

$M(R) = ?$

$$\frac{1}{\xi^2}\underbrace{\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right)} = -\theta \quad (*)$$

Uvedemo novo spremenljivko:

$$\chi = \xi\theta$$

$$\frac{d\chi}{d\xi} = \theta + \xi\frac{d\theta}{d\xi}$$

$$\frac{d}{d\xi}\left(\frac{d\chi}{d\xi}\right) = \frac{d\theta}{d\xi} + \frac{d\theta}{d\xi} + \xi\frac{d^2\theta}{d\xi^2}$$

$$= 2\frac{d\theta}{d\xi} + \xi\frac{d^2\theta}{d\xi^2}$$

$$\Rightarrow \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = 2\xi\frac{d\theta}{d\xi} + \xi^2\frac{d^2\theta}{d\xi^2} =$$

$$= \xi\left(2\frac{d\theta}{d\xi} + \xi\frac{d^2\theta}{d\xi^2}\right) = \xi\frac{d^2\chi}{d\xi^2}$$

$$\Rightarrow (*) \quad \frac{d^2\chi}{d\xi^2} = -\xi\theta = -\chi$$

$$\Rightarrow \chi = C\sin(\xi - D)$$

$$\theta = \frac{\chi}{\xi} = \frac{C\sin(\xi-D)}{\xi}$$

Robni pogoji:

$\xi = 0 \to \theta = 1$, $\frac{d\theta}{d\xi} = 0$ $\Rightarrow$ $D = 0$, $C = 1$ (iz limite)

$$\Rightarrow \theta(\xi) = \frac{\sin\xi}{\xi} \qquad \theta(\xi_1) = 0 \Rightarrow \underline{\underline{\xi_1 = \pi}}$$

$$M = -4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} =$$

$$= -4\pi\left(\frac{R}{\xi_1}\right)^3\rho_c\xi_1^2\left(\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right)_{\xi=\xi_1} = -4\pi\frac{R^3}{\xi_1^3}\rho_c\xi_1^2\left(-\frac{1}{\pi}\right) =$$

$$= \underline{\underline{\frac{4R^3}{\pi}\rho_c}}$$

Poigramo se še z:

$$\rho_c = D_n\bar{\rho} \qquad \rho_c \to \frac{M\pi}{4R^3} = D_n\frac{M\cdot 3}{4\pi R^3}$$

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} \qquad \Rightarrow D_1 = \frac{\pi^2}{3}$$

7. [Alternativen zapis enačbe hidrostatičnega ravnovesja]

$$\frac{dP}{d\tau} = \frac{g}{\varkappa}$$

g → Lokalna grav.

$\varkappa$ → Koef. neprozornosti

τ ↑ Optična globina

$$\frac{dP}{dr} = -G\frac{\rho m}{r^2} \qquad \tau = -\varkappa\rho r$$

$$g = \frac{Gm}{r^2}$$

$$\frac{dP}{dr} = -g\rho \quad / : (-\varkappa\rho)$$

$$\frac{dP}{-\varkappa\rho\, dr} = \frac{g}{\varkappa} \Rightarrow \frac{dP}{d\tau} = \frac{g}{\varkappa}$$

$\varkappa_R = \text{konst.}$

$$\int \varkappa_R dP = \int g\, d\tau = g \underbrace{\int_\infty^R d\tau}_{\substack{\text{Prozorno do zvezde} \\ =1}} = g$$

$$\varkappa_R P_R = g$$

V fotosferi bo potem tlak zvezde: $P_R = \dfrac{GM}{R^2\varkappa_R}$ $\qquad P_c > \dfrac{GM^2}{8\pi R^4}$

$$\frac{P_c}{P_R} > \frac{GM^2}{8\pi R^4}\frac{R^2\varkappa_R}{GM} = \frac{M\varkappa_R}{8\pi R^2} \sim 10^{11} - 10^{12} \text{ (Za sonce)}$$

8. [Najmanjša masa zvezde za zlitje jeder]

$\mu = 0{,}61$
$\mu_e = 1{,}17$

$$\rho(r) = \rho_c\left(1-(r/R)^2\right)$$

$$m(r) = \int_0^r 4\pi r^2 \rho(r)\,dr = 4\pi\rho_c \int_0^r r^2\left(1-\frac{r^2}{R^2}\right)dr = 4\pi\rho_c\left[\frac{r^3}{3}-\frac{r^5}{5R^2}\right]$$

Pri $r = R \rightarrow m(R) = M$

$$M = 4\pi\rho_c R^3\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{4\pi\rho_c R^3}{3}\,\frac{2}{5} \Rightarrow \rho_c = \frac{15}{8\pi}\frac{M}{R^3}$$

Za tlak uporabimo enačbo hidrostatičnega ravnovesja:

$$\frac{dp}{dr} = -\frac{Gm\rho}{r^2}$$

$$\Rightarrow \int_{p_c}^{0} dp = -G\int_0^R \frac{m(r)\rho(r)}{r^2}\,dr$$

$$p_c = G\int_0^R \frac{m(r)\rho(r)}{r^2}\,dr = 4\pi\rho_c^2 G\int_0^R \frac{r^3}{r^2}\left[\frac{1}{3}-\frac{r^2}{5R^2}\right]\left[1-\frac{r^2}{R^2}\right]dr =$$

$$= \frac{4\pi G\rho_c^2}{15R^4}\int_0^R r\left[5R^2-3r^2\right]\left[R^2-r^2\right]dr =$$

$$= \frac{4\pi G\rho_c^2}{15R^4}\int_0^R\Big[5R^4 r \underbrace{-5R^2r^3-3r^3R^2}_{-8R^2r^3}+3r^5\Big]dr =$$

$$= \frac{4\pi G\rho_c^2}{15R^4}\left[\frac{5R^6}{2}-\frac{8R^6}{4}+\frac{3R^6}{6}\right] = \frac{4\pi G\rho_c^2}{15}R^2\left[\frac{5}{2}-2+\frac{1}{2}\right] =$$

$$= \frac{4\pi G\rho_c^2}{15}R^2$$

$$\Rightarrow p_c = \frac{15G}{16\pi}\frac{M^2}{R^4}$$

Ker je plin nedegeneriran uporabimo enačbo idealnega plina:

$$P_c = \frac{\rho_c}{\mu m_H} k T_c$$

$$\Rightarrow \quad T_c = \frac{GM}{2R} \frac{\mu m_H}{k} \quad \rightarrow R = \frac{GM \mu m_H}{2 k T_c}$$

Upoštevamo še

$$P_{gas} > P_{e,deg}$$

$$\downarrow$$

$$P_{ion} + P_e > P_{e,deg}$$

$$P_e > P_{e,deg}$$

$$\frac{\rho_c k T_c}{\mu_e m_H} > K \left(\frac{\rho_c}{\mu_e}\right)^{5/3} \Big/ : \left(\frac{\rho_c}{\mu_e}\right)$$

$$\frac{kT}{m_H} > K \left(\frac{\rho_c}{\mu_e}\right)^{2/3} \quad \rightarrow \cancel{T_c > K \left(\frac{\rho_c}{\mu_e}\right)^{2/3} \cdot \frac{m_H}{k}}$$

Vstavimo ρ_c

$$\frac{kT_c}{m_H} > K \left(\frac{15}{8\pi}\right)^{2/3} \frac{M^{2/3}}{R^2} \cdot \left(\frac{1}{\mu_e}\right)^{2/3}$$

Vstavimo R

$$\frac{kT_c}{m_H} > K \left(\frac{15}{8\pi}\right)^{2/3} M^{2/3} \frac{(2k)^2 T_c^2}{G^2 M^2 \mu^2 m_H^2} \left(\frac{1}{\mu_e}\right)^{2/3}$$

$$\Rightarrow \quad T_c < \frac{G M^2 \mu^2 m_H \mu_e^{2/3} k}{M^{2/3} \, 4 k^2 \, K} \left(\frac{8\pi}{15}\right)^{2/3} \quad ; \; K = 10^7 \, m^4 \, kg^{-2/3} \, s^{-2}$$

$$T_c < A M^{4/3} \quad \Rightarrow \quad M > \left(\frac{T_c}{A}\right)^{3/4}$$

9. [Konvekcija v zvezdi z uniformno neprozornostjo]

- $\varkappa$ neprozornost uniformna
- β uniformen
- jedro zvezde je konvektivno
- jedrske reakcije so edini vir energije in potekajo v središču/jedru

$$\Downarrow$$

$$L(r) = L(R)$$

masa jedra

$$\frac{M_c^K}{M} = \frac{\gamma_a}{4(\gamma_a - 1)}$$

masa zvezde

$$\frac{dT}{dr} = \left(\frac{\gamma_a - 1}{\gamma_a}\right) \frac{T}{P} \left(\frac{dP}{dr}\right)_{jedro}$$; konvektivno jedro

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\varkappa \rho}{T^3} \left(\frac{L(r)}{4\pi r^2}\right)_{ovojnica}$$

$$\frac{P_{rad}}{P} = 1 - \beta$$

Želimo, da je na stičišču zvezno:

$$\left(\frac{\gamma_a - 1}{\gamma_a}\right) \frac{T}{P} \left(\frac{dP}{dr}\right)_{jedro} = -\frac{3}{4ac} \frac{\varkappa \rho}{T^3} \left(\frac{L(r)}{4\pi r^2}\right)_{ovojnica}$$

$$+\left(\frac{\gamma_a - 1}{\gamma_a}\right) \frac{T}{P} \frac{G M_c \cancel{\rho}}{\cancel{r^2}} = +\frac{3}{4ac} \frac{\varkappa \cancel{\rho}}{T^3} \frac{L(r)}{4\pi \cancel{r^2}}$$

$$\Rightarrow \quad M_c = \frac{3\varkappa}{16\pi a c} \frac{L(R)}{T^4} \frac{P}{G} \left(\frac{\gamma_a}{\gamma_a - 1}\right) =$$

P_{rad}

$$M_c = \frac{\varkappa}{16\pi c} \frac{L}{(1-\beta)} \frac{1}{G} \left(\frac{\gamma_a}{\gamma_a - 1}\right) \Big/ : M$$

$$\frac{M_c}{M} = \frac{L}{\underbrace{\frac{4\pi c G M}{\varkappa}}_{L_{Edd}}} \frac{1}{4(1-\beta)} \left(\frac{\gamma_a}{\gamma_a - 1}\right)$$

$$\frac{L}{L_{Edd}} = (1-\beta)$$

$$\Rightarrow \quad \frac{M_c}{M} = \frac{\gamma_a}{4(\gamma_a - 1)}$$

10. [Kritična temperatura za dinamično stabilnost delno ioniziranega plina]

$x = \frac{n_+}{n_0 + n_+}$ stopnja ionizacije $\quad \gamma_a(x,T) = \frac{5 + A^2 x(1-x)}{3 + Bx(1-x)}$; $A = \frac{5}{2} + \frac{\chi}{kT}$

$B = \frac{3}{2} + \left(\frac{\chi}{kT} + \frac{3}{2}\right)^2$

χ ionizacijski potencial

Preko odvoda $\gamma_a(x,T)$ bi lahko ugotovili, da ima funkcija minimum pri $x = 0,5$.

$$\gamma_a(0{,}5, T) = \frac{5 + A^2 \cdot \frac{1}{4}}{3 + B\frac{1}{4}} = \frac{4}{3}$$

Pogoj ravnovesja / stabilnosti (ravno meja)

$$A = \frac{5}{2} + \frac{\chi}{kT} = \frac{5}{2} + u$$

$$B = \frac{3}{2} + \left(u + \frac{3}{2}\right)^2.$$

$$\Rightarrow \gamma_a = \frac{5 + \frac{1}{4}\left(\frac{25}{4} + 5u + u^2\right) /\cdot 4}{3 + \frac{1}{4}\left(\frac{3}{2} + u^2 + \frac{9}{4} + 3u\right) /\cdot 4} = \frac{20 + \left(\frac{25}{4} + 5u + u^2\right)}{12 + \frac{15}{4} + 3u + u^2} = \frac{4}{3}$$

$$\Rightarrow 20 + \frac{25}{4} + 5u + u^2 = \frac{4}{3}\left[12 + \frac{15}{4} + 3u + u^2\right] /:4$$

$$5 + \frac{25}{16} + \frac{5}{4}u + \frac{u^2}{4} = \frac{1}{3}[-11-] /\cdot 3$$

$$\frac{15/\cdot 4}{4/\cdot 4} = \frac{60}{16} - \frac{75}{16} = -\frac{15}{16}$$

$$\cancel{15}^{3} + \frac{75}{16} + \frac{15}{4}u + \frac{3u^2}{4} = \cancel{12} + \frac{15}{4} + 3u + u^2$$

$$\frac{3/\cdot 2}{2/\cdot 2} = \frac{6}{4} + \frac{9}{4} = \frac{15}{4}$$

$$\frac{30}{8} - \frac{75}{8}$$

$$\left(1 - \frac{3}{4}\right)u^2 + \left(3 - \frac{15}{4}\right)u + \frac{15/\cdot 2}{4/\cdot 2} - \frac{75}{16} = 0$$

$$\Rightarrow \frac{u^2}{4} - \frac{3}{4}u - \frac{15}{16} = 0$$

$$\frac{u^2}{4} - \frac{3}{4}u - \frac{45}{8} = 0 \;/\cdot 4$$

$$4u^2 - 12u - 15 = 0$$

$$u^2 - 3u - \frac{45}{2} = 0 \;/\cdot 2$$

Nekje zafrknil moralo bi biti: $4u^2 - 12u - 63 = 0$

$$2u^2 - 6u - 45 = 0$$

$u \to T = 2{,}75 \cdot 10^4\,K \quad \chi = 13{,}6\,eV$

[Kako daleč bi lahko videli če bi bila neprozornost zemljine atmosfere 11. tolikšna kot od sonca]

$\rho = 1{,}2\ kg/m^3$ $\tau_\lambda = \frac{2}{3}$

$\lambda = 500nm$

$\varkappa_{\lambda=500} = 0{,}03\ \frac{m^2}{kg}$

$$l = \frac{1}{\varkappa_{500}\rho} = \frac{1}{0{,}03\frac{m^2}{kg}\cdot 1{,}2\ kg/m^3} = \underline{\underline{27{,}8m}}$$

$$s = \tau_\lambda l = \frac{2}{3} l = \underline{\underline{18{,}5m}}$$

12. [Model sonca]

$\rho_{\odot,c} = 1{,}53\cdot 10^5\ kg/m^3$

$\bar{\varkappa} = 0{,}217\ \frac{m^2}{kg}$

a) $l = ?$

b) $t = ?$

da zapusti sonce

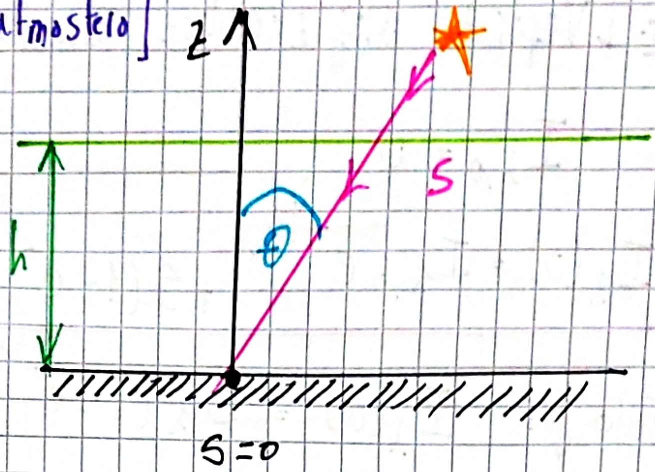
a) $$l = \frac{1}{\bar{\varkappa}\rho_{\odot,c}} = \frac{1}{0{,}217\ \frac{m^2}{kg}\cdot 1{,}53\cdot 10^5\ \frac{kg}{m^3\ n}} = \simeq \underline{\underline{3\cdot 10^{-5}m}}$$

b) $t = ?$ $d^2 = Nl^2$ $N = \frac{R_\odot^2}{l^2}$

$$\Rightarrow \quad t = \frac{Nl}{c} = \frac{R_\odot^2}{l^2}\frac{l}{c} = \frac{R_\odot^2}{lc} =$$

$$= \frac{(7\cdot 10^8 m)^2}{3\cdot 10^{-5}m\cdot 3\cdot 10^8\ m/s} \cong 4\cdot 10^{13}s \approx \underline{\underline{10^6\ let}}$$

13. [Specifična intenziteta nad zemljino atmosfero]



Naredimo dve meritvi

$$\left.\begin{array}{l} 1.\ I_{\lambda,1}\ ,\ \theta_1 \\ 2:\ I_{\lambda,2}\ ,\ \theta_2 \end{array}\right\}\ I_{\lambda,0}\ ?$$

$$ds = -\frac{dz}{\cos\theta}$$

$$\tau_\lambda = \int_0^s \kappa_\lambda \rho\, ds = -\int_h^0 \kappa_\lambda \rho \frac{dz}{\cos\theta} = \int_0^h \kappa_\lambda \rho \frac{dz}{\cos\theta} = \frac{1}{\cos\theta}\int_0^h \kappa_\lambda \rho\, dz$$

$$= \frac{\tau_v}{\cos\theta}$$

$$I_\lambda = I_{\lambda,0}\, e^{-\tau_\lambda} = I_{\lambda,0}\, e^{-\frac{\tau_v}{\cos\theta}}$$

$$\left.\begin{array}{l} \dfrac{I_{\lambda,1}}{I_{\lambda,0}}\ e^{-\frac{\tau_v}{\cos\theta_1}} \\[2ex] \dfrac{I_{\lambda,2}}{I_{\lambda,0}} = e^{-\frac{\tau_v}{\cos\theta_2}} \end{array}\right\}\ \tau_v = -\frac{\ln(I_{\lambda,1}/I_{\lambda,2})}{\left(\frac{1}{\cos\theta_1} - \frac{1}{\cos\theta_2}\right)}$$

$$\Rightarrow I_{\lambda,0} = I_{\lambda,1}\, e^{\frac{\tau_v}{\cos\theta}} = I_{\lambda,1}\, e^{\left(-\frac{\ln(I_{\lambda,1}/I_{\lambda,2})}{\cos^{-1}\theta_1 - \cos^{2}\theta_2}\right)}$$

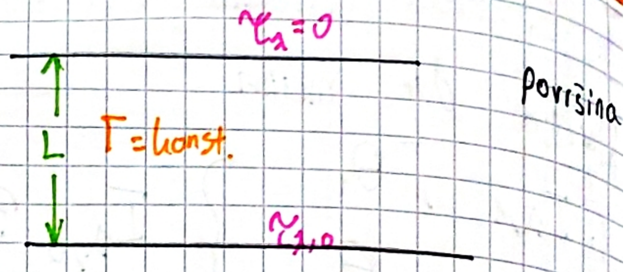
$$\Rightarrow -\tau_v = \ln(I_{\lambda,1}/I_{\lambda,0})\cdot\cos\theta_1 = \ln(I_{\lambda,2}/I_{\lambda,0})\cdot\cos\theta_2$$

In končno:

$$I_{\lambda,0} = \left(\frac{I_{\lambda,2}^{\frac{1}{\cos\theta_1}}}{I_{\lambda,1}^{\frac{1}{\cos\theta_2}}}\right)^{\frac{1}{\left(\frac{1}{\cos\theta} - \frac{1}{\cos\theta}\right)}}$$

$$\Rightarrow \ln I_\lambda = \ln I_{\lambda,0} - \frac{\tau_0}{\cos\theta}$$

14. [Vzporeden sloj zvezdine atmosfere]



$$I_{\lambda,0} = 0$$

$$I_\lambda(0) = \cancel{I_{\lambda,0}} e^{-\tau_{\lambda,0}} + S(1 - e^{-\tau_{\lambda,0}})$$

Tako je: $$I_\lambda(0) = S_\lambda(1 - e^{-\tau_{\lambda,0}})$$

- $\tau_{\lambda,0} \gg 1$ Predpostavimo TD ravnovesje: $S_\lambda = B_\lambda$

$$I_\lambda(0) = B_\lambda$$ -> Vidimo da zgornji plast seva kot črno telo

- $\tau_{\lambda,0} \ll 1$

$$e^{-\tau_{\lambda,0}} \simeq 1 - \tau_{\lambda,0}$$

$$I_\lambda(0) = S_\lambda(1 - 1 + \tau_{\lambda,0}) = S_\lambda \tau_{\lambda,0} = \star$$

$$\tau_{\lambda,0} = \kappa_\lambda \rho L$$

$$S_\lambda = \frac{j_\lambda}{\kappa_\lambda}$$

$$\star = \frac{j_\lambda}{\kappa_\lambda} \kappa_\lambda \rho L = j_\lambda \rho L$$

15. [Kot 14. samo $I_{\lambda,0} \neq 0$]

Naša enačba je torej:

$$I_\lambda(0) = I_{\lambda,0} e^{-\tau_{\lambda,0}} + S_\lambda(1 - e^{-\tau_{\lambda,0}})$$

- $\tau_{\lambda,0} \gg 1$

TDR predpostavimo

$$I_\lambda(0) = S_\lambda = B_\lambda$$ -> Zopet zgornja plast črno telo

- $\tau_{\lambda,0} \ll 1$: $e^{-\tau_{\lambda,0}} \simeq 1 - \tau_{\lambda,0}$

$$I_\lambda(0) = I_{\lambda,0}(1-\tau_{\lambda,0}) + S_\lambda \tau_{\lambda,0}$$

$$= I_{\lambda,0} - \tau_{\lambda,0}(I_{\lambda,0} - S_\lambda)$$

$$= I_{\lambda,0} - \kappa_\lambda \rho L (I_{\lambda,0} - S_\lambda)$$

a) $I_{\lambda,0} > S_\lambda$

$I_\lambda(0) < I_{\lambda,0}$ $\Rightarrow$ imamo absorpcijske črte (kont. + abs. črte)

b) $I_{\lambda,0} < S_\lambda$

$I_\lambda(0) > I_{\lambda,0}$ $\Rightarrow$ imamo emisijske črte (kont. + em. črte)

16. [Uporaba Eddingtonove aproksimacije]

$I_{in}(\tau_\nu) = ?$

$I_{out}(\tau_\nu) = ?$

$$\langle I \rangle = \frac{1}{2}(I_{in} + I_{out})$$

$$F_{rad} = \pi (I_{out} - I_{in}) = \sigma T_{eff}^4$$

$$P_{rad} = \frac{4\pi}{3c}\langle I \rangle$$

vertikalna optička dubina

$$\langle I \rangle = \frac{3\sigma}{4\pi} T_{eff}^4 \left(\tau_\nu + \frac{2}{3}\right)$$

$$\frac{1}{2}(I_{in} + I_{out}) = \frac{3\sigma}{4\pi} T_{eff}^4 \left(\tau_\nu + \frac{2}{3}\right)$$

$$\Rightarrow I_{in} + I_{out} = \frac{3\sigma}{2\pi} T_{eff}^4 \left(\tau_\nu + \frac{2}{3}\right) \quad (A)$$

$$\Rightarrow I_{out} - I_{in} = \frac{\sigma T^4}{\pi} \quad (B)$$

$$(A) + (B): \quad I_{out} = \frac{\sigma T_{eff}^4}{2\pi}\left(1 + \frac{3}{2}\tau_\nu + 1\right)$$

$$(A) - (B): \quad I_{in} = \frac{\sigma T_{eff}^4}{2\pi} \frac{3}{2}\tau_\nu$$

Kje se razlikuje za 1 odstotek od izotropnega?

$$\frac{|I_{out} - I_{in}|}{\langle I \rangle} = 0.01$$

$$\frac{\frac{\cancel{\sigma T_{eff}^4}}{\cancel{\pi}}}{\frac{\cancel{\sigma T_{eff}^4}}{4\cancel{\pi}}(2+3\tau_\nu)} = 0{,}01$$

$$\frac{4}{0{,}01} = 2 + 3\tau_\nu$$

$$\tau_\nu = \frac{4}{0{,}03} - \frac{2}{3} = \frac{398}{3} \simeq \underline{\underline{133}}$$

[Ponovitev]

Boltzmannova enačba:

$E_a = -13,6\,eV$

$S_a = \{n=1, l=0, m_l=0, m_s=+\frac{1}{2}\}$

Drugo stanje: S_b, E_b

$$\frac{P(S_b)}{P(S_a)} = \frac{g_b e^{-E_b/kT}}{g_a e^{-E_a/kT}} = \frac{g_b}{g_a} e^{-\frac{(E_b-E_a)}{kT}} = \frac{N_b}{N_a}$$

g ... statistične uteži (deg. stanja)

Sahova enačba:

particijska funkcija

$$Z = \sum_{j=1}^{\infty} g_j e^{-(E_j - E_1)/kT}$$

j = nivoji

$$\frac{N_{i+1}}{N_i} = \frac{2 Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT} =$$

$P_e = n_e kT$

$$= \frac{2kT}{P_e}\frac{Z_{i+1}}{Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/kT}$$

17. [Particijska funkcija vodika]

$E_1 = -13,6\,eV \qquad E_n = \frac{E_1}{n^2} \qquad g_n = 2n^2 \qquad T = 10\,000\,K$

$n=1 \quad E_1 = -13,6\,eV \qquad g_1 = 2$

$n=2 \quad E_2 = -3,4\,eV \qquad g_2 = 8$

$n=3 \quad E_3 = -1,5\,eV \qquad g_3 = 18$

Tako je

$$Z = g_1 + g_2 e^{-(E_2-E_1)/kT} + g_3 e^{-(E_3-E_1)/kT} =$$

$$= 2 + 5,8\cdot 10^{-5} + 1,45\cdot 10^{-5} \simeq 2$$

18. [Ocena moči absorpcijskih črt]

Sončeva atmosfera

$T = 5770\,K$

Vemo: 500 000 vodika : 1 kalcij

Zanima nas relativna moč črt.

a) $$\left.\frac{N_2}{N_{tot}}\right|_{vodik} = \left(\frac{N_2}{N_1+N_2}\right)\left(\frac{N_I}{N_{tot}}\right)$$

Saha $$\frac{N_{II}}{N_I} = \frac{2kT}{P_e}\frac{Z_{II}}{Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT} = \frac{1}{13400} \simeq 7,1\cdot 10^{-5}$$

Boltzman

$$\left.\frac{N_2}{N_1}\right|_{HI} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT} =$$

$$= \frac{2\cdot 2^2}{2} \exp\left(\left(-\frac{13{,}6eV}{4} + 13{,}6eV\right)/kT\right) =$$

$$= 4{,}96\cdot 10^{-9} = \frac{1}{202\,000\,000}$$

Torej:
$$\frac{N_2}{N_{tot}} = \frac{N_2}{N_1+N_2}\left(\frac{N_I}{N_{tot}}\right) = \left(\frac{N_2/N_1}{1+N_2/N_1}\right)\left(\frac{N_I}{N_I+N_{II}}\right) =$$

$$= \left(\frac{N_2/N_1}{1+N_2/N_1}\right)\cdot\left(\frac{1}{1+N_{II}/N_I}\right) \simeq \underline{\underline{4{,}9\cdot 10^{-9}}}$$

b) $\chi_I = 6{,}11eV \quad Z_I = 1{,}32 \quad Z_{II} = 2{,}3$

$$\left.\frac{N_{II}}{N_I}\right|_{Ca} = \frac{2kT Z_{II}}{P_e \frac{1}{Z_1}}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT} = \underline{\underline{903}}$$

c) $$\left.\frac{N_2}{N_1}\right|_{CaII} = \frac{g_2}{g_1} e^{-(E_2-E_1)/kT} = 3{,}7\cdot 10^{-3} = \frac{1}{265}$$

$$\left.\frac{N_1}{N_{tot}}\right|_{CaII} \cong \left(\frac{N_1}{N_1+N_2}\right)_{CaII}\left(\frac{N_{II}}{N_{tot}}\right)_{CaII} = \left(\frac{1}{1+N_2/N_1}\right)_{CaII}\cdot\left(\frac{N_{II}/N_I}{1+N_{II}/N_I}\right) =$$

$$= 0{,}995$$

Vodik: $500 \cdot 4{,}9 \cdot 10^{-9} \simeq \frac{1}{400}$

Torej so lakejeve črte 400x močnejše od Balmerjevih črt.

11. [Schönberg-Chandrasekharjeva relacija]

Zvezda z maso M in radijem R. Jedro zvezde ima maso M_1 in radij R_1.

Porazdelitev gostote:

$$\rho(r) = \rho_c - (\rho_c - \rho_1)\left(\frac{r}{R_1}\right)^2 \quad \text{v jedru} \quad 0 \leq r \leq R_1$$

$$\rho(r) = \rho_1 \frac{(R_1/r)^3 - (R_1/R)^3}{1 - \left(\frac{R_1}{R}\right)^3} \quad \text{v ovojnici} \quad R_1 \leq r \leq R$$

$$\rho_1 = \rho(R_1)$$

a) izračunaj $\frac{R}{R_1}$, ki bo odvisno od $x_1 = \frac{\rho_c}{\rho_1}$ in $y_1 = \frac{M}{M_1}$

b) izračunaj razmerje R/R_1 za $x_1 = 10$ in $y_1 = 7{,}5$ z uporabo SC relacije $\left(\frac{M_c}{M} \lesssim 0{,}37\left(\frac{\mu_{env}}{\mu_c}\right)^2\right)$

Poiščimo prvo maso jedra M_1:

$$M_1 = \int_0^{R_1} 4\pi r^2 \rho(r)\,dr = \int 4\pi r^2 \left[\rho_c - (\rho_c - \rho_1)\left(\frac{r^2}{R_1^2}\right)\right] dr =$$

$$= \frac{4\pi R_1^3 \rho_c}{3} - 4\pi(\rho - \rho_1)\frac{R_1^5}{5R_1^2} =$$

$$= 4\pi R_1^3\left(\frac{\rho_c}{3} - \frac{\rho_c}{5} + \frac{\rho_1}{5}\right) =$$

$$= \frac{4\pi R_1^3}{5}\left(\frac{2}{3}\rho_c + \rho_1\right)$$

Poiščimo maso zvezde

$$M - M_1 = \int_{R_1}^{R} 4\pi r^2 \rho(r)\, dr = 4\pi \int_{R_1}^{R} r^2 \rho_1 \frac{\left(\frac{R_1}{r}\right)^3 - \left(\frac{R_1}{R}\right)^3}{1-\left(\frac{R_1}{R}\right)^3}\, dr =$$

$$= \frac{4\pi\rho_1}{1-(R_1/R)^3} \int \left[\left(\frac{R_1}{r}\right)^3 - \left(\frac{R_1}{R}\right)^3\right] r^2\, dr = \frac{4\pi\rho_1}{1-(R_1/R)^3} \left[R_1^3 \ln\left(\frac{R}{R_1}\right) - \frac{r^3}{3}\left(\frac{R_1}{R}\right)^3\right]_{R_1}^{R}$$

$$= \frac{4\pi\rho_1}{1-(R_1/R)^3}\left[R_1^3 \ln\left(\frac{R}{R_1}\right) - \frac{R^3}{3}\left(\frac{R_1}{R}\right)^3 + \frac{R_1^3}{3}\left(\frac{R_1}{R}\right)^3\right] =$$

$$= \frac{4\pi\rho_1}{1-(R_1/R)^3}\left[R_1^3 \ln\left(\frac{R}{R_1}\right) - \left(\frac{R_1}{R}\right)\left(\frac{R^3 - R_1^3}{3}\right)\right] =$$

$$= \frac{4\pi\rho_1}{\left(1-(R_1/R)^3\right)} R_1^3 \left[\ln\left(\frac{R}{R_1}\right) - \frac{1}{3}\left(1-\left(\frac{R_1}{R}\right)^3\right)\right] =$$

$$= 4\pi\rho_1 R_1^3 \left[\frac{\ln(R/R_1)}{1-(R_1/R)^3} - \frac{1}{3}\right]$$

$$\frac{M - M_1}{M_1} = \frac{M}{M_1} - 1 = \frac{4\pi\rho_1 R_1^3 \left[\frac{\ln(R/R_1)}{1-(R_1/R)^3} - \frac{1}{3}\right]}{\frac{4\pi R_1^3}{5}\left[\frac{2}{3}\rho_c + \rho_1\right]}$$

$$y_1 - 1 = \frac{\left[\frac{\ln(R/R_1)}{1-(R_1/R)^3} - \frac{1}{3}\right]}{\frac{1}{5}\left[\frac{2}{3}x_1 + 1\right]} \Rightarrow \frac{\ln(R/R_1)}{1-(R_1/R)^3} = (y_1 - 1)\frac{1}{5}\left(\frac{2}{3}x_1 + 1\right) + \frac{1}{3}$$

Velja $R_1 \ll R$. Ocenimo:

$$\frac{R}{R_1} \cong \exp\left[\frac{1}{5}(x_1-1)\left(\frac{2}{3}x_1+1\right)+\frac{1}{3}\right]$$

b) $x_1 = 10 \quad y_1 = 7{,}5$

$$\Rightarrow \frac{R}{R_1} \approx 3 \cdot 10^4$$

$R_1 \sim 0{,}01 R_\odot$ (~ kot Zemlja / tipična bela pritlikavka)

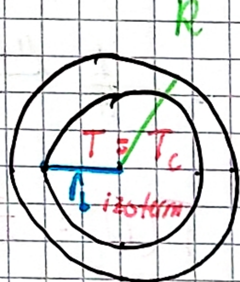
$R \sim 300 R_\odot$

$$\frac{1}{7{,}5} = 0{,}37 \cdot \frac{\mu_{env}^2}{1} \Rightarrow \mu_{env} = \sqrt{0{,}36} = 0{,}6$$

20. [Temperaturni profil bele pritlikavke]

M, R

$$T(r) = \frac{4}{17} \frac{\mu m_H}{k} GM \left(\frac{1}{r} - \frac{1}{R}\right)$$



$\underline{l = R - r_b^*}$

a) Izpelji enačbo:

- Idealni plin
- Hidrostatično ravnovesje

$$\frac{dp}{dr} = \frac{dp}{dT} \cdot \frac{dT}{dr} = -\rho \frac{GM}{r^2} \quad \text{(hidrostatično ravnovesje)}$$

$$p = \frac{\rho}{\mu m_H} kT \rightarrow \rho = \frac{p \mu m_H}{kT}$$

$$\Rightarrow \frac{dp}{dT}\frac{dT}{dr} = -\frac{p\mu m_H}{kT}\frac{GM}{r^2} \quad / \cdot \left(\frac{T}{p}\right)$$

$$\frac{dp}{p}\frac{T}{dT}\cdot\frac{dT}{dr} = -\frac{\mu m_H}{k}\frac{GM}{r^2}$$

$$\Rightarrow \frac{d\ln p}{d\ln T}\ ; \quad p(T) = \underbrace{\left(\frac{64\pi a c k G}{51 \varkappa_0 \mu m_H}\right)\left(\frac{M}{L}\right)^{1/2}}_{A} T^{17/4} =$$

$$d\ln P = \ln A + \frac{17}{4}\ln T$$

$$d(\ln p) = \frac{17}{4}\, d(\ln T) \quad \rightarrow \quad \frac{d\ln p}{d\ln T} = \frac{17}{4}$$

$$\frac{17}{4}\cdot\frac{dT}{dr} = -\frac{\mu m_H G M}{\varkappa_0 r^2}$$

$$\frac{dT}{dr} = -\frac{4}{17}\frac{\mu m_H GM}{k}\cdot\frac{1}{r^2} \quad / \cdot \int_r^R$$

$$\Rightarrow T(r) = \frac{4}{17}\frac{\mu m_H GM}{k}\left(\frac{1}{r} - \frac{1}{R}\right)$$

b) Pokaži, da je $l = R - r_b \ll 0$

$$\int_{T(r_b)}^{T(R)} dT = -\frac{4}{17}\frac{\mu m_H}{k} GM \int_{r_b}^{R}\frac{1}{r^2}\,dr$$

Robna pogoja:

$T(R) = 0$

$T(r_b) = T_c$

$$T_c = \frac{4}{17}\frac{\mu m_H}{k} GM\left(\frac{1}{r_b} - \frac{1}{R}\right)$$

$$\frac{kT_c}{\mu m_H} = \boxed{\frac{4}{17}\frac{GM}{R}\left(\frac{R-r_b}{r_b}\right)}$$

$-P/g$

$\frac{P_{ions}}{g}$

$P_i \ll 1$

$\frac{R-r_b}{r_b} \ll 1 \Rightarrow R-r_b \ll r_b < R$

$R - r_b \ll R$

$l \ll R$

$$\frac{kT_c}{\mu m_H} = \frac{4}{17}\frac{GM}{R}\left(\frac{R-r_b}{r_b}\right)$$

c) $\frac{l_1}{l_2}$ će izsev pade iz $L_1 = 10^{-2} M_\odot$ na $L_2 = 10^{-4} L_\odot$

$$\frac{L}{M} = \frac{64\pi\alpha c G M_1^3 \mu m_H}{51\, k^4\, \kappa_0\, \mu_e^2}$$

$l = R - r_b \ll R$

$r_b \approx R$

$T_c \propto l$

$L \propto T^{7/2}$

$L \propto l^{7/2}$

$l \propto L^{2/7}$

$$\left(\frac{l_1}{l_2}\right) = \left(\frac{L_1}{L_2}\right)^{2/7} = 100^{2/7} = 3{,}73$$